Regular potentials of additive functionals in semidynamical systems

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Abstract. We consider a semidynamical system $(X, \mathcal{B}, \Phi, w)$. We introduce the cone $\mathbb{A}$ of continuous additive functionals defined on X and the cone $\mathcal{P}$ of regular potentials. We define an order relation "$\leq$" on $\mathbb{A}$ and a specific order "$\prec$" on $\mathcal{P}$. We will investigate the properties of $\mathbb{A}$ and $\mathcal{P}$ and we will establish the relationship between the two cones.

Keywords: additive functional, excessive functions, regular potential, semidynamical system, specific order

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1. Introduction

Many applications involve semidynamical systems in non locally compact infinite dimensional spaces, for example semidynamical systems generated by partial differential equations.
So starting from a semidynamical system $(X, \mathcal{B}, \Phi, w)$ (cf. [3], [6] and [13]), we associate the concepts of additive functionals and regular potentials with respect to the inherent topology $\mathcal{T}_\Phi^0$ defined on $(X, \mathcal{B}, \Phi, w)$ (cf. [3] and [12]). Note that the space X is not assumed to be an artificial topological space (L.C.D or not) nor a Radonian space (cf. [14]) and that the inherent topology is not in general locally compact neither having a countable base (cf. [2]). Indeed, we assume only that $(X, \mathcal{B})$ is a separable measurable space and that the semidynamical system $(X, \mathcal{B}, \Phi, w)$ is transient.
The concepts used in this paper were already introduced in the case of a standard Markov Process $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \Theta_t, P^x)$ with state space $(E, \mathcal{E})$ which is locally compact with countable base (cf. [7]).
It is worth mentioning that there is correlation between the inherent topology $\mathcal{T}_\Phi^0$ and the continuity of additive functionals.

In the preliminary, we will introduce preliminary material and we will establish some results that will be used in this paper, particularly the fine topology $\mathcal{T}_\Phi$ and the inherent topology $\mathcal{T}_\Phi^0$ which will be used extensively in the sequel.
We will give the definition of additive functionals and regular potentials defined

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on $(X, \mathcal{B}, \Phi, w)$, then we will illustrate with some examples. We show particularly that any continuous additive functional gives rise to a regular potential on X_0 (Theorem 3.1) and conversely, every regular potential is associated to a continuous additive functional (Theorem 3.2). In Section 4 we will introduce an order relation on the cone $\mathbb{A}$ of continuous additive functionals, then we will prove in Theorem 4.1 that two elements $\mathcal{A}, \mathcal{B} \in \mathbb{A}$ are comparable if and only if their associated potentials are comparable with respect to the specific order defined on $\mathcal{E}(\Lambda)$. Moreover we will show in Theorem 4.5 and Corollary 4.2 that the Riesz decomposition holds in the cone of regular potentials $\mathcal{P}$ with respect to the natural and specific order and holds in $\mathbb{A}$ (Theorem 4.6). Also, we show in Theorem 4.5 and 4.6 that for any bounded increasing family $(f_i)_i$ in $(\mathcal{P}, \prec)$ $((\mathcal{A}^i)_i$ in $(\mathbb{A}, \leq)$ resp.) we have $\sup_i f_i \in \mathcal{P}$ ($\sup_i \mathcal{A}^i \in \mathbb{A}$ resp.). Similarly, for a decreasing family $(f_i)_i$ in $(\mathcal{P}, \prec)$ we show that $\wedge_i f_i \in \mathcal{P}$ and that for any decreasing family $(\mathcal{A}^i)_i$ in $(\mathbb{A}, \leq)$ the element $\wedge_i \mathcal{A}^i$ which is the greatest lower bound in $\mathbb{A}$ exists.

2. Preliminary

Definition 2.1. Let $(X, \mathcal{B})$ be a separable measurable space with a distinguished point ω. A measurable map $\Phi : \mathbb{R}_+ \times X \longrightarrow X$ is called a *semidynamical system with cofinal point* ω if the following conditions are fulfilled:

(S_1) for any x in X, there exists an element $\rho(x)$ in $[0, \infty]$ such that $\Phi(t, x) \neq \omega$ for all $t \in [0, \rho(x))$ and $\Phi(t, x) = \omega$ for all $t \geq \rho(x)$,

(S_2) for any $s, t \in \mathbb{R}_+$ and any $x \in X$ we have

$$\Phi(s, \Phi(t, x)) = \Phi(s + t, x),$$

(S_3) $\Phi(0, x) = x$ for all $x \in X$,

(S_4) if $\Phi(t, x) = \Phi(t, y)$ for all $t > 0$, then $x = y$.

Note that ρ is called the life time of the semidynamical system $(X, \mathcal{B}, \Phi, \omega)$. Next, we will denote by $X_0 = X \setminus \{w\}$ and by $\mathcal{B}_0$ the trace of the σ-algebra $\mathcal{B}$ on X_0. For any $x \in X_0$ we denote by Γ_x the trajectory of x, i.e.:

$$\Gamma_x = \{\Phi(t, x); t \in [0, \rho(x))\}$$

and we define the function Φ_x on $[0, \rho(x))$ by $\Phi_x(t) = \Phi(t, x)$. So for any $x, y \in X_0$ we put

$$x \underset{\Phi}{\leq} y \Leftrightarrow y \in \Gamma_x.$$

A maximal trajectory is a totally ordered subset Γ of $X \setminus \{\omega\}$ with respect to the above order, such that there is no $x_0 \in X_0 \setminus \Gamma$ which is minorant of Γ and such for any $x \in \Gamma$, we have $\Gamma_x \subset \Gamma$.

In what follows, we shall suppose that $(X, \mathcal{B}, \Phi, \omega)$ is a transient semidynamical system (cf. [3], [11]). It is proved that the map Φ_x is a measurable isomorphism between $[0, \rho(x))$ and Γ_x endowed with trace measurable structures.

Let Λ be the Lebesgue measure associated with the semidynamical system $(X, \mathcal{B}, \Phi, \omega)$ given by $\Lambda(A) = \lambda(\Phi_x^{-1}(A))$ for any $x \in X_0$, $A \in \mathcal{B}_0$ and $A \subset \Gamma_x$, where λ is the Lebesgue measure on $\mathbb{R}$ (cf. [4]). We recall (cf. [1]) that in the same way Λ can be defined on the σ-algebra $\mathcal{B}_0(\Lambda)$ which is the set of all subsets A of X_0 such that $A \cap M \in \mathcal{B}_0$ for any countable union M of trajectories of X_0. The resolvent family $\mathbb{V}$ associated to $(X, \mathcal{B}, \Phi, w)$ on $(X_0, \mathcal{B}_0)$ is given by

$$V_\alpha f(x) = \int_0^{\rho(x)} e^{-\alpha t} f(\Phi(t,x))\, dt$$

for any $\mathcal{B}_0$-measurable function f.

We consider also the arrival time function $\Psi : X_0 \times X_0 \longrightarrow \mathbb{R}_+$ given by

$$\Psi(x,y) = \begin{cases} t \ \text{ if } \ \Phi(t,x) = y, \ t \in [0, \rho(x)[\\ +\infty \ \text{ if not} \end{cases}$$

(cf. [6, Chapter III]).
It is shown that the arrival time function Ψ is measurable if we endow $X_0 \times X_0$ with the product measurable structure of the σ-algebra $\mathcal{B}_0(\Lambda)$ (cf. [1], [4], [5]).

For each $x \in X_0$, let us denote by

$$\mathcal{V}_x = \{V \subset X_0 : \exists\, \alpha \in]0, \rho(x)[\ \text{ such that } \ \Phi(t,x) \in V, \ \forall\, t \in [0, \alpha[\}$$

and let $\mathcal{T}_\Phi$ be the topology for which $\mathcal{V}_x$ generates all the neighborhoods of x. This topology is called the fine topology (see [3]).

In the sequel, we define the inherent topology $\mathcal{T}_\Phi^0$ as the set of all subsets D of X_0 satisfying the following condition (see [3], [12]):

$$(\forall\, x \in X_0, \ \forall\, t_0 \in [0, \rho(x)[\ \text{ such that } \ \Phi(t_0, x) \in D)$$
$$(\exists\, \epsilon > 0, \ \text{ such that } \ \forall\, t \in]t_0 - \varepsilon, t_0 + \varepsilon[\cap [0, \rho(x)[, \Phi(t,x) \in D).$$

In the next, let $\mathcal{E}$ be the set of excessive functions on X_0 with respect to $\mathbb{V}$.

By [3], we have that $\mathcal{E}$ is the set of all measurable functions $f : X_0 \to \mathbb{R}_+$ which are nonincreasing with respect to "$\underset{\Phi}{\leq}$" and continuous with respect to $\mathcal{T}_\Phi$.

Remark. A function $f : X_0 \to \mathbb{R}$ is $\mathcal{T}_\Phi$-continuous ($\mathcal{T}_\Phi^0$-continuous resp.) if and only if for each $x \in X_0$, the function $t \to f(\Phi(t,x))$ is right continuous (continuous resp.) on $[0, \rho(x)[$.

Notation. In the sequel, we will denote by $\mathcal{F}(X_0, \Lambda)$ the set of all nonnegative $\mathcal{B}_0(\Lambda)$-measurable functions on X_0.

For any nonnegative $\mathcal{B}_0$-measurable function f on X_0 and $\forall\, \alpha \geq 0$, the restriction of $V_\alpha f$ on Γ_x depends only on the restriction of f on Γ_x and someone can establish the following results (see [5]).

Proposition 2.1. *For any $f \in \mathcal{F}(X_0, \Lambda)$ and any $\alpha \geq 0$ let*

$$\widetilde{V}_\alpha f(x) = \int_0^{\rho(x)} e^{-\alpha t} f(\Phi(t,x))\, dt = \int_{\Gamma_x} e^{-\alpha \Psi(x,y)} G(x,y) f(y)\, d\Lambda(y),$$

where

$$G(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if not.} \end{cases}$$

Then, $\widetilde{V}_\alpha f$ is $\mathcal{B}_0(\Lambda)$-measurable, the family $\widetilde{\mathbb{V}} = (\widetilde{V}_\alpha)_{\alpha \geq 0}$ is a resolvent of kernels on the measurable space $(X_0, \mathcal{B}_0(\Lambda))$ and $\widetilde{\mathbb{V}}$ is an extension of $\mathbb{V}$.

Theorem 2.1. *The set $\mathcal{E}(\Lambda)$ of the $\widetilde{\mathbb{V}}$-excessive functions on $(X_0, \mathcal{B}_0(\Lambda))$ is identical to the set of all positive decreasing functions on X_0 with respect to the order "$\underset{\Phi}{\leq}$", continuous with respect to the fine topology $\mathcal{T}_\Phi$ and finite at the points $x \in X_0$ which are not minimal with respect to the same order.*

Thus, the following result holds.

Proposition 2.2. *Any function $f \in \mathcal{E}(\Lambda)$ is lower semicontinuous with respect to $\mathcal{T}_\Phi^0$.*

PROOF: Since $\widetilde{\mathbb{V}}$ is submarkovian on $(X_0, \mathcal{B}_0(\Lambda))$, by Hunt's approximation theorem (cf. [8]) there exists a sequence $(f_n)_n \in \mathcal{F}(X_0, \Lambda)$ such that

$$\sup_n \widetilde{V}_0 f_n = f.$$

Since $\widetilde{V}_0 f_n$ is $\mathcal{T}_\Phi^0$-continuous (cf. [3]), f is lower semicontinuous with respect to $\mathcal{T}_\Phi^0$. $\square$

Next, we shall prove the following theorem which will be needed later.

Theorem 2.2. *The following properties hold:*

(1) *every open set in $\mathcal{T}_\Phi$ is $\mathcal{B}_0(\Lambda)$-measurable,*
(2) *every decreasing function f with respect to "$\underset{\Phi}{\leq}$" is $\mathcal{B}_0(\Lambda)$-measurable.*

PROOF: (1) Let $O \in \mathcal{T}_\Phi$. Using a result in [3], $\Gamma_x \in \mathcal{T}_\Phi$, we get that $O \cap \Gamma_x \in \mathcal{T}_\Phi$ which means that $\Phi_x^{-1}(O \cap \Gamma_x)$ is an open set with respect to the fine trace topology on $[0, \rho(x)[$. Thus, it is measurable with respect to trace Borel σ-algebra. Using the fact that Φ_x is a measurable isomorphism, we get that $O \cap \Gamma_x \in \mathcal{B}_0$ and therefore $O \in \mathcal{B}_0(\Lambda)$.

(2) The function g defined by $g(t) := f o \Phi_x(t)$ is decreasing on $[0, \rho(x)[$ which is measurable with respect to trace Borel σ-algebra on $[0, \rho(x)[$. Using the fact that Φ_x is a measurable isomorphism, we get that $f = g o \Phi_x^{-1}$ is $\mathcal{B}_0$-measurable and then f is $\mathcal{B}_0(\Lambda)$-measurable. $\square$

In the sequel, the extension $\widetilde{\mathbb{V}}$ will be denoted simply by $\mathbb{V}$.

3. Regular potentials

In this section, let $(X, \mathcal{B}, \Phi, \omega)$ be a fixed data transient semidynamical system and denote by ρ the life time associated defined on X and taking values in $[0, \infty]$.

Definition 3.1. A family $\mathcal{A} = \{A_t, t \in [0, \rho[\}$ of functions defined from X to $[0, +\infty]$ is called an *additive functional of* $(X, \mathcal{B}, \Phi, \omega)$ provided the following conditions are satisfied:

(A_1) for each $x \in X_0$, the mapping : $t \to A_t(x)$ is nondecreasing, right continuous and satisfies $A_0(x) = 0$ for all $x \in X$,

(A_2) for each $t \geq 0$, the mapping $x \to A_t(x)$ is measurable with respect to $\mathcal{B}_0(\Lambda)$,

(A_3) for each $x \in X_0$, $t, s \geq 0$,

$$A_{t+s}(x) = A_t(x) + A_s(\Phi(t, x)),$$

(A_4) $A_t(w) = 0$, $\forall\, t \geq 0$.

If the mapping $t \to A_t$ is continuous, then $\mathcal{A}$ is said to be a *continuous additive functional*.

In the sequel, we assume that the map $t \to A_t(x)$ is continuous.

Notation. We will denote by $\mathbb{A}$ the set of all continuous additive functionals on X.

Remark 3.1. Since the map $t \to A_t$ is increasing, we denote

$$A_t(x) = \lim_{t \to \rho(x)} A_t(x)$$

for all $t \geq \rho(x)$. Thus, we can set $A_\infty(x) = A_{\rho(x)}(x) = \lim_{t \to \rho(x)} A_t(x)$.
For any measurable function f defined on X_0, we set

$$\lim_{t \to \infty} f(\Phi(t, x)) = \lim_{t \to \rho(x)} f(\Phi(t, x))$$

when it exists.

Definition 3.2. Let $\mathcal{A}$ be in $\mathbb{A}$. Then, we define

$$R(x) = \inf\{t : A_t(x) > 0\}$$

provided the set in braces is not empty and $R(x) = \infty$ if it is empty and

$$\varphi^{\mathcal{A}}(x) = 1_{[0, \rho(x)[}(R(x)) e^{-R(x)}.$$

It is obvious that $R(x) = \sup\{t : A_t(x) = 0\}$.

Proposition 3.1. $\varphi^{\mathcal{A}}(x) = e^{-R(x)}$.

Proof: Suppose that $\rho(x) \le R(x) < \infty$. By ($A_4$), we have

$$A_t(\Phi(R(x), x)) = 0$$

for $t \ge \rho(x)$. Hence for each $t \ge 0$, we have

$$A_{t+R(x)}(x) = A_{R(x)}(x) + A_t(\Phi(R(x), x)) = A_{R(x)}(x).$$

On the other hand, by the definition of R, we have that $A_t(x) = 0$ for every $t < R(x)$, which gives us that $A_t(x) = 0$, $\forall\, t \ge 0$. Hence $R(x) = \infty$. □

Definition 3.3. Let $\mathcal{A}$ be an additive functional. Then, we define

$$\operatorname{supp} \mathcal{A} = \{x \in X : \varphi^{\mathcal{A}}(x) = 1\}.$$

Proposition 3.2. *If $\mathcal{A}$ is a continuous additive functional, then*

$$\operatorname{supp} \mathcal{A} = \{x \in X : A_s(x) > 0, \forall\, s > 0\}.$$

Definition 3.4. We say that a real valued map f defined on X_0 is a *Liapunov* (*strict Liapunov* resp.) *function* if f is decreasing (strictly decreasing resp.) and continuous on each trajectory $\Gamma \subset X$ with respect to $\mathcal{T}^0_\Phi$.

Definition 3.5. We say that a real valued map f defined on X_0 is a *regular potential* if f is a Liapunov function such that $\lim_{t\to\infty} f(\Phi(t,x)) = 0$ for every $x \in X_0$. We will denote by $\mathcal{P}$ the set of regular potentials.

Definition 3.6. We say that a function f is a *potential* if $f \in \mathcal{E}(\Lambda)$ and $\lim_{t\to\infty} f(\Phi(t,x)) = 0$ for every $x \in X_0$.

Proposition 3.3. *Let $f \in \mathcal{F}(X_0, \Lambda)$. Then, the potential of f defined by*

$$V_0(f)(x) = \int_0^{\rho(x)} f(\Phi(t,x))\, dt$$

is a Liapunov function when the integral is finite.

Proof: For the proof see Theorem 8 in [3]. □

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{A}$. If $A_\infty < \infty$, then the function A_∞ is a regular potential. Moreover it is a strict Liapunov function on* $\operatorname{supp} A$ *and we have that* $\operatorname{supp} \mathcal{A}$ *is the set of strict monotony of A_∞.*

Proof: Let $x, y \in X$ such that $x \leq y$. Then, there exists $t \geq 0$ such that $y = \Phi(t, x)$. Since

$$A_{s+t}(x) = A_t(x) + A_s(\Phi(t, x)),$$

by letting $s \to \infty$, we get

$$A_\infty(x) = A_t(x) + A_\infty(\Phi(t, x)). \tag{3.1}$$

Hence, we get that

$$A_\infty(y) = A_\infty(\Phi(t, x)) \leq A_\infty(x)$$

and

$$A_\infty(\Phi(t, x)) < A_\infty(x)$$

if and only if $x \in \operatorname{supp} \mathcal{A}$. Hence, A_∞ is decreasing on X_0 and strictly decreasing on $\operatorname{supp} A$. Moreover, by (A_1) and (3.1), we get

$$\lim_{t \to 0} A_\infty(\Phi(t, x)) = A_\infty(x)$$

which yields that A_∞ is right continuous with respect to $\mathcal{T}_\Phi^0$. Now, let us consider $x_0 \in X_0$ not minimal and let $y < x_0$, then there exists $t_0 \in [0, \rho(y)[$ such that $x_0 = \Phi(t_0, y)$. Thus, from (3.1), we get that for every $0 \leq t < t_0$

$$A_\infty(\Phi(t, y)) - A_\infty(\Phi(t_0 - t, \Phi(t, y))) = A_{t_0 - t}(\Phi(t, y)), \tag{3.2}$$

i.e.,

$$A_\infty(\Phi(t, y)) - A_\infty(\Phi(t_0, y)) = A_{t_0 - t}(\Phi(t, y)). \tag{3.3}$$

On the other hand, by (A_3) in Definition 3.1, we have

$$A_{t_0 - t}(\Phi(t, y)) = A_{t_0}(y) - A_t(y).$$

Since the map $t \to A_t$ is continuous, we get that

$$\lim_{t \to t_0} A_{t_0 - t}(\Phi(t, y)) = 0. \tag{3.4}$$

Hence using (3.3) we obtain that

$$\lim_{t \to t_0^-} A_\infty(\Phi(t, y)) - A_\infty(\Phi(t_0, y)) = 0$$

which implies that A_∞ is left continuous with respect to $\mathcal{T}_\Phi^0$.
Again using (3.1), we obtain $\lim_{t \to \infty} A_\infty(\Phi(t, x)) = 0$. Hence A_∞ is a regular potential which is a strict Liapunov function on $\operatorname{supp} A$. □

Example 3.1. Let $f > 0$ be a measurable function defined on X_0 and set

$$A_t(x) = \int_0^t f(\Phi(s,x))\,ds.$$

If $\int_0^{\rho(x)} f(\Phi(s,x))\,ds < \infty$ for each x, then the function

$$A_\infty(x) = \int_0^\infty f(\Phi(s,x))\,ds$$

is a regular potential which is a strict Liapunov function.

Definition 3.7. For every $f \in \mathcal{F}(X_0,\Lambda)$ we define the *potential of f relative to $\mathcal{A}$* by

$$U_{\mathcal{A}}(f)(x) = \int_0^{\rho(x)} f(\Phi(t,x))\,dA_t(x)$$

when the integral is finite. When $f = 1$ we write $u_{\mathcal{A}}$.

Proposition 3.4. *Let $f \in \mathcal{F}(X_0,\Lambda)$ and let $\mathcal{A}$ be a continuous additive functional. Then, $U_{\mathcal{A}}(f)$ is a regular potential if $U_{\mathcal{A}}(f) < \infty$.*

PROOF: Let us denote $B_t(x) = \int_0^t f(\Phi(s,x))\,dA_s(x)$. We shall prove that $(B_t)_{t\geq 0}$ is a continuous additive functional. First, we shall prove the property (A_3). Indeed, let $t, s \geq 0$.

$$\begin{aligned}
B_{t+s}(x) &= \int_0^{t+s} f(\Phi(u,x))\,dA_u(x) \\
&= \int_0^t f(\Phi(u,x))\,dA_u(x) + \int_t^{t+s} f(\Phi(u,x))\,dA_u(x) \\
&= B_t(x) + \int_0^s f(\Phi(u+t,x))\,dA_{u+t}(x) \\
&= B_t(x) + \int_0^s f(\Phi(u,\Phi(t,x)))\,dA_u(\Phi(t,x)) \\
&= B_t(x) + B_s(\Phi(t,x)).
\end{aligned}$$

Next, we claim that for every $x \in X_0$ the map $t \to B_t(x)$ is continuous.
Let $t, t_0 \geq 0$, then

$$B_t(x) - B_{t_0}(x) = \int_t^{t_0} f(\Phi(u,x))\,dA_u(x).$$

The result follows by using the continuity of $\mathcal{A}$ and the fact that $U_{\mathcal{A}}(f) < \infty$. Now, we see that $U_{\mathcal{A}}(f) = B_\infty$ and the proof is achieved by using Theorem 3.1. $\square$

Corollary 3.1. *Let $\mathcal{A}$ be a continuous additive function and set*

$$u_{\mathcal{A}}(x) = \int_0^{\rho(x)} 1\, dA_t$$

the potential of $\mathcal{A}$. If $u_{\mathcal{A}} < \infty$, then, $u_{\mathcal{A}}$ is a regular potential which is a strict Liapunov function on $\operatorname{supp} A$.

Proof: We see that $u_{\mathcal{A}} = A_\infty$ and hence by Theorem 3.1 it is a Liapunov function which is strict on $\operatorname{supp} A$. □

Theorem 3.2. *If f is a regular potential, then there exists an unique continuous additive functional $\mathcal{A}$ such that $f = A_\infty$.*

Proof: Let $\mathcal{A}$ be an additive functional such that $f = A_\infty$. Then we get by (3.1) that $A_t(x) = f(x) - f(\Phi(t,x))$ which implies that $\mathcal{A}$ is unique if it exists.
Next, set $A_t(x) = f(x) - f(\Phi(t,x))$. Note that the continuity of $\mathcal{A}$ follows from the continuity of f with respect to $\mathcal{T}_\Phi^0$. The properties (A$_2$) and (A$_4$) are obvious. It is easy to check that $A_0(x) = 0, f(x) = A_\infty(x)$ and that for every $t \geq 0$ and for every $x \in X$ we have $A_t(x) \geq 0$.
The property (A$_1$) holds since for $t \geq s$ we have

$$A_t(x) - A_s(x) = -f(\Phi(t,x)) + f(\Phi(s,x)) \geq 0.$$

Finally, we shall prove (A$_3$). Indeed, for every $s, t \geq 0$ we have

$$\begin{aligned} A_{s+t}(x) &= f(x) - f(\Phi(t+s,x)) \\ &= f(x) - f(\Phi(t,\Phi(s,x))) \\ &= f(x) - f(\Phi(s,x)) + (f(\Phi(s,x)) - f(\Phi(t,\Phi(s,x)))) \\ &= A_s(x) + A_t(\Phi(s,x)). \end{aligned}$$

□

Remark 3.2. Note that this result is not unique in its formulation. In fact, such result was given for a special class of potentials (see [10]). Moreover, in our case we formulate the result for regular potentials with respect to the inherent topology and we proved the continuity of the additive functionals associated.

Definition 3.8. We say that a function h is *harmonic* if

$$h(\Phi(t,x)) = h(x)$$

for every $t \geq 0$.
We denote by $\mathcal{H}$ the set of all harmonic functions on X_0.

Theorem 3.3. *If $h \in \mathcal{H}$, then h is constant on every connected component of X_0 with respect to $\mathcal{T}_\Phi^0$.*

Proof: Let C be a connected component of X_0 and let $x, y \in C$. Then there exists $z \in C$ such that $x \underset{\Phi}{\leq} z$ and $y \underset{\Phi}{\leq} z$, i.e., there exist $s, t \geq 0$ such that $z = \Phi(s, x)$ and $z = \Phi(t, y)$. Thus

$$h(x) = h(z) = h(y).$$

□

Theorem 3.4. *For each $s \in \mathcal{E}(\Lambda)$, there exist $h \in \mathcal{H}$, $h \geq 0$ and a potential p such that $s = h + p$.*

Proof: Let $x \in X_0$ and let C_x be a connected component of X_0 such that $x \in C_x$. Since s is decreasing on Γ_x and $s \geq 0$, we have $\lim_{t\to\infty} s(\Phi(t, x)) = l_x$. So, for any $z \in \Gamma_x$ we have

$$\lim_{t\to\infty} s(\Phi(t, z)) = \lim_{t\to\infty} s(\Phi(t + \Psi(x, z), x)) = l_x.$$

Now, let $y, \in C_x$, then there exists $z \in \Gamma_x$ such that $y \underset{\Phi}{\leq} z$. It follows that

$$l_x = l_y = l_z = h.$$

We set then $h(x) := \lim_{t\to\infty} s(\Phi(t, x))$. The proof is achieved by setting $p = s - h$.

□

4. Specific order for additive functionals

Let $\mathbb{A}$ be the cone of all continuous additive functionals on X. Under the usual pointwise definitions of $\mathcal{A} + \mathcal{B}$ and $\alpha\mathcal{A}$ for $\alpha \geq 0$ the set $\mathbb{A}$ becomes a cone.

Definition 4.1. We define an order relation "$\leq$" in $\mathbb{A}$ as follows:
$\mathcal{A} \leq \mathcal{B}$ provided there exists $\mathcal{C} \in \mathbb{A}$ such that $\mathcal{A} + \mathcal{C} = \mathcal{B}$.

Definition 4.2. We define a specific order relation "$\prec$" on $\mathcal{E}(\Lambda)$ as follows:
For every $u, v \in \mathcal{E}(\Lambda)$,

$$u \prec v$$

if and only if there exists $s \in \mathcal{E}(\Lambda)$ such that $u + s = v$.

Theorem 4.1. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{A}$ be such that $u_\mathcal{A}$ and $u_\mathcal{B}$ are finite. Then*

$$\mathcal{A} \leq \mathcal{B} \Leftrightarrow u_\mathcal{A} \prec u_\mathcal{B}.$$

Proof: Suppose that $\mathcal{A} \leq \mathcal{B}$. Then, there exists $\mathcal{C} \in \mathbb{A}$ such that $\mathcal{A} + \mathcal{C} = \mathcal{B}$ which implies that

$$u_{\mathcal{A}} + u_{\mathcal{C}} = u_{\mathcal{B}}.$$

Since by Corollary 3.1 $u_{\mathcal{C}} \in \mathcal{E}(\Lambda)$, we get that $u_{\mathcal{A}} \prec u_{\mathcal{B}}$.
Conversely, assume that $u_{\mathcal{A}} \prec u_{\mathcal{B}}$. Thus, there exists $s \in \mathcal{E}(\Lambda)$ such that $u_{\mathcal{A}} + s = u_{\mathcal{B}}$. Note that he relationship $s = u_{\mathcal{B}} - u_{\mathcal{A}}$ implies that s is continuous with respect to $\mathcal{T}^0_\Phi$. On the other hand, by Corollary 3.1

$$\lim_{t\to\infty} u_{\mathcal{A}}(\Phi(t,x)) = \lim_{t\to\infty} u_{\mathcal{B}}(\Phi(t,x)) = 0$$

which gives us that $\lim_{t\to\infty} s(\Phi(t,x)) = 0$.
Consequently, s is a regular potential and by Theorem 3.2, there exists an unique additive functional $\mathcal{C}$ satisfying $s = C_\infty = u_{\mathcal{C}}$. Since

$$u_{\mathcal{A}+\mathcal{C}} = u_{\mathcal{A}} + u_{\mathcal{C}} = u_{\mathcal{B}},$$

the uniqueness gives us $\mathcal{A} + \mathcal{C} = \mathcal{B}$. □

Definition 4.3. Let $f \in \mathcal{F}(X_0, \Lambda)$ and let $\mathcal{A}$ be a continuous additive functional such that $U_{\mathcal{A}} f < \infty$. We write $f\mathcal{A}$ for the family of random variables

$$(f\mathcal{A})_t = \int_0^t f(\Phi(s,.))\, dA_s.$$

By Proposition 3.4 $f\mathcal{A}$ is a continuous additive functional.

We recall the following result, namely the Lebesgue differentiation theorem.

Theorem 4.2. *Let μ be a Radon measure on $\mathbb{R}_+$ and let λ be the Lebesgue measure on $\mathbb{R}_+$ such that μ is absolutely continuous with respect to λ. Then the family of functions $(\varphi_h)_{h>0}$, given by*

$$\varphi_h(t) = \frac{\mu([t, t+h])}{\lambda([t, t+h])}$$

converges λ-a.e. to a Borel function w such that

$$\mu([a,b]) = \int_a^b w(t)\, d\lambda(t).$$

The less conventional forms assert that the same result is true if λ is replaced by any Radon measure on $\mathbb{R}_+$ not charging points and not vanishing on non-empty open intervals.

Proposition 4.1. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{A}$ be such that $B_t \leq A_t$ for all $t \geq 0$. Then there exists $0 \leq f \leq 1$ such that $\mathcal{B} = f\mathcal{A}$.*

Proof: For each $n \in \mathbb{N}^*$, the family of functions

$$\mathcal{C}^n = \{C_t^n = A_t + \frac{t}{n}, t \geq 0\}$$

defines a strict continuous additive functional.
Let $0 \leq a < b < \infty$. Then

$$\begin{aligned}
\int_a^b dB_t(x) &= B_b(x) - B_a(x) \\
&= B_{b-a}(\Phi(a,x)) \\
&\leq A_{b-a}(\Phi(a,x)) \\
&\leq C_{b-a}^n(\Phi(a,x)) \\
&= C_b^n(x) - C_a^n(x) \\
&= \int_a^b dC_t^n(x)
\end{aligned}$$

which implies that

$$dB_t \leq dC_t^n, \forall\, n \in \mathbb{N}^*.$$

Hence, for every $x \in X_0$ and for every $n \in \mathbb{N}^*$, there exists a Borel function defined on $\mathbb{R}_+$ by

$$\varphi_x^n(t) = \liminf_{m \to \infty} \frac{B_{\frac{1}{m}}(\Phi(t,x))}{C_{\frac{1}{m}}^n(\Phi(t,x))}$$

with $B_t(x) = \int_0^t \varphi_x^n(s)\, dC_s^n(x)$.
Since $B_t \leq C_t^n$, we get that $\varphi_x^n(t) \in [0,1]$, $\forall\, t \geq 0$.
Let us denote

$$f_n(x) = \varphi_x^n(0) = \liminf_{m \to \infty} \frac{B_{\frac{1}{m}}(x)}{A_{\frac{1}{m}}(x) + \frac{1}{nm}}, \quad x \in X_0.$$

Then by (A$_2$), f_n is a $\mathcal{B}_0(\Lambda)$-measurable function on X_0. On the other hand, using the additivity of $\mathcal{A}$ and $\mathcal{B}$, we get $\varphi_{\Phi(s,x)}^n(t) = \varphi_x^n(s+t)$ which implies that

$$B_t(x) = \int_0^t f_n(\Phi(s,x))\, dA_s(x) + \frac{1}{n}\int_0^t f_n(\Phi(s,x))\, dx,\ \forall\, x \in X_0, \forall\, t \geq 0.$$

Since $(f_n)_n$ is nondecreasing and is dominated by 1, we conclude that $(f_n)_n$ converges on X_0 to a $\mathcal{B}_0(\Lambda)$-measurable function f. By the Lebesgue dominated convergence theorem, we deduce that $B_t(x) = \int_0^t f(\Phi(s,x))\, dA_s(x)$, i.e., $\mathcal{B} = f\mathcal{A}$. $\square$

Corollary 4.1. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{A}$. Then, $\mathcal{B} \leq \mathcal{A}$ if and only if there exists a $\mathcal{B}_0(\Lambda)$-measurable function f on X_0 such that $0 \leq f \leq 1$ and $\mathcal{B} = f\mathcal{A}$.*

Proposition 4.2. *Let $\mathcal{A} \in \mathbb{A}$ be such that $u_{\mathcal{A}} < \infty$ and let $f \in \mathcal{E}(\Lambda)$ be such that*

(1) $f \leq u_{\mathcal{A}}$,
(2) $f(x) - f(\Phi(t,x)) \leq A_t(x)$ *for all* $t \geq 0$, $x \in X_0$.

Then f is a regular potential on X_0.

PROOF: Since $f \in \mathcal{E}(\Lambda)$, it is continuous with respect to $\mathcal{T}_\Phi$ and decreasing. Moreover, by (1) we get that $\lim_{t\to\infty} f(\Phi(t,x)) = 0$. Now, we should show that it is continuous with respect to $\mathcal{T}_\Phi^0$. So, let $y < x$ and t_0 be such that $x = \Phi(t_0, y)$. Then, for every $t \in [0, t_0]$,

$$0 \leq f(\Phi(t,y)) - f(\Phi(t_0,y)) = f(\Phi(t,y)) - f(\Phi(t_0 - t, \Phi(t,y))) \leq A_{t_0-t}(\Phi(t,y)).$$

Now, using (3.4) we get

$$\lim_{t\to t_0} f(\Phi(t,y)) = f(\Phi(t_0,y)).$$

□

Theorem 4.3. *Let $\mathcal{A} \in \mathbb{A}$ be such that $u_{\mathcal{A}} < \infty$ and let $f \in \mathcal{E}(\Lambda)$. Then, the following assertions are equivalent:*

(1) *there exists a $\mathcal{B}_0(\Lambda)$-measurable function $0 \leq g \leq 1$ such that $f = U_{\mathcal{A}}(g)$,*
(2) $f \leq u_{\mathcal{A}}$ *and* $f(x) - f(\Phi(t,x)) \leq A_t(x), \forall\, t \geq 0, \forall\, x \in X_0$.

PROOF: We start to prove (1) ⇒ (2).
Indeed, we have $f(x) = \int_0^{\rho(x)} g(\Phi(t,x))\, dA_t(x) \leq \int_0^{\rho(x)} dA_t(x) = u_{\mathcal{A}}(x)$. Now, using the fact that $t + \rho(\Phi(t,x)) = \rho(x)$ we get

$$\begin{aligned}
(u_{\mathcal{A}} - f)(\Phi(t,x)) &= \int_0^{\rho(\Phi(t,x))} (1-g)(\Phi(t+s,x))\, dA_s(\Phi(t,x)) \\
&= \int_t^{t+\rho(\Phi(t,x))} (1-g)(\Phi(u,x))\, dA_{u-t}(\Phi(t,x)) \\
&= \int_t^{t+\rho(\Phi(t,x))} (1-g)(\Phi(u,x))\, dA_u(x) \\
&\leq \int_0^{\rho(x)} (1-g)(\Phi(u,x))\, dA_u(x) \\
&= (u_{\mathcal{A}} - f)(x).
\end{aligned}$$

Hence, $u_{\mathcal{A}} - f$ is decreasing. Using Proposition 4.2 and Corollary 3.1, we get that $u_{\mathcal{A}} - f$ it is continuous with respect to $\mathcal{T}_\Phi^0$ and $\lim_{t\to\infty}(u_{\mathcal{A}} - f)(\Phi(t,x)) = 0$.

Hence, it is a regular potential. Since $u_{\mathcal{A}}$ is a regular potential, the relationship $f + (u_{\mathcal{A}} - f) = u_{\mathcal{A}}$ and Theorem 3.2 imply the existence of continuous additive functionals $\mathcal{B}$ and $\mathcal{C}$ such that $f = u_{\mathcal{B}}$, $u_{\mathcal{A}} - f = u_{\mathcal{C}}$ and $\mathcal{B} + \mathcal{C} = \mathcal{A}$.
Consequently

$$f(x) - f(\Phi(t,x)) = B_t(x) \leq A_t(x)$$

$\forall\, t \geq 0$, $\forall\, x \in X_0$.
Conversely, we shall show that (2) $\Rightarrow$ (1). Under condition (2) and using Proposition 4.2, f is a regular potential, thus by Theorem 3.2 there exists a continuous additive functional $\mathcal{B}$ such that

$$f(x) - f(\Phi(t,x)) = B_t(x)$$

for each $t \geq 0$, $x \in X_0$.
Using Proposition 4.1, we get the existence of a $\mathcal{B}_0(\Lambda)$ measurable function g on X_0, $0 \leq g \leq 1$ such that

$$f(x) - f(\Phi(t,x)) = B_t(x) = \int_0^t g(\Phi(s,x))\, dA_s(x).$$

Now since $f(\Phi(t,x)) \to 0$ as $t \to \infty$, we obtain

$$f(x) = \int_0^{\rho(x)} g(\Phi(t,x))\, dA_t(x).$$

□

Next, we recall the following result from [9].

Proposition 4.3. *The set $\mathcal{E}(\Lambda)$ is an H-cone of functions on X_0 with respect to the pointwise algebraic operations and order relation in the sense of* [8].

Proposition 4.4 (cf. Proposition 2.1.6 in [8])**.**

(1) *Let $F \subset \mathcal{E}(\Lambda)$ be specifically increasing and dominated in $\mathcal{E}(\Lambda)$. Then*

$$\curlyvee F = \vee F$$

where $\curlyvee F$ ($\vee F$ resp.) is the least upper bound for the specific order (natural order resp.) in $\mathcal{E}(\Lambda)$.

(2) *Let $F \subset \mathcal{E}(\Lambda)$ be specifically decreasing. Then, we have*

$$\curlywedge F = \wedge F$$

where $\curlywedge F$ ($\wedge F$ resp.) is the greatest lower bound for the specific order (natural order resp.) in $\mathcal{E}(\Lambda)$.

Theorem 4.4. (1) *Let $(f_i)_{i\in I}$ be a family of elements of $\mathcal{E}(\Lambda)$ which is increasing to a function f. Then $f \in \mathcal{E}(\Lambda)$.*

(2) *Let $(f_i)_{i\in I}$ be a family of elements of $\mathcal{E}(\Lambda)$. Then, $f = \inf_{i\in I} f_i$ is supermedian. Moreover, if $\widehat{f} = \sup_{\alpha\geq 0} \alpha V_\alpha f$ is the excessive regularization of f, then $\widehat{f} = \wedge_{i\in I} f_i$.*

Proof: (1) By Proposition 2.2, $f = \sup_{i\in I} f_i$ is lower semicontinuous with respect to $\mathcal{T}^0_\Phi$ and decreasing which yields that it is continuous with respect to $\mathcal{T}_\Phi$. Thus, $f \in \mathcal{E}(\Lambda)$.

(2) Since $f = \inf_{i\in I} f_i$ is decreasing, by Theorem 2.2 it is $\mathcal{B}_0(\Lambda)$-measurable and therefore $\lambda V_\lambda f$ is well defined. Since

$$\lambda V_\lambda(\inf_{i\in I} f_i) \leq \lambda V_\lambda(f_j) \leq f_j, \forall\, j,$$

we get that

$$\lambda V_\lambda(\inf_{i\in I} f_i) \leq \inf_{i\in I} f_i$$

which yields that f is supermedian. On the other hand, the fact that $\widehat{\inf_{i\in I} f_i} \leq \widehat{f_j} = f_j$ gives us that

$$\widehat{f} \leq \bigwedge_{i\in I} f_i. \tag{4.1}$$

Now, let $t \in \mathcal{E}(\Lambda)$ be such that $t \leq f_i, \forall\, i$. Thus, $t \leq \inf_{i\in I} f_i$ and therefore

$$t = \widehat{t} \leq \widehat{f}. \tag{4.2}$$

Now combining (4.1) and (4.2), we get that $\wedge_{i\in I} f_i = \widehat{f}$. □

Theorem 4.5. (1) *Let $(u_i)_{i\in I}$ be a specifically increasing family dominated in $\mathcal{P}$, then $\sup_i u_i \in \mathcal{P}$ and $\sup_i u_i = \curlyvee_i u_i$.*

(2) *Let $(u_i)_{i\in I}$ be a specifically decreasing family in $\mathcal{P}$, then $\wedge_i u_i \in \mathcal{P}$ and $\wedge_i u_i = \curlywedge_i u_i$.*

(3) *The Riesz decomposition holds in $\mathcal{P}$. i.e. if $u, v_1, v_2 \in \mathcal{P}$ are such that $u \leq v_1 + v_2$, then there exist $u_1, u_2 \in \mathcal{P}$ satisfying*

$$u = u_1 + u_2,\ u_1 \leq v_1,\ u_2 \leq v_2.$$

Proof: (1) Let $v \in \mathcal{P}$ be such that $u_i \prec v$ for all $i \in I$. Then there exists $v_i \in \mathcal{E}(\Lambda)$ such that $u_i + v_i = v$. It is obvious that $\sup_i u_i$ is lower semicontinuous with respect to $\mathcal{T}^0_\Phi$ and decreasing on each trajectory. Since $\sup_i u_i + \inf_i v_i = v$, using

(1) and (2) in Theorem 4.4, we get that $\widehat{\inf_i v_i} = \inf_i v_i = \wedge_i v_i$ and $\sup_i u_i \in \mathcal{E}(\Lambda)$ which implies that

$$\sup_i u_i + \bigwedge_i v_i = v.$$

By Proposition 2.2, we get that $\wedge_i v_i$ is lower semicontinuous with respect to $\mathcal{T}_\Phi^0$. Hence, we get that $\sup_i u_i = v - \wedge_i v_i$ is continuous. In addition, since $\sup_i u_i \leq v$ we have that

$$\lim_{t\to\infty} \sup_i u_i(\Phi(t,x)) = 0.$$

Thus $\sup_i u_i \in \mathcal{P}$.

(2) Set $u = \wedge_i u_i$. It is obvious that u is upper semicontinuous with respect to $\mathcal{T}_\Phi^0$ and decreasing on each trajectory. On the other hand, by (2) in Theorem 4.4 and (2) in Proposition 4.4, we have $u = \wedge_i u_i = \widehat{\inf_i u_i} \in \mathcal{E}(\Lambda)$. Since $u \prec u_i$, there exists $v_i \in \mathcal{E}(\Lambda)$ such that $u = u_i - v_i$. By Proposition 2.2 again, v_i is lower semicontinuous with respect to $\mathcal{T}_\Phi^0$ and thus we get that u is continuous with respect to $\mathcal{T}_\Phi^0$ which yields that $u \in \mathcal{P}$.

(3) Let $u, v_1, v_2 \in \mathcal{P}$ such that $u \leq v_1 + v_2$. Then, by [8] there exist $u_1, u_2 \in \mathcal{E}(\Lambda)$ such that

$$u = u_1 + u_2,\ u_1 \leq v_1,\ u_2 \leq v_2.$$

By Proposition 2.2, u_1, u_2 are lower semicontinuous with respect to $\mathcal{T}_\Phi^0$. Since $u_1 = u - u_2$ and u is continuous, we conclude that u_1 and u_2 are continuous with respect to $\mathcal{T}_\Phi^0$. Now, using the fact that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u_1 = u - u_2$, we get that $u_1, u_2 \in \mathcal{P}$. $\square$

Next, we set

$$\mathbb{A}_\Phi = \{\mathcal{A} \in \mathbb{A} : u_\mathcal{A} < \infty\}.$$

Theorem 4.6. *The following assertions hold.*

1. *Let $(\mathcal{A}^i)_{i\in I}$ be an increasing family in $\mathbb{A}_\Phi$ and upper bounded. Then $\mathcal{A} = \sup_i \mathcal{A}^i \in \mathbb{A}_\Phi$.*
2. *Let $(\mathcal{A}^i)_{i\in I}$ be a decreasing family in $\mathbb{A}_\Phi$. Then $\mathcal{A} = \wedge_i \mathcal{A}^i$, the greatest lower bound in $\mathbb{A}_\Phi$, exists.*
3. *The Riesz decomposition holds in $\mathbb{A}_\Phi$.*

Proof: (1) By Theorem 4.1, the family $(u_{\mathcal{A}^i})_i$ is specifically increasing and specifically upper bounded, hence by (1) in Theorem 4.5 $\sup_i u_{\mathcal{A}^i} \in \mathcal{P}$. Moreover, by Theorem 3.2 we can find $\mathcal{A} \in \mathbb{A}_\Phi$ such that $\sup_i u_{\mathcal{A}^i} = u_\mathcal{A} = A_\infty$. Since by (3.1) we have

$$u_{\mathcal{A}^i}(x) = A_t^i(x) + u_{\mathcal{A}^i}(\Phi(t,x))$$

we get

$$u_\mathcal{A}(x) = \sup_i A_t^i(x) + u_\mathcal{A}(\Phi(t,x))$$

which yields that

$$\sup_i A_t^i(x) = A_t(x).$$

(2) By Theorem 4.1, the family $(u_{\mathcal{A}^i})_i$ is specifically decreasing, hence by (2) in Theorem 4.5 the function $\wedge_i u_{\mathcal{A}^i} \in \mathcal{P}$ and $\wedge_i u_{\mathcal{A}^i} = \curlywedge_i u_{\mathcal{A}^i}$. Thus, by Theorem 3.2 there exists $\mathcal{A} \in \mathbb{A}_\Phi$ such that $\wedge_i u_{\mathcal{A}^i} = u_{\mathcal{A}} = A_\infty$. Hence, it holds by Theorem 4.1 that

$$\mathcal{A} \leq \mathcal{A}^i, \ \forall\, i. \tag{4.3}$$

Now, let $\mathcal{B} \in \mathbb{A}_\Phi$ be such that $\mathcal{B} \leq \mathcal{A}^i$ for all $i \in I$. It follows that $u_{\mathcal{B}} \prec u_{\mathcal{A}^i}$ for all $i \in I$ and therefore $u_{\mathcal{B}} \prec u_{\mathcal{A}}$ which implies that

$$\mathcal{B} \leq \mathcal{A}. \tag{4.4}$$

Finally, combining (4.3) and (4.4), we get that $\mathcal{A} = \wedge_i \mathcal{A}^i$.

(3) Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{A}$ be such that $\mathcal{A} \leq \mathcal{B} + \mathcal{C}$. Thus, there exists $0 \leq f \leq 1$ such that $\mathcal{A} = f(\mathcal{B} + \mathcal{C})$. Hence $f\mathcal{B} \leq \mathcal{B}$ and $f\mathcal{C} \leq \mathcal{C}$. □

Corollary 4.2. *The Riesz decomposition in $\mathcal{P}$ is valid for the specific order.*

Proof: Let $u, v, w \in \mathcal{P}$ be such that $u \prec v + w$. Thus, there exist $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{A}$ such that $u = u_{\mathcal{A}}$, $v = u_{\mathcal{B}}$, $w = u_{\mathcal{C}}$. Hence, by Theorem 4.6, there exist $\mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{A}_1 \leq \mathcal{B}$, $\mathcal{A}_2 \leq \mathcal{C}$ and $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$. It follows that $u_{\mathcal{A}} = u_{\mathcal{A}_1} + u_{\mathcal{A}_2}$ and $u_{\mathcal{A}_1} \prec u_{\mathcal{B}}, u_{\mathcal{A}_2} \prec u_{\mathcal{C}}$. □

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