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An example of a nonlinear second order elliptic system in three dimension

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Abstract. We provide an explicit example of a nonlinear second order elliptic system of two equations in three dimension to compare two $C^{0,\gamma}$-regularity theories. We show that, for certain range of parameters, the theory developed in *Daněček, Nonlinear Differential Equations Appl.* **9** (2002), gives a stronger result than the theory introduced in *Koshelev, Lecture Notes in Mathematics,* **1614**, 1995. In addition, there is a range of parameters where the first theory gives Hölder continuity of solution for all $\gamma < 1$, while the *Koshelev* theory is not applicable at all.

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1. Introduction

In this paper we consider a second-order nonlinear elliptic system of the type

$$-D_\alpha \left(A_i^\alpha (Du)\right) = 0, \qquad i = 1, \dots, N,\ N > 1, \tag{1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Here $x = (x_1, \dots, x_n)$, $u : \Omega \to \mathbb{R}^N$, $u(x) = (u^1(x), \dots, u^N(x))$, is a vector-valued function, $Du = (D_1 u, \dots, D_n u)$, $D_\alpha = \partial/\partial x_\alpha$, $\alpha \in \{1, \dots, n\}$. Let further $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^m$ and $B_r(x) = \{y \in \mathbb{R}^m :| y - x |< r\}$, $m \geq 1$, $r > 0$, $x \in \mathbb{R}^m$. Throughout the whole text, we use the summation convention over repeated indices.

The system (1) is considered under the following assumptions:

(H1) $A_i^\alpha(p)$ are continuously differentiable functions in p on $\mathbb{R}^{nN}$ for which $A_i^\alpha(0) = 0$ and

$$\left| \frac{\partial A_i^\alpha}{\partial p_j^\beta}(p) \right| \leq M, \quad \forall\, p \in \mathbb{R}^{nN},\ M > 0,$$

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(H2) there exists $\nu > 0$ such that, for every p, $\xi \in \mathbb{R}^{nN}$,

$$\frac{\partial A_i^\alpha}{\partial p_j^\beta}(p)\, \xi_\alpha^i \xi_\beta^j \geq \nu |\xi|^2.$$

We will consider a weak solution of Dirichlet problem for (1), with fixed boundary function $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, i.e. the function $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, for which

$$\text{(D)} \qquad \begin{cases} \displaystyle\int_\Omega A_i^\alpha(Du) D_\alpha \varphi^i \, dx = 0\,, & \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N), \\ u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N). \end{cases}$$

It is well known that, under the conditions (H1) and (H2), the Dirichlet problem (D) has a unique solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ and it holds

$$\int_\Omega |Du|^2 \, dx \leq \left(\frac{M}{\nu}\right)^2 \int_\Omega |Dg|^2 \, dx.$$

In this paper the regularity of the system means that the weak solutions of (1) belong to $C^{0,\gamma}(\Omega, \mathbb{R}^N)$, $0 < \gamma < 1$.

It is possible to say that the first systematic study of $C^{0,\gamma}$-regularity of the weak solution to (D) can be found in [C2], [Gia] and [Ne], however these results are applicable to the dimensions $n = 2, 3$ and 4 only. Other conditions (e.g. the Liouville condition) guaranteeing smoothness of solutions of nonlinear elliptic systems were studied in [D1], [Ne], [Gia], and for higher order systems, in [BV].

In this paper we construct an example of a nonlinear elliptic system of type (1) for $n = 3$ and $N = 2$ with coefficients

$$\text{(2)} \qquad A_i^\alpha(p) = \Big(a\delta_{ij}\delta_{\alpha\beta} + \delta_{i\alpha}\delta_{j\beta}\, b \arctan(\mu + |p|^2)\Big)\, p_j^\beta\,,$$

where α, $\beta = 1, 2, 3$, i, $j = 1, 2$, the numbers $a > 0$, $b \geq 0$, $\mu \geq 0$ are parameters and δ_{ij} is the Kronecker δ-symbol. In what follows, (Section 2 and 3), we use this example to compare two different regularity results.

The first approach presented here is due to A.I. Koshelev, see [Ko] for detailed information. He showed that, if $M/\nu \leq K(n)$ ($K(n) \searrow 1$ as $n \to \infty$), then all weak solutions of (1) are Hölder continuous with an exponent $\gamma(n)$, $n \geq 3$ such that $0.781 > \gamma(n) \searrow 0$ as $n \to \infty$.

In [D2] it was showed that if the quotient of the coefficient of boundedness M to the coefficient of the positiveness ν of the so-called ellipticity matrix $\mathcal{A} = (\partial A_i^\alpha / \partial p_j^\beta)$ (see the conditions (H1) and (H2) of (1)) is less than or equal to an arbitrary constant $P > 1$, then, for a sufficiently big constant of positiveness ν, the gradient of a weak solution to (1) is from a $\mathcal{L}^{2,n}_{\rm loc}$-space (BMO-space) so that the weak solution is from the Hölder space $C^{0,\gamma}(\Omega, \mathbb{R}^N)$ for every $\gamma < 1$. For a more general result, see [DJS], where the $C^{1,\gamma}$-regularity is proved.

It is worth noticing that, even if the results of [Ko] and [D2] are of a similar nature, they have been proved by different techniques and they do not include each other. Our system does not satisfy the Koshelev sufficient condition while it fulfills the condition of [D2]. It is also necessary to remark that the result from [D2] can be proved in a way simpler than that contained in [Ko].

For our system, we have the following fundamental result.

Proposition 1. *Let the coefficients of the system* (1) *be given by* (2). *Then the conditions* (H1) *and* (H2) *hold with*

$$M = a\left[1 + \left(\frac{\pi}{2} + 3\vartheta(\mu)\right) r\right], \qquad \nu = a\left[1 - 4\vartheta(\mu) r\right] \tag{3}$$

where

$$r = \frac{b}{a}, \qquad \vartheta(\mu) = \frac{1}{2\left(\mu + \sqrt{1+\mu^2}\right)}.$$

Remark 1. As the matrix of ellipticity $\mathcal{A}$ is not symmetric, the weak solution of (1) with (2) is not the minimiser of any functional.

Proof: In our case, the coefficients of the matrix of ellipticity $\mathcal{A}$ are of the following form

$$\frac{\partial A_i^\alpha}{\partial p_j^\beta}(p) = A_{ij}^{\alpha\beta}(p) = a\left(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\alpha}\delta_{j\beta}T(p) + \delta_{i\alpha}p_j^\beta L(p)\right)$$

where

$$T(p) = r \arctan\left(\mu + |p|^2\right), \qquad L(p) = \frac{2(p_1^1 + p_2^2)}{1 + (\mu + |p|^2)^2}\, r\,.$$

Now we can prove (H2) for all p, $\xi \in \mathbb{R}^6$.

$$\begin{aligned} A_{ij}^{\alpha\beta}(p)\xi_\alpha^i\xi_\beta^j &= a\left[|\xi|^2 + T(p)(\xi_1^1 + \xi_2^2)^2 + L(p)p_j^\beta\xi_\beta^j(\xi_1^1 + \xi_2^2)\right] \\ &\geq a\left[|\xi|^2 + L(p)p_j^\beta\xi_\beta^j(\xi_1^1 + \xi_2^2)\right] \geq a\left(1 - \frac{4|p|^2}{1 + (\mu + |p|^2)^2}\, r\right)|\xi|^2 \\ &\geq a\left(1 - \frac{4\sqrt{1+\mu^2}}{1 + \left(\mu + \sqrt{1+\mu^2}\right)^2}\, r\right)|\xi|^2 = a\left(1 - 4\vartheta(\mu) r\right)|\xi|^2. \end{aligned}$$

Next, we will prove the boundedness $A_{ij}^{\alpha\beta}$ for all $p \in \mathbb{R}^6$ as follows

$$|A_{ij}^{\alpha\beta}(p)| = a\left|\delta_{ij}\delta_{\alpha\beta} + \delta_{i\alpha}\delta_{j\beta}T(p) + \delta_{i\alpha}p_j^\beta L(p)\right| \le a\,(1 + |T(p)| + |p||L(p)|)$$
$$\le a\left(1 + \frac{\pi}{2}r + \frac{2\sqrt{2}|p|^2}{1+(\mu+|p|^2)^2}r\right) < a\left[1 + \left(\frac{\pi}{2} + 3\vartheta(\mu)\right)r\right] = M.$$

□

2. Koshelev's approach to regularity

Following the idea of [Ko], we can decompose the ellipticity matrix $\mathcal{A} = \left(\partial A_i^\alpha/\partial p_j^\beta\right)$, $\alpha, \beta = 1,2,3$, $i,j = 1,2$ as

$$\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^- = \frac{1}{2}\left(\mathcal{A} + \mathcal{A}^T\right) + \frac{1}{2}\left(\mathcal{A} - \mathcal{A}^T\right)$$

where $\mathcal{A}^+$and $\mathcal{A}^-$ are symmetric and skew-symmetric parts of $\mathcal{A}$, respectively.

We denote by $\lambda_i(p)$ the eigenvalues of the matrix $\mathcal{A}^+$ and by $\sigma_i(p)$ the eigenvalues of the matrix
$$\mathcal{Q} = \mathcal{A}^+\mathcal{A}^- - \mathcal{A}^-\mathcal{A}^+ - (\mathcal{A}^-)^2.$$

Denote

$$K^2 = \begin{cases} \dfrac{\sigma}{\lambda^2+\sigma}, & \sigma \ge \frac{1}{2}\lambda\,(\Lambda - \lambda)\,, \\ \dfrac{(\Lambda-\lambda)^2 + 4\sigma}{(\Lambda+\lambda)^2}, & \sigma \le \frac{1}{2}\lambda\,(\Lambda-\lambda) \end{cases} \tag{4}$$

where $\lambda = \inf_{i,p}\lambda_i(p)$, $\Lambda = \sup_{i,p}\lambda_i(p)$ and $\sigma = \sup_{i,p}\sigma_i(p)$.

We define the function

$$H(\gamma) = \sqrt{\frac{3+2\gamma}{2-\gamma-2\gamma^2}}, \quad \gamma \in I = (\tfrac{1}{2}, \tfrac{\sqrt{17}-1}{4}).$$

Remark 2. It is well known that, under the conditions (H1) and (H2), the weak solutions of (1) belong to $W^{2,2+\eta}_{\text{loc}}(\Omega, \mathbb{R}^N)$ (for some small $\eta > 0$) and, in our case $n = 3$, we have $W^{2,2+\eta}_{\text{loc}} \hookrightarrow C^{0,(1+\eta)/2}$ (see [C1], [Gia] and [Ne]). We can also consider $\gamma > 1/2$ only.

Now we rewrite Koshelev's theorem for dimension $n = 3$. For detailed information see [Ko].

Theorem A (see [Ko], p. 53)**.** *Let the nonlinear system* (1) *satisfy the conditions* (H1) *and* (H2)*. If the inequality*

$$K\,H\,(\gamma) < 1$$

holds for some $\gamma \in I$, then the weak solution of the system (1) *belongs to the Hölder space $C^{0,\gamma}(\overline{\Omega}_0, \mathbb{R}^2)$ where $\Omega_0 \subset\subset \Omega$ is arbitrary.*

For our system, we have

$$\begin{aligned} \left(A^+(p)\right)_{ij}^{\alpha\beta} &= a\left(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\alpha}\delta_{j\beta}T(p) + \frac{1}{2}(\delta_{i\alpha}p_j^\beta + \delta_{j\beta}p_i^\alpha)L(p)\right), \\ Q_{ij}^{\alpha\beta}(p) &= \frac{1}{4}a^2L^2(p)\left[\delta_{i\alpha}\left(\delta_{j\beta}|p|^2 - p_j^\beta(p_1^1+p_2^2)\right) + p_i^\alpha\left(2p_j^\beta - \delta_{j\beta}(p_1^1+p_2^2)\right)\right]. \end{aligned}$$

As it is difficult to establish the exact values of the constants Λ, λ, σ, we will look for their interval estimates. In the following, we will assume that

$$\Lambda \in [\Lambda_M, \overline{\Lambda}]\,, \qquad \lambda \in [\overline{\lambda}, \lambda_M]\,, \qquad \sigma \in [\sigma_M, \overline{\sigma}] \tag{5}$$

where $0 < \overline{\lambda} \le \lambda_M \le \Lambda_M \le \overline{\Lambda}$ and $\overline{\sigma} \ge \sigma_M \ge 0$. We define the following constants

$$\overline{K}^2 = \begin{cases} (\overline{\lambda}^2 + 1)\,\overline{\sigma}\,, & \sigma_M \ge \frac{1}{2}\lambda_M\left(\overline{\Lambda} - \overline{\lambda}\right)\,, \\ \dfrac{(\overline{\Lambda} - \overline{\lambda})^2 + 4\overline{\sigma}}{(\Lambda_M + \overline{\lambda})^2}\,, & \overline{\sigma} \le \frac{1}{2}\overline{\lambda}\left(\Lambda_M - \lambda_M\right)\,. \end{cases}$$

$$K_M^2 = \begin{cases} \dfrac{\sigma_M}{\lambda_M^2 + \sigma_M}\,, & \sigma_M \ge \frac{1}{2}\lambda_M(\overline{\Lambda} - \overline{\lambda})\,, \\ \dfrac{(\Lambda_M - \lambda_M)^2 + 4\sigma_M}{(\overline{\Lambda} + \lambda_M)^2}\,, & \overline{\sigma} \le \frac{1}{2}\overline{\lambda}\left(\Lambda_M - \lambda_M\right)\,. \end{cases} \tag{6}$$

Now we can formulate

Corollary A. *Let the nonlinear system* (1) *satisfy the conditions* (H1), (H2) *and, for Λ, λ, and σ, the interval estimates* (5) *hold.*

(a) *If the inequality*

$$\overline{K}\cdot H(\gamma) < 1\,,$$

holds for some $\gamma \in I$, then the weak solution of (1) *belongs to the Hölder space $C^{0,\gamma}(\overline{\Omega}_0, \mathbb{R}^2)$, $\overline{\Omega}_0 \subset \Omega$.*

(b) *If*

$$K_M \ge \tfrac{1}{2}$$

holds, then Theorem A does not guarantee any regularity.

Proof: Let, for λ, Λ and σ, the assumptions (5) be satisfied.

Let

$$\sigma_M \geq \tfrac{1}{2}\lambda_M\left(\overline{\Lambda} - \overline{\lambda}\right).$$

From (5) we have $\lambda(\Lambda-\lambda)/2 \leq \lambda_M(\overline{\Lambda}-\overline{\lambda})/2 \Rightarrow \sigma \geq \sigma_M \geq \lambda_M(\overline{\Lambda}-\overline{\lambda})/2 \geq \lambda(\Lambda-\lambda)/2 \Rightarrow \sigma \geq \lambda(\Lambda-\lambda)/2$. Further

$$K_M^2 = \frac{\sigma_M}{\lambda_M^2+\sigma_M} = \frac{1}{1+\frac{\lambda_M^2}{\sigma_M}} \leq \frac{1}{1+\frac{\lambda^2}{\sigma}} = \frac{\sigma}{\lambda^2+\sigma} = K^2.$$

From $K_M \geq 1/2$, it follows that $KH(\gamma) \geq H(\gamma)/2 \geq 1$ and, by Theorem A, the regularity a of weak solution is not guaranteed.

We have

$$\overline{K}^2 = \frac{\overline{\sigma}}{\overline{\lambda}^2+\overline{\sigma}} = \frac{1}{1+\frac{\overline{\lambda}^2}{\overline{\sigma}}} \geq \frac{1}{1+\frac{\lambda^2}{\sigma}} = \frac{\sigma}{\lambda^2+\sigma} = K^2.$$

From $\overline{K}H(\gamma) < 1$ it follows that $KH(\gamma) < 1$ and, by Theorem A, the regularity of a weak solution is guaranteed.

Let

$$\overline{\sigma} \leq \tfrac{1}{2}\overline{\lambda}\left(\Lambda_M - \lambda_M\right).$$

From (5) we have $\lambda(\Lambda-\lambda)/2 \geq \overline{\lambda}(\Lambda_M-\lambda_M)/2 \Rightarrow \sigma \leq \overline{\sigma} \leq \overline{\lambda}(\Lambda_M-\lambda_M)/2 \leq \lambda(\Lambda-\lambda)/2 \Rightarrow \sigma \leq \lambda(\Lambda-\lambda)/2$. Further

$$K_M^2 = \frac{(\Lambda_M-\lambda_M)^2+4\sigma_M}{(\overline{\Lambda}+\lambda_M)^2} \leq \frac{(\Lambda-\lambda)^2+4\sigma}{(\Lambda+\lambda)^2} = K^2.$$

From $K_M \geq 1/2$, it follows that $KH(\gamma) \geq H(\gamma)/2 \geq 1$ and, by Theorem A, the regularity of a weak solution is not guaranteed.

We have

$$\overline{K}^2 = \frac{(\overline{\Lambda}-\overline{\lambda})^2+4\overline{\sigma}}{(\Lambda_M+\overline{\lambda})^2} \geq \frac{(\Lambda-\lambda)^2+4\sigma}{(\Lambda+\lambda)^2} = K^2.$$

From $\overline{K}H(\gamma) < 1$ it follows, that $KH(\gamma) < 1$ and by Theorem A the regularity of a weak solution is guaranteed. □

Proposition 2. *The numbers λ, Λ, and σ satisfy the following inequalities*

$$\begin{aligned}
&a\,(1+2r\arctan\mu) = \Lambda_M \leq \Lambda \leq \overline{\Lambda} \\
&\quad = a\left[1+\left(\frac{\pi}{2}+2\vartheta(\mu)+\frac{1}{2}\sqrt{\pi^2+8\pi\vartheta(\mu)+8s(\mu)}\right)r\right], \\
&a\left[1+\left(\arctan\mu-\frac{1}{2}\sqrt{\pi^2+8\pi\vartheta(\mu)+8s(\mu)}\right)r\right] = \overline{\lambda} \leq \lambda \leq \lambda_M = a,
\end{aligned}$$

$$0 = \sigma_M \leq \sigma \leq \overline{\sigma} = 2b^2 s(\mu)$$

where $s(\mu) = (1+\mu^2)/[1+(\mu+\sqrt{1+\mu^2})^2]^2$.

PROOF: The eigenvalues of the matrices $\mathcal{A}^+$ and $\mathcal{Q}$ are real because $\mathcal{A}^+$ and $\mathcal{Q}$ are symmetric. The eigenvalues of the matrix $\mathcal{A}^+$ are $\lambda_{1,2,3,4} = a$ and $\lambda_{5,6}$

$$a\left(1 + T(p) + \frac{1}{2}(p_1^1 + p_2^2)L(p) \pm \frac{1}{2}\sqrt{4T^2(p) + 4(p_1^1+p_2^2)T(p)L(p) + 2|p|^2L^2(p)}\right).$$

Further, we can estimate the eigenvalues $\lambda_{5,6}$ by means of the following inequalities

$$r\arctan\mu \le T(p) \le \frac{\pi}{2}r\,, \quad 0 \le \frac{1}{2}(p_1^1+p_2^2)L(p) \le \frac{2|p|^2}{1+(\mu+|p|^2)^2}r \le 2r\vartheta(\mu)\,,$$
$$4(p_1^1+p_2^2)T(p)L(p) \le 8\pi r^2\vartheta(\mu)\,, \quad 2|p|^2L^2(p) \le \frac{8|p|^4}{(1+(\mu+|p|^2)^2)^2}r^2 \le 8r^2 s(\mu)$$

and we get the values $\overline{\lambda}$ and $\overline{\Lambda}$.

If we, for simplicity, choose $\tilde{p} = (0,0,0,0,0,0)$, we have the matrix $\mathcal{A}^+$ depending on the parameters a, b, μ and we get

$$\left(A^+(\tilde{p})\right)_{ij}^{\alpha\beta} = a\left(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\alpha}\delta_{j\beta}\, r\arctan\mu\right)\,.$$

The eigenvalues of $\mathcal{A}^+(\tilde{p})$ are $\lambda_{1,2,3,4,5} = a$, $\lambda_6 = a\,(1+2r\arctan\mu)$ and we have

$$\lambda_M = a, \qquad \Lambda_M = a\,(1+2r\arctan\mu)\,.$$

The matrix $\mathcal{Q}$ has the eigenvalues $\lambda_{1,2,3,4} = 0$ and

$$\lambda_{5,6} = \frac{a^2}{4}\left(2|p|^2 - (p_1^1+p_2^2)^2\right)L^2(p).$$

Also

$$\sigma \le \sup_{i,p}\sigma_i(p) = \sup_p \frac{a^2}{4}\left(2|p|^2 - (p_1^1+p_2^2)^2\right)L^2(p)$$
$$\le 2a^2r^2\sup_p \frac{|p|^4}{(1+(\mu+|p|^2)^2)^2} \le 2b^2 s(\mu).$$

□

Proposition 3. *We put*

$$\mathcal{M} = \Big\{(r,\mu) \in [0,\infty)\times[0,\infty) : r < \frac{1}{4\vartheta(\mu)},$$
$$Z_1(r,\mu) = [1 + (\arctan\mu - h(\mu))\,r]\arctan\mu - 2rs(\mu) \ge 0\Big\}\,,$$

$$\mathcal{M}_K = \{(r,\mu,\gamma) \in \mathcal{M} \times I :$$
$$\left. Z_2(r,\mu)H(\gamma) = \frac{r\sqrt{\left[\frac{\pi}{2} - \arctan\mu + 2\vartheta(\mu) + 2h(\mu)\right]^2 + 18\vartheta^2(\mu)}}{2 + [3\arctan\mu - h(\mu)]\, r} H(\gamma) < 1 \right\},$$

$$\mathcal{M}_N = \left\{ (r,\mu) \in \mathcal{M} : Z_3(r,\mu) = \frac{r\arctan\mu}{1 + \frac{1}{2}\left[\frac{\pi}{2} + 2\vartheta(\mu) + h(\mu)\right] r} > \frac{1}{2} \right\}$$

where $h(\mu) = \frac{1}{2}\sqrt{\pi^2 + 8\pi\vartheta(\mu) + 8s(\mu)}$*. Then*

(a) $\mathcal{M} \neq \emptyset$,

(b) $\mathcal{M}_K \neq \emptyset$ *and, for each* $(r,\mu,\gamma) \in \mathcal{M}_K$*, a weak solution of* (1) *with coefficients* (2) *belongs to* $C^{0,\gamma}(\overline{\Omega}_0,\mathbb{R}^2)$, $\overline{\Omega}_0 \subset \Omega$,

(c) $\mathcal{M}_N \neq \emptyset$ *and, for each* $(r,\mu) \in \mathcal{M}_N$*, Koshelev's theorem does not guarantee any regularity.*

Proof: (a) The definition of the set $\mathcal{M}$ follows from the condition of ellipticity, Proposition 2, and the condition (see (6))

$$\overline{\sigma} \leq \tfrac{1}{2}\overline{\lambda}\left(\Lambda_M - \lambda_M\right).$$

The parameters $(r,\mu) \in \mathcal{M}$ for every $\mu \geq 0$ and arbitrary

$$0 \leq r \leq W(\mu) = \min\left\{ \frac{1}{4\vartheta(\mu)}, \frac{\arctan\mu}{2s(\mu) + [h(\mu) - \arctan\mu]\arctan\mu} \right\}.$$

(b) Taking into account that $\lim_{\mu\to\infty}\vartheta(\mu) = \lim_{\mu\to\infty} s(\mu) = 0$, $\lim_{\mu\to\infty} h(\mu) = \pi/2$ and $\lim_{\mu\to\infty} W(\mu) = \infty$, we have

$$\lim_{\mu\to\infty} Z_2(r,\mu)H(\gamma) = \frac{\pi}{\frac{2}{r} + \pi} H(\gamma), \quad \forall\, \gamma \in I.$$

Now, for every $r < \frac{2}{\pi[H(\gamma)-1]}$, there is $\mu_0(r) > 0$ such that $(r,\mu) \in \mathcal{M}_K$ for every $\mu > \mu_0$.

(c) In a way similar to (b) we have

$$\lim_{\mu\to\infty} Z_3\,(r,\mu) = \lim_{\mu\to\infty} \frac{2\arctan\mu}{\frac{2}{r} + \frac{\pi}{2} + 2\vartheta(\mu) + h(\mu)} = \frac{\pi}{\frac{2}{r} + \pi}.$$

For each $r > 2/\pi$, there is $\mu_1(r) > 0$ such that, for each $\mu > \mu_1$, we get

$$Z_3(r,\mu) > \tfrac{1}{2} \;\Rightarrow\; (r,\mu) \in \mathcal{M}_N.$$

$\square$

3. Another approach to regularity

Now we will study the problem of regularity from [D2] and for more general result, see [DJS]. In the sequel, we need a slightly stronger version of (H1):

(H1*) includes condition (H1) but, moreover, we assume that the derivatives $\partial A_i^\alpha / \partial p_j^\beta$ are uniformly continuous.

On the basis of assumption (H1*), we can define on $[0, \infty)$ a real function ω (modulus of continuity) as follows

$$\omega(t) = \sup_{i,j,\alpha,\beta} \sup_{\substack{|p-q|^2 \le t \\ p,q \in \mathbb{R}^{nN}}} \left| \frac{\partial A_i^\alpha(p)}{\partial p_j^\beta} - \frac{\partial A_i^\alpha(q)}{\partial p_j^\beta} \right| .$$

From the assumption (H1*) and the definition of the function ω, it follows that ω is continuous, nondecreasing, bounded, and $\omega(0) = 0$. We can moreover suppose that the function ω is concave and absolutely continuous on every compact subinterval of $(0, \infty)$.

Theorem B ([D2]). *For every $P > 1$ and $L > 0$, there exists $\nu_0 = \nu_0(n, P, L) > 0$ such that, for every $\overline{\Omega}_0 \subset \Omega$, every nonlinear system* (1) *satisfying the hypotheses* (H1*), (H2) *such that $\nu \ge \nu_0$, $M/\nu \le P$, $\sup_{t>0}(\omega(t)\,\omega'(t)) \le P\nu$ and, if every weak solution u for which $\|Du\|^2_{L^2(\Omega,\mathbb{R}^{nN})}/[\operatorname{dist}(\Omega_0, \partial\Omega)]^n \le L$, we have*

(a) $Du \in \mathcal{L}^{2,n}(\Omega_0, \mathbb{R}^{nN})$,

(b) *the estimate*

$$[Du]_{\mathcal{L}^{2,n}(\Omega_0,\mathbb{R}^{nN})} \le c\,(n, P, L, \operatorname{dist}(\Omega_0, \partial\Omega))$$

holds.

Corollary B. *Let the assumptions of Theorem B be satisfied. Then $u \in C^{0,\gamma}(\overline{\Omega}_0, \mathbb{R}^N)$ for every $\gamma < 1$.*

Proof: See [C1]. □

To apply Theorem B to our system we need the following two Lemmas.

Lemma 1. *For the system* (1) *with coefficients* (2) *we have*

$$\left| \frac{\partial A_i^\alpha(q_1)}{\partial p_j^\beta} - \frac{\partial A_i^\alpha(q_2)}{\partial p_j^\beta} \right| \le \overline{\omega}_\infty = b\left(\frac{\pi}{2} - \arctan\mu + 6\vartheta(\mu)\right), \qquad \forall\, q_1, q_2 \in \mathbb{R}^6,$$

$$\sup_{q \in \mathbb{R}^6} \left| DA_{ij}^{\alpha\beta}(q) \right| \le C(b, \mu) = 72\, b\, f(\mu)$$

where

$$f(\mu)=\frac{4\sqrt{\sqrt{3+4\mu^2}-\mu}}{9+(2\mu+\sqrt{3+4\mu^2})^2}+\frac{(1+\mu)(\mu+\sqrt{3+4\mu^2})^{3/2}}{(1+\mu^2)\left[1+(2\mu+\sqrt{3+4\mu^2})^2\right]}\,.$$

Proof: The first inequality follows easily:

$$\left|A_{ij}^{\alpha\beta}(p)-A_{ij}^{\alpha\beta}(q)\right|\le a\,|T(p)-T(q)|+a\left|p_j^{\beta}L(p)-q_j^{\beta}L(q)\right|$$
$$\le b\left(\arctan\frac{1}{\mu}+\frac{2\sqrt{2}|p|^2}{1+(\mu+|p|^2)^2}+\frac{2\sqrt{2}|q|^2}{1+(\mu+|q|^2)^2}\right)\le b\left(\frac{\pi}{2}-\arctan\mu+6\vartheta(\mu)\right).$$

Now we estimate the second inequality. For $\alpha,\beta=1,2,3$, $i,j=1,2$, the vectors $DA_{ij}^{\alpha\beta}(p)=(\partial A_{ij}^{\alpha\beta}/\partial p_k^{\gamma})$, $\gamma=1,2,3$, $k=1,2$ have the following components

$$\frac{\partial A_{ij}^{\alpha\beta}(p)}{\partial p_k^{\gamma}}=a\delta_{i\alpha}\left(\delta_{j\beta}\frac{2rp_k^{\gamma}}{1+w^2}+\delta_{jk}\delta_{\beta\gamma}L(p)+2rp_j^{\beta}\frac{\delta_{k\gamma}(1+w^2)-4w(p_1^1+p_2^2)p_k^{\gamma}}{(1+w^2)^2}\right)$$

where $w=\mu+|p|^2$. These components can be estimated as follows:

$$\begin{aligned}\sup_p\left|\frac{\partial A_{ij}^{\alpha\beta}(p)}{\partial p_k^{\gamma}}\right|&\le 2b\left(\sup_p\frac{|p|}{1+w^2}+\frac{1}{2}\sup_p L(p)+\sup_p\frac{|p|(1+w^2)+4\sqrt{2}w|p|^3}{(1+w^2)^2}\right)\\&\le 2b\left((2+\sqrt{2})\sup_p\frac{|p|}{1+w^2}+4\sqrt{2}\sup_p\frac{w|p|^3}{(1+w^2)^2}\right)\\&<4b\left(2\sup_p V_1(p)+3\sup_p V_2(p)\right)\,.\end{aligned}$$

Further

$$\sup_p V_1(p)\le\frac{3\sqrt{3}\sqrt{\sqrt{3+4\mu^2}-\mu}}{9+(2\mu+\sqrt{3+4\mu^2})^2}\,,$$
$$\sup_p V_2(p)\le\left(\sup_p\frac{w}{1+w^2}\right)\left(\sup_p\frac{|p|^3}{1+w^2}\right)\le\frac{1+\mu}{1+\mu^2}\,\frac{(\mu+\sqrt{3+4\mu^2})^{3/2}}{1+(2\mu+\sqrt{3+4\mu^2})^2}$$

and together we obtain

$$\sup_p\left|\frac{\partial A_{ij}^{\alpha\beta}(p)}{\partial p_k^{\gamma}}\right|$$
$$\le 12b\left(\frac{4\sqrt{\sqrt{3+4\mu^2}-\mu}}{9+(2\mu+\sqrt{3+4\mu^2})^2}+\frac{(1+\mu)(\mu+\sqrt{3+4\mu^2})^{3/2}}{(1+\mu^2)\left[1+(2\mu+\sqrt{3+4\mu^2})^2\right]}\right)\,.$$

□

Lemma 2. *We define the function $\tilde{\omega}$ as follows:*

$$\tilde{\omega}(t) = \begin{cases} C\sqrt{t}, & t \in [0, \left(\frac{\overline{\omega}_\infty}{C}\right)^2], \\ \overline{\omega}_\infty + \delta + \dfrac{2\overline{\omega}_\infty \delta^2}{\overline{\omega}_\infty^2 - 2\overline{\omega}_\infty \delta - C^2 t}, & t \in [\left(\frac{\overline{\omega}_\infty}{C}\right)^2, \infty) \end{cases}$$

where C, $\overline{\omega}_\infty$ are the constants from Lemma 1 and $0 < \delta \leq \overline{\omega}_\infty$ is an arbitrary real number. Then the function $\tilde{\omega}$ has all properties stated before Theorem B, $\omega(t) \leq \tilde{\omega}(t)$ for all $t \in (0, \infty)$ and

$$\sup_{t>0} [\tilde{\omega}(t)\tilde{\omega}'(t)] \leq C^2(b, \mu). \tag{7}$$

Proof: We will prove only property (7). The other properties easily follow from the definition of the function $\tilde{\omega}$. From Lemma 1 and the mean value theorem, we get for $t \in [0, (\overline{\omega}_\infty/C)^2]$

$$\omega(t) = \sup_{i,j,\alpha,\beta} \sup_{\substack{|p-q|^2 \leq t \\ p,q \in \mathbb{R}^{nN}}} |A_{ij}^{\alpha\beta}(p) - A_{ij}^{\alpha\beta}(q)| \leq \sup_{\substack{|p-q|^2 \leq t \\ p,q \in \mathbb{R}^{nN}}} \left(\sup_{\substack{\xi \in \mathbb{R}^6 \\ i,j,\alpha,\beta}} |DA_{ij}^{\alpha\beta}(\xi)||p-q| \right)$$
$$\leq C\sqrt{t} = \tilde{\omega}(t)$$

and, for $t > (\overline{\omega}_\infty/C)^2$, the inequalities $\omega(t) \leq \tilde{\omega}(t)$ and (7) follow easily from the definitions of the functions ω and $\tilde{\omega}$. □

Now we can formulate the main result

Theorem. *Let $\overline{\Omega}_0 \subset \Omega \subset \mathbb{R}^3$ and $P > 3$, $L > 0$ be constants. There are parameters $(r, \mu) \in \mathcal{M}_N$ such that the weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ of Dirichlet problem* (D), *$\|Dg\|_{L^2(\Omega,\mathbb{R}^2)}/[\operatorname{dist}(\Omega_0, \partial\Omega)]^3 \leq L$ for* (1) *with coefficients* (2) *and the parameters r, μ, belongs to $C^{0,\gamma}(\overline{\Omega}_0, \mathbb{R}^2)$ for every $\gamma < 1$.*

Proof: It is sufficient to verify the assumptions of Theorem B for some parameters $(b/a, \mu) = (r, \mu) \in \mathcal{M}_N$. By Proposition 3, we have that $(1, \mu) \in \mathcal{M}_N$ for every $\mu > \tilde{\mu}$.

From Proposition 1, the assumptions of Theorem B, and Lemma 2, it follows

$$\nu \geq \nu_0 \iff a \geq \nu_0 + 4\vartheta(\mu)\, b\,,$$
$$\frac{M}{\nu} \leq P \iff a \geq \frac{\frac{\pi}{2} + (3+4P)\vartheta(\mu)}{P-1}\, b\,,$$
$$\sup_{t>0}[\tilde{\omega}(t)\tilde{\omega}'(t)] \leq P\nu \iff a \geq \left(\frac{72 f(\mu)}{\sqrt{P}}\right)^2 b^2 + 4\vartheta(\mu)\, b.$$

Now for each $a > \nu_0$, taking into consideration that $a = b$, there is $\overline{\mu} > \tilde{\mu}$ such that the preceding three inequalities on the right hand sides are satisfied for all $\mu > \overline{\mu}$. $\square$

Concluding remarks. If Theorems A and B are applicable simultaneously ($\mathcal{M} \neq \emptyset$ must hold), then Theorem A is giving us only $C^{0,\gamma}$-regularity ($1/2 < \gamma < (\sqrt{17} - 1)/4 < 0.781$, and it is necessary to recall that $H(\gamma) \to \infty$ as $\gamma \to (\sqrt{17} - 1)/4$) while, in the case of Theorem B, we have $\mathcal{L}^{2,n}$-regularity of the gradient from which follows $C^{0,\gamma}$-regularity for all $\gamma < 1$ by Corollary B. But it may be useful to remember that in Theorem B, we need a smallness of the norm of the gradient of a weak solution opposite to Theorem A.

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