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On weakly projective and weakly injective modules

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Abstract. The purpose of this paper is to further the study of weakly injective and weakly projective modules as a generalization of injective and projective modules. For a locally q.f.d. module M, there exists a module $K \in \sigma[M]$ such that $K \oplus N$ is weakly injective in $\sigma[M]$, for any $N \in \sigma[M]$. Similarly, if M is projective and right perfect in $\sigma[M]$, then there exists a module $K \in \sigma[M]$ such that $K \oplus N$ is weakly projective in $\sigma[M]$, for any $N \in \sigma[M]$. Consequently, over a right perfect ring every module is a direct summand of a weakly projective module. For some classes $\mathcal{M}$ of modules in $\sigma[M]$, we study when direct sums of modules from $\mathcal{M}$ satisfy property $\mathbb{P}$ in $\sigma[M]$. In particular, we get characterizations of locally countably thick modules, a generalization of locally q.f.d. modules.

Keywords: tight, weakly tight, weakly injective, weakly projective, countably thick, locally q.f.d., weakly semisimple

Classification: 16D50, 16D60, 16D70, 16P40

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [29]). Given a module X_R, the injective hull of X in Mod-R (resp., in $\sigma[M]$) is denoted by $E(X)$ (resp., $\widehat{X}$). The M-injective hull $\widehat{X}$ is the trace of M in $E(X)$, i.e. $\widehat{X} = \sum\{f(M), f \in \mathrm{Hom}(M, E(X))\}$.

The purpose of this paper is to further the study of the concepts of weak injectivity (projectivity) in $\sigma[M]$ studied in [4], [9], [21], [24], [25], [26], [27], [30], [31]. In view of Theorem 2.9, if a right module M is projective and right perfect in $\sigma[M]$, then there exists a module $K \in \sigma[M]$ such that $K \oplus X$ is a weakly projective module, for every module $X \in \sigma[M]$. Consequently, over a right perfect ring every module is a direct summand of a weakly projective module which was proved by S.K. Jain, S.R. López-Permouth and M. Saleh. Similarly, every module X in $\sigma[M]$ is a direct summand of a weakly injective module in $\sigma[M]$, a result that generalizes 2.12, 2.13 and 2.14 in [17], 2.1, 2.2. and 2.3 in [19]. For a locally q.f.d. module M, there exists a module $K \in \sigma[M]$ such that $K \oplus N$ is weakly injective in $\sigma[M]$, for any $N \in \sigma[M]$. For some classes $\mathcal{M}$ of modules in $\sigma[M]$

we study when direct sums of modules from $\mathcal{M}$ are weakly tight in $\sigma[M]$. In particular, we get necessary and sufficient conditions for $\sum$-weak tightness of the injective hull of a simple module. As a consequence, we get characterizations of q.f.d. rings by means of weakly injective (tight) modules given by A. Al-Huzali, S.K. Jain and S.R. López-Permouth.

Given two modules Q and $N \in \sigma[M]$, we call Q *weakly N-injective* in $\sigma[M]$ if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to \widehat{Q}$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly injective* in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Equivalently, if Q is weakly N-injective for all finitely generated modules N in $\sigma[M]$. A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull of X is embeddable in X. A module is *tight* (*R-tight*) in $\sigma[M]$ if it is tight relative to all finitely generated (cyclic) submodules of its M-injective hull, and Q is *weakly tight* (*weakly R-tight*) in $\sigma[M]$ if every finitely generated (cyclic) submodule N of $\widehat{Q}$ is embeddable in a direct sum of copies of Q. It is clear that every weakly injective module in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness (see [4], [31]).

Given two modules $Q, N \in \sigma[M]$, we call Q *weakly N-projective* in $\sigma[M]$ if Q has a $\sigma[M]$-projective cover $P(Q)$ (see [29, Section 19.4]) and for every homomorphism $\varphi : P(Q) \to N$ there exists a homomorphism $\widehat{\varphi} : Q \to N$ and an epimorphism $\sigma : P(Q) \to Q$ such that $\varphi = \widehat{\varphi}\sigma$. Equivalently, Q is weakly N-projective in $\sigma[M]$ if for every homomorphism $\varphi : P(Q) \to N$, there exists a submodule X of $\ker(\varphi)$ such that $P(Q)/X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly projective* in $\sigma[M]$ if it is weakly N-projective for all finitely M-generated modules N in $\sigma[M]$. A module M_R is called *locally q.f.d.* ([3], [7], [18]) if every finitely generated (or cyclic) module $N \in \sigma[M]$ has finite uniform dimension. For a right R-module and N in $\sigma[M]$, N is called *perfect* in $\sigma[M]$ if for any index set Λ, the sum $N^{(\Lambda)}$ is semiperfect in $\sigma[M]$ (see [29, Section 43]). A module Q is called *weakly* (N-)*projective* (*resp., weakly* (N-)*injective, tight*) ([17], [14], [15], [16]) if it is weakly (N-)projective (resp., weakly (N-)injective, tight) in $\sigma[R_R] = $ Mod-R.

2. A large class of modules

The class of weakly injective (tight, weakly tight) modules in $\sigma[M]$ is closed under finite direct sums, and essential extensions, and the class of weakly projective modules in $\sigma[M]$ is closed under finite direct sums. Also, the domains of the class of weakly injective (tight, weakly tight, weakly projective) modules in $\sigma[M]$ are closed under submodules and quotients.

First, we list below some known results on weak projectivity and weak injectivity (tightness) in $\sigma[M]$ that will be needed through this paper (cf. [4], [24], [23], [25], [30], [31]).

Lemma 2.1 ([24, Lemma 2.2]). *Let $\{X_i\}_I$ be a class of weakly N-projective modules in $\sigma[M]$ and let $\bigoplus_I X_i$ have a projective cover in $\sigma[M]$. Then $\bigoplus_I X_i$ is weakly N-projective in $\sigma[M]$.*

Proof: The proof follows directly from the fact that in this case $P(\bigoplus_I X_i) = \bigoplus_I P(X_i)$. □

Lemma 2.2. *Given modules $N, Q \in \sigma[M]$, the following hold true.*

(a) *If Q is self-projective and weakly N-projective in $\sigma[M]$, then Q is N-projective in $\sigma[M]$.*

(b) *If Q is self-injective and N-tight in $\sigma[M]$, then Q is N-injective in $\sigma[M]$.*

Lemma 2.3. *A finite direct sum of weakly injective (tight, weakly tight) modules in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$, and an essential extension of a weakly injective (tight, weakly tight) module in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$.*

Lemma 2.4. *A uniform module $X \in \sigma[M]$ is weakly tight in $\sigma[M]$ iff X is weakly injective in $\sigma[M]$.*

Proof: Let X be uniform and weakly tight in $\sigma[M]$, and let N be a finitely generated submodule of $\widehat{X}$. Then N is embeddable in $X^{(\alpha)}$ via a monomorphism, say, ϕ. Let $\pi_i : X^{(\alpha)} \to X$ be the ith projection map. Then $\bigcap_{i\in\alpha} \ker(\pi_i\phi) \subseteq \ker\phi = 0$. Since X is uniform, we have $\ker(\pi_i\phi) = 0$, and thus N embeds in X, proving that X is tight. By Lemma 2.3, X is weakly injective in $\sigma[M]$. □

Example 2.5. (i) [17, Example 2.11], [19]. Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then $M = \mathrm{Soc}(R_R) \oplus R$ is tight but not weakly injective.

(ii) [4]. Let $R = Z$ and $X = (Q/Z) \oplus (Z/pZ)$. Then X is weakly tight in $\sigma[M]$ but not tight.

(iii) [17, Example 4.4(d)]. Let F be a field. Then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly injective but the summand $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly injective.

(iv) [15, Example 2.14(1)]. Let R be a uniserial ring which is not a division ring (e.g. $Z/(p^n), p$ is prime), and $S = \mathrm{Soc}(R)$. Then, as a right R-module, $R/S \times R$ is weakly R-projective but not R-projective (see [14, Proposition 2.11]).

It has been shown in [17, Theorem 2.8] that if $\mathcal{K}$ is a class of modules that is closed under direct sums and under injective hulls, and if every cyclic module in $\mathcal{K}$ has finite uniform dimension, then every tight module in $\mathcal{K}$ is weakly injective. Taking $\mathcal{K} = \sigma[M]$, we obtain the following interesting corollary.

Corollary 2.6. *Let M be a locally q.f.d. module. Then every tight module in $\sigma[M]$ is weakly injective in $\sigma[M]$.*

Theorem 2.7. *Let M be a locally q.f.d. module. Then there exists a module $K \in \sigma[M]$ such that $Q = K \oplus N$ is a weakly injective module for every module $N \in \sigma[M]$.*

PROOF: Let $\mathcal{F}$ be the family of all indecomposable injectives up to isomorphism in $\sigma[M]$ and let $K = \bigoplus \sum_{F \in \mathcal{F}} F^{(\alpha)}$ where α is an infinite cardinal number greater than both the cardinality of M and the cardinality of the ring R. Let $Q = K \oplus N$. Then Q is weakly injective in $\sigma[M]$ for every module $N \in \sigma[M]$, since every finitely generated module over a locally q.f.d. module is embeddable in a finite direct sum of indecomposable injectives and thus embeddable in Q. Thus Q is tight in $\sigma[M]$ and thus, Q is weakly injective in $\sigma[M]$. $\square$

In [19], it is shown that any semisimple module is a direct summand of a weakly injective module. The next lemma shows that in fact any module is a direct summand of a weakly injective module.

Lemma 2.8 ([26, Lemma 2.3])**.** *Every module in $\sigma[M]$ is a direct summand of a weakly injective module in $\sigma[M]$.*

PROOF: Follows from the fact that for any module X in $\sigma[M]$, $X \oplus \widehat{(X)^{(\alpha)}}$ is weakly injective in $\sigma[M]$ where α is an infinite cardinal number. $\square$

The above result generalizes 2.12, 2.13 and 2.14 in [17], and 2.1, 2.2. and 2.3 in [19].

The next is a dual to Lemma 2.8 for weak projectivity.

Theorem 2.9. *Let M be projective and right perfect in $\sigma[M]$. Then there exists a module $K \in \sigma[M]$ such that $K \oplus X$ is a weakly projective module in $\sigma[M]$ for every module $X \in \sigma[M]$.*

PROOF: Let L be the direct sum of all finitely M-generated modules (up to isomorphism) in $\sigma[M]$. Let $K = L \oplus [P(L)]^{(\alpha)}$ where α is an infinite cardinal number greater than both the cardinality of M and the cardinality of the ring R. We claim that $Q = X \oplus K$ is weakly projective in $\sigma[M]$ for every module $X \in \sigma[M]$. Let $\varphi : P(Q) \to N$ be an epimorphism, where N is a finitely M-generated module in $\sigma[M]$. Let $\pi : P(N) \to N$ be the M-projective cover map. By the projectivity of $P(Q)$, there exists a homomorphism $\widehat{\varphi} : P(Q) \to P(N)$ such that $\pi\widehat{\varphi} = \varphi$. Since $\operatorname{Ker}\pi \ll P(N)$, $\widehat{\varphi}$ is onto. Since $P(N)$ is projective, $\widehat{\varphi}$ splits, and therefore we may write $P(Q) = P \oplus \operatorname{Ker}\widehat{\varphi}$, for some submodule $P \subset P(Q)$ isomorphic to $P(N)$. Also since N is finitely M-generated, $P(N)$ is also finitely M-generated and thus $P(N)$ is a direct summand of $P(Q)$. Since M is perfect, $M \cong \bigoplus_{\lambda=1}^{n} L_\lambda$, where each L_λ is local. It follows by [29, Section 41.17] that every projective module in $\sigma[M]$ is a projective cover of a simple module and thus a direct sum of the indecomposable local projective modules L_λ in $\sigma[M]$. Write $P(N) \cong \bigoplus_{\lambda=1}^{n} L_\lambda^{\alpha_i}$, $\operatorname{Ker}\widehat{\varphi} \cong \bigoplus_{\lambda=1}^{n} L_\lambda^{\beta_i}, P(L) \cong \bigoplus_{\lambda=1}^{n} L_\lambda^{\gamma_i}$, and $P(Q) \cong \bigoplus_{\lambda=1}^{n} L_\lambda^{\lambda_i}$. Since $P(N)$ is finitely generated, α_i are finite. Now, it follows easily that $P(Q) \cong \operatorname{Ker}\widehat{\varphi}$, and one may think of $\widehat{\varphi}$ as the projection map $p : P(Q) \oplus P(N) \to P(N)$. It follows that $\operatorname{Ker}\varphi \cong P(Q) \oplus \operatorname{Ker}\pi$. Now Q is a homomorphic image of $P(Q)$ and, by definition of L, there exists a submodule $Q' \subset Q$ such that $N \oplus Q' = Q$. Thus there exists a submodule $K' \subset P(Q)$ such that $P(Q)/K' \cong Q'$. Let $X = K' \oplus \operatorname{Ker}\pi \subset \operatorname{Ker}\varphi$. Then $P(Q)/X = [P(Q) \oplus P(N)]/[K' \oplus \operatorname{Ker}\pi] \cong Q' \oplus N \cong Q$, as desired. □

The above results show that the classes of weakly injective and weakly projective modules are quite large.

We call a module M_R *weakly semisimple* (*weakly R-semisimple*) iff every module $N \in \sigma[M]$ is weakly injective (weakly R-injective) in $\sigma[M]$. As a direct application of the above results, we state the following characterizations of semisimple and weakly (R-)semisimple modules in terms of weak injectivity, tightness, weak tightness, and weak projectivity. The proofs are straightforward, for the sake of convenience of the reader we provide proofs to some of these implications.

Theorem 2.10. *For a module M_R, the following are equivalent:*

(a) *M is semisimple;*
(b) *M is projective and perfect in $\sigma[M]$ and every weakly projective module in $\sigma[M]$ is (quasi-)discrete;*
(c) *M is projective and perfect in $\sigma[M]$ and every discrete module is weakly projective in $\sigma[M]$;*
(d) *M is projective and perfect in $\sigma[M]$ and every weakly projective module in $\sigma[M]$ is (quasi-)continuous;*
(e) *every weakly injective module in $\sigma[M]$ is (quasi-)discrete;*
(f) *every weakly injective module in $\sigma[M]$ is (quasi-)continuous;*
(g) *every continuous module is weakly projective in $\sigma[M]$;*
(h) *every (direct summand of a) weakly injective module in $\sigma[M]$ is (injective) projective in $\sigma[M]$;*
(i) *M is projective and perfect and every weakly projective module in $\sigma[M]$ is injective (projective) in $\sigma[M]$;*
(j) *M is projective and perfect in $\sigma[M]$ and every direct summand of a weakly projective module in $\sigma[M]$ is weakly projective in $\sigma[M]$;*
(k) *M is projective and perfect in $\sigma[M]$ and every (direct summand of a) weakly projective module in $\sigma[M]$ is quasi-projective in $\sigma[M]$;*
(l) *every direct summand of a weakly injective module in $\sigma[M]$ is quasi-injective in $\sigma[M]$;*
(m) *M is projective and perfect in $\sigma[M]$ and every direct summand of a weakly projective module in $\sigma[M]$ is injective in $\sigma[M]$.*

Proof: (h) $\Rightarrow$ (a). Let $X \in \sigma[M]$. By Theorem 2.8, $X \oplus \widehat{(X)^{(\alpha)}}$ is weakly injective in $\sigma[M]$, where α is an infinite cardinal number. Thus X is injective, proving that M is semisimple.

(i) $\Rightarrow$ (a). Let $N \in \sigma[M]$. By Theorem 2.9, there exists a module $Q \in \sigma[M]$ such that $Q \oplus N$ is weakly projective and thus $Q \oplus N$ is injective, and thus N is injective, proving that M is semisimple.

Clearly (a) implies all other items. The other implications are similar and thus are left to the reader. □

Theorem 2.11. *For a module M_R, the following are equivalent:*

(a) *M is weakly semisimple (resp., weakly R-semisimple);*
(b) *M is projective and perfect in $\sigma[M]$ and every direct summand of a weakly projective module in $\sigma[M]$ is weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) in $\sigma[M]$;*
(c) *every direct summand of a weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) module in $\sigma[M]$ is weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) in $\sigma[M]$.*

Proof: (b) $\Rightarrow$ (a). Let $N \in \sigma[M]$. By Theorem 2.9, there exists a module $Q \in \sigma[M]$ such that $Q \oplus N$ is weakly projective and thus N is weakly injective, proving that M is weakly semisimple.

(c) $\Rightarrow$ (a). Let $X \in \sigma[M]$. By Theorem 2.8, $X \oplus \widehat{(X)^{(\alpha)}}$ is weakly injective in $\sigma[M]$, where α is an infinite cardinal number. Thus X is weakly injective, proving that M is weakly semisimple. The other cases are similar and are left to the reader. □

In case $M = R$ in the above two theorems we get the following characterization of semisimple, weakly semisimple, and weakly R-semisimple rings.

Corollary 2.12. *For a ring R, the following are equivalent:*

(a) *R is semisimple;*
(b) *R is perfect and every weakly projective module is (quasi-)discrete;*
(c) *R is perfect and every discrete module is weakly projective;*
(d) *R is perfect and every weakly projective module is (quasi-)continuous;*
(e) *every weakly injective module is (quasi-)discrete;*
(f) *every weakly injective module is (quasi-)continuous;*
(g) *every continuous module is weakly projective;*
(h) *every (direct summand of a) weakly injective module is (injective) projective;*
(i) *R is perfect and every weakly projective module is injective (projective);*
(j) *R is perfect and every direct summand of a weakly projective module is weakly projective;*

(k) *R is perfect and every* (*direct summand of* a) *weakly projective module is quasi-projective;*
(l) *every direct summand of a weakly injective module is quasi-injective;*
(m) *R is perfect and every direct summand of a weakly projective module is injective.*

Corollary 2.13. *For a ring R, the following are equivalent:*
(a) *R is weakly semisimple* (*resp., weakly R-semisimple*)*;*
(b) *R is perfect and every direct summand of a weakly projective module is weakly injective* (*or tight, weakly tight*) (*resp., weakly R-injective*) (*or R-tight, weakly R-tight*)*;*
(c) *every direct summand of a weakly injective* (*or tight, weakly tight*) (*resp., weakly R-injective*) (*or R-tight, weakly R-tight*) *module is weakly injective* (*or tight, weakly tight*) (*resp., weakly R-injective*) (*or R-tight, weakly R-tight*).

3. Direct sums of classes of modules

Let M_R be a fixed module and $\mathcal{K}$ a class of simple modules in $\sigma[M]$. We denote

$$\mathrm{Soc}_{\mathcal{K}}(X) = \sum\{A \subseteq X | A \simeq P \ \text{ for some } \ P \in \mathcal{K}\}.$$

Recall ([4], [5], [6]) that $X \in \sigma[M]$ is said to be *countably thick relative to* $\mathcal{K}$ if $\mathrm{Soc}_{\mathcal{K}}(X/A)$ is finitely generated for all $A \subseteq X$. In particular, if $\mathcal{K}$ is the class of all simple modules in $\sigma[M]$ then $X \in \sigma[M]$ is countably thick relative to $\mathcal{K}$ if and only if all factor modules of X have finite uniform dimension, that is X is q.f.d. (see [4, Lemma 1], [5], [6]).

For a module X_R and a module property $\mathbb{P}$, X is said to be $\sum -\mathbb{P}$ in case every direct sum of copies of X enjoys the property $\mathbb{P}$. Also we call X *locally* $\mathbb{P}$ in case every finitely generated submodule of X enjoys the property $\mathbb{P}$ (see [1], [3], [18]).

Theorem 3.1. *For a module M_R, the following implications* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) *always hold:*
(a) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is weakly injective in $\sigma[M]$;*
(b) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is tight in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is tight in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly tight in $\sigma[M]$;*

(f) *every direct sum* $\bigoplus_\Lambda M_\lambda$ *of weakly tight modules in* $\sigma[M]$*, where each* M_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$*, is weakly tight in* $\sigma[M]$.

Proof: (a) $\Rightarrow$ (b). Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly injective modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X}$. By (a) the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly injective in $\sigma[M]$ and $X = \bigoplus_\Lambda M_\lambda \subseteq' \bigoplus_\Lambda \widehat{M_\lambda} \subseteq' \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. Thus by (a) there exists a submodule $Y \subseteq \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$ such that $N \subseteq Y \cong \bigoplus_\Lambda \widehat{M_\lambda}$. Write $Y = \bigoplus_\Lambda \widehat{Y}_\lambda$, where $Y_i \cong M_i$, $i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma = \{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$ such that $N \subseteq \bigoplus_\Gamma \widehat{Y_\lambda} = \widehat{\bigoplus_\Gamma Y_\lambda}$. Since $Y_{\lambda_1}, \ldots, Y_{\lambda_m}$ are weakly injective in $\sigma[M]$, the finite direct sum $Y_{\lambda_1} \oplus \cdots \oplus Y_{\lambda_m}$ is weakly injective in $\sigma[M]$. Therefore, there exists $X_1 \cong \bigoplus_\Gamma Y_\lambda \cong \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq \widehat{\bigoplus_\Gamma Y_\lambda}$. Thus $N \subseteq X_1 \oplus \bigoplus_{\lambda \notin \Gamma} Y_\lambda \simeq X$, proving that X is weakly injective.

(c) $\Rightarrow$ (d). Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By (c), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$. Thus N embeds in $\bigoplus_\Lambda \widehat{M_\lambda}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m} = \bigoplus_{\lambda=1}^m \widehat{M_\lambda}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight, $N \simeq \varphi(N)$ embeds in finite direct sums $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$, proving that X is tight.

(e) $\Rightarrow$ (f). Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By (e), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly tight in $\sigma[M]$. Thus N embeds in $(\bigoplus_\Lambda \widehat{M_\lambda})^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m} = \bigoplus_{\lambda=1}^m \widehat{M_\lambda}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight, $N \simeq \varphi(N)$ embeds in a direct sums of copies of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sums of X, proving that X is weakly tight.

Clearly, (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e). □

The next theorem provides several characterizations of countably thick (consequently, locally q.f.d.) modules which extends the main result in [26]. Consequently, we get the main result in [2] as a corollary to the main results of this section.

Theorem 3.2. *For a module* M_R *and any class* $\mathcal{K}$ *of simple modules in* $\sigma[M]$*, the following conditions are equivalent:*

(a) M *is locally countably thick relative to* $\mathcal{K}$;
(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of injectives in* $\sigma[M]$*, where each* E_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$*, is tight in* $\sigma[M]$;

(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of tight modules in* $\sigma[M]$*, where each* E_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$*, is tight in* $\sigma[M]$*;*
(d) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly tight modules in* $\sigma[M]$*, where each* E_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$*, is weakly tight in* $\sigma[M]$*;*
(e) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly tight modules in* $\sigma[M]$*, where each* E_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$*, is weakly* N*-tight, for every cyclic module* N *in* $\sigma[M]$*;*
(f) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where* $P_\lambda \in \mathcal{K}$*, is weakly* N*-tight, for every cyclic module* N *in* $\sigma[M]$*.*

Proof: (a) $\Rightarrow$ (b). Consider $X = \bigoplus_\Lambda E_\lambda$, where E_λ is injective in $\sigma[M]$ for every $\lambda \in \Lambda$ and $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$ is essential in E_λ. Let N be a finitely generated submodule of $\widehat{X}$. By the hypothesis, $\mathrm{Soc}_{\mathcal{K}}(N)$ is finitely generated that is

$$\mathrm{Soc}_{\mathcal{K}}(N) = P_1 \oplus \cdots \oplus P_n \quad \text{with} \quad P_i \simeq P_i' \quad \text{for some} \quad P_i' \in \mathcal{K} \quad (1 \leq i \leq n).$$

So $\mathrm{Soc}_{\mathcal{K}}(N) \subseteq \mathrm{Soc}_{\mathcal{K}}(\widehat{X}) = \mathrm{Soc}_{\mathcal{K}}(X) \subseteq X$ and hence $\mathrm{Soc}_{\mathcal{K}}(N) \subseteq E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. This implies that $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ contains an injective hull E of $\mathrm{Soc}_{\mathcal{K}}(N)$. Thus N embeds in X, proving that X is tight.

(b) $\Rightarrow$ (c). Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X}$. By (b), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$ and thus N embeds in $\bigoplus_\Lambda \widehat{M}_\lambda$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight, $N \simeq \varphi(N)$ embeds in $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ and thus embeds in X, proving that X is tight.

(c) $\Rightarrow$ (d). Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X}$. By (c) the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$. Thus N embeds in $\bigoplus_\Lambda \widehat{M}_\lambda$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight, $N \simeq \varphi(N)$ embeds in a finite direct sums of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a finite direct sums of X, proving that X is weakly tight.

Clearly, (d) $\Rightarrow$ (e) $\Rightarrow$ (f).

(f) $\Rightarrow$ (a). Let K be a cyclic submodule of M. If $\mathrm{Soc}_{\mathcal{K}}(K) = 0$, we are done. Suppose $0 \neq \mathrm{Soc}_{\mathcal{K}}(K) = \bigoplus_\Lambda P_\lambda$, where $P_\lambda \simeq P_\lambda'$ for some $P_\lambda' \in \mathcal{K}$. We shall show that $\mathrm{Soc}_{\mathcal{K}}(K)$ is finitely generated. To this end, consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_\Lambda P_\lambda & \xrightarrow{\gamma} & K \\ & & \big\downarrow{\scriptstyle\varphi} & & \\ & & \widehat{\bigoplus_\Lambda \widehat{P}_\lambda} & & \end{array}$$

where φ and γ are the inclusion homomorphisms. By M-injectivity of $\widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$, there exists $\psi : K \to \widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$ such that $\psi\gamma = \varphi$. By our hypothesis, $\bigoplus_\Lambda \widehat{P}_\lambda$ is weakly K-tight in $\sigma[M]$, hence $\operatorname{Im}\varphi \subset \operatorname{Im}\psi$ is embeddable in $(\bigoplus_\Lambda \widehat{P}_\lambda)^{(\aleph_0)}$ and thus embeddable in a finite sum. Therefore, $\mathrm{Soc}_{\mathcal{K}}(K)$ is embeddable in $\widehat{P}_{\lambda_1} \oplus \cdots \oplus \widehat{P}_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since each $\widehat{P}_{\lambda_i}$ is uniform, $\mathrm{Soc}_{\mathcal{K}}(K)$ has finite uniform dimension and is therefore finitely generated. □

Combining Theorem 3.1 and Theorem 3.2 we get the following

Theorem 3.3. *For a module M_R, and any class $\mathcal{K}$ of simple modules in $\sigma[M]$, the following conditions are equivalent:*

(a) *M is locally countably thick relative to $\mathcal{K}$;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is tight in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly injective in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is tight in $\sigma[M]$;*
(f) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is tight in $\sigma[M]$;*
(g) *every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly tight in $\sigma[M]$;*
(h) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly tight in $\sigma[M]$;*
(i) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$, is weakly N-tight, for every cyclic module N in $\sigma[M]$;*
(j) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where $P_\lambda \in \mathcal{K}$, is weakly N-tight for every cyclic module N in $\sigma[M]$;*
(k) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where $P_\lambda \in \mathcal{K}$, is weakly R-tight in $\sigma[M]$.*

Taking $\mathcal{K}$ to be all simple R-modules in $\sigma[M]$ in Theorem 3.3 we get [26, Theorem 2.7] as a corollary.

Corollary 3.4. *For a module M_R, the following conditions are equivalent:*

(a) *M is locally q.f.d.;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly injective in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$;*
(d) *every direct sum of tight modules in $\sigma[M]$ is tight (weakly tight) in $\sigma[M]$;*

(e) *every direct sum of weakly tight modules in* $\sigma[M]$ *is weakly tight* (*weakly* R*-tight*) *in* $\sigma[M]$*;*
(f) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* N*-tight for every cyclic module* N *in* $\sigma[M]$*;*
(g) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* R*-tight in* $\sigma[M]$*.*

In case $M = R_R$ in Corollary 3.4, we obtain characterizations of q.f.d. rings that generalize Theorem 2.6 and Corollary 2.7 in [30] and the main theorem in [2].

Corollary 3.5. *For a ring* R*, the following conditions are equivalent:*

(a) R *is q.f.d.;*
(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of injectives is weakly injective* (*or tight, weakly tight*)*;*
(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly injective is weakly injective* (*or tight, weakly tight*)*;*
(d) *every direct sum of tight modules is tight* (*or weakly tight*)*;*
(e) *every direct sum of weakly tight module is weakly tight* (*or weakly* R*-tight*)*;*
(f) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ *is simple, is weakly* N*-tight for every cyclic module* N*;*
(g) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ*is simple, is weakly* R*-tight.*

Theorem 3.6. *A locally right q.f.d. module* M_R *over which every uniform cyclic right module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$ *is right weakly semisimple.*

PROOF: Let $N \in \sigma[M]$. Then N contains an essential submodule $X = \bigoplus_I X_i$ which is a direct sum of cyclic uniform submodules. It follows by our hypothesis that each X_i is weakly injective in $\sigma[M]$ and thus by Lemma 2.3, $\bigoplus_I X_i$ is weakly injective in $\sigma[M]$. Thus N is weakly injective in $\sigma[M]$, proving that M is weakly semisimple. □

Theorem 3.7. *For a module* M_R*, the following conditions are equivalent:*

(a) M *is weakly semisimple;*
(b) M *is locally q.f.d. and every finitely generated module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$*;*
(c) M *is locally q.f.d. and every cyclic module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$*;*
(d) M *is locally q.f.d. and every uniform cyclic module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$*;*
(e) M *is locally q.f.d. and every finitely generated module in* $\sigma[M]$ *is compressible.*

Proof: (a) $\Rightarrow$ (b). Follows from Corollary 3.5.

Clearly, (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (e). Let N be a finitely generated module in $\sigma[M]$ and let $K \subseteq' N$. Since M is locally q.f.d., N has finite uniform dimension. Thus there exist cyclic uniform submodules U_i, $i = 1, \ldots, n$, of N such that $\bigoplus_{i=1}^{i=n} U_i \subseteq' K \subseteq N$. Since each U_i is uniform it follows that each U_i is weakly injective in $\sigma[M]$ and thus by Lemma 2.4, $\bigoplus_{i=1}^{i=n} U_i$ is weakly injective in $\sigma[M]$. Thus, by Lemma 2.3, K is weakly injective in $\sigma[M]$ and thus N embeds in K, proving that N is compressible.

(e) $\Rightarrow$ (a). Let $0 \neq X$ in $\sigma[M]$ and let N be a finitely generated submodule of $\widehat{X}$. Let $0 \neq x \in X$. Then $xR \cap N \subseteq' N$. By our hypothesis N is compressible and thus N embeds in $xR \cap N$ and thus embeds in X, proving that X is tight in $\sigma[M]$. Thus, M is weakly semisimple by Theorem 3.6. □

As a consequence of Theorem 3.7 we get [9, Theorem 3.1].

In case $M = R$ we obtain characterizations of weakly semisimple rings that generalize those known results.

Corollary 3.8. *For a ring R, the following conditions are equivalent:*

(a) *R is weakly semisimple;*
(b) *R is q.f.d. and every finitely generated module is weakly injective (tight, weakly tight);*
(c) *R is q.f.d. and every cyclic module is weakly injective (tight, weakly tight);*
(d) *R is q.f.d. and every uniform cyclic module is weakly injective (tight, weakly tight);*
(e) *R is q.f.d. and every finitely generated module is compressible.*

We conclude with the following open questions:

(1) Can we replace the assumption of perfectness of a ring in Theorem 2.9 by semiperfectness?
(2) Can we remove locally q.f.d. throughout Theorem 3.7 and in Theorem 3.6?

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