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ω_1-generated uniserial modules over chain rings

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Abstract. The purpose of this paper is to provide a criterion of an occurrence of uncountably generated uniserial modules over chain rings. As we show it suffices to investigate two extreme cases, nearly simple chain rings, i.e. chain rings containing only three two-sided ideals, and chain rings with "many" two-sided ideals. We prove that there exists an ω_1-generated uniserial module over every non-artinian nearly simple chain ring and over chain rings containing an uncountable strictly increasing (resp. decreasing) chain of right (resp. two-sided) ideals. As a consequence we describe right steady serial rings.

Keywords: chain rings, serial rings, uniserial modules

Classification: Primary 16L30; Secondary 16D20, 16D80

Introduction

Obviously, every right ideal of a chain ring is a uniserial right module. The question is when there exists a uniserial module generated by more than countably many elements over a chain ring with at most countably generated right ideals. Note that by definition a chain ring R has the linearly ordered lattice of left ideals as well as the lattice of right ideals. So R is a right Ore ring and the right ring of fractions of every completely prime factor of R is uniserial as a right R-module.

The class of chain rings is a natural generalization of the well known class of valuation rings. Basic properties of general chain rings are described in [BBT] and in [F, Chapter 6]. In [D1], [D2] and [BBT] there are presented non-trivial examples of a chain ring which contains precisely three two-sided ideals, i.e. of a *nearly simple* chain ring (non-trivial means that it has the infinitely generated Jacobson radical). Surprisingly, it is possible to construct an ω_1-generated uniserial module over every non-trivial nearly simple chain ring (Theorem 1.8). The "strange" properties of this class of chain rings are partially described in [P1] and [P2].

The occurrence or, on the other hand, non-existence of a big uniserial right module over a chain ring is equivalent to the question whether the ring is right steady or non-steady. Recall that a ring R is called *right steady* if every right R-module which is not a union of a countable infinite strictly increasing chain of

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submodules is finitely generated. The basic results concerning steady rings are published in [EGT], [ZT] and [Z].

A ring-theoretic characterization of commutative steady chain rings (valuation rings) is proved in [ZT]. We characterize the steadiness of chain rings in general in the present paper (Theorem 2.4).

An application of the results obtained for chain rings gives us a characterization of steadiness of a natural generalization of the notion of a chain ring, serial rings (for more about serial rings see [F, Chapter 6] and [P4]). Namely, a serial ring is not right steady if and only if there exists an ω_1-generated uniserial module (Theorem 3.5).

Throughout the paper a *ring* R means an associative ring with unit, a *module* means a right R-module, and a *bimodule* is an R-R-bimodule. If U is a bimodule, $\mathrm{r.ann}(x)$ (resp. $\mathrm{l.ann}(x)$), denotes the right (resp. left) annihilator of an element $x \in U$. The minimal cardinality of a set of generators of an R-module M is denoted by $\mathrm{gen}_R(M)$. The Jacobson radical of the ring R is denoted by $J(R)$ and the prime radical is denoted by $\mathrm{rad}(R)$.

A bimodule ${}_RB_R$ is said to be *right* (resp. *left*) *chain* provided B_R (resp. ${}_RB$) is a uniserial right (resp. left) module, i.e. its lattice of right (resp. left) submodules is linearly ordered. A bimodule is said to be a *chain* if it is both right and left chain. A ring is said to be *chain* provided it is a chain bimodule ${}_RR_R$. A ring R is called *serial* if R contains a set of orthogonal idempotents $\{e_i,\ i \leq n\}$ such that $1 = \sum_{i \leq n} e_i$ and for every index $i \leq n$ the ideals e_iR and Re_i are uniserial modules. The set $\{e_i,\ i \leq n\}$ is called a *complete set* of orthogonal idempotents. A module is said to be *dually slender* if it is not a union of a countable strictly increasing chain of submodules and a ring is called *right steady* if each dually slender right R-module is finitely generated. As we have remarked, a *nearly simple ring* is a ring which contains precisely three ideals, i.e. 0, $J(R)$ and R.

1. Nearly simple chain rings

We start the section with studying properties of a generalization of a two-sided ideal.

Definition 1.1. Let U be a bimodule. Then $S \subseteq U$ is a *subbimodule* of U if S is both a right and left submodule of U. A non-zero bimodule is said to be *simple* provided it contains only trivial subbimodules. A bimodule U is said to be *exceptional* if for every non-invertible element $r \in R$ there exist non-zero elements $u, v \in U$ such that $ur = rv = 0$.

Note that the bimodule $R/J(R)$ is an exceptional simple bimodule whenever R is local i.e. $R/J(R)$ is a skew-field. The notion of an exceptional chain bimodule generalizes the Dubrovin's term of an *exceptional nearly simple chain ring* (cf. [P2]). A natural example of an exceptional chain bimodule is the Jacobson radical of an exceptional nearly simple chain ring. Later we will show that there exists a non-artinian (as a right module) exceptional simple bimodule over every non-artinian nearly simple chain ring.

For every element u of a ring denote by $\mathrm{r.ker}(u)$ (resp. $\mathrm{l.ker}(u)$) a right (resp. left) submodule of a bimodule U containing all elements of U annihilated from left (resp. right) by u.

Remark 1.2. Let U be an exceptional chain bimodule over a chain ring R and $a, b \in U$.

(1) If $a,\ b$ have the same right (resp. left) annihilator, there exists an invertible element $u \in R$ for which $ua = b$ (resp. $au = b$).
(2) If $aR \subset bR$, then $\mathrm{l.ann}(b) \subset \mathrm{l.ann}(a)$.

Proof: (1) It is sufficient to prove the right hand version of the remark.

If $a = 0$, the claim is trivial. Suppose $a \neq 0$. Since the bimodule U is left chain, there exists an element $u \in R$ such that $ua = b$ or $ub = a$. Without loss of generality suppose that $ua = b$. Assume that $u \in J_R$. Then $\mathrm{r.ker}(u) \neq 0$ because U is exceptional. As $\mathrm{r.ker}(u) \cap aR \neq 0$, there is $r \in R$ such that $ar \neq 0$ and $uar = br = 0$, i.e., $r \in \mathrm{r.ann}(b) \setminus \mathrm{r.ann}(a)$, a contradiction. Thus $u \in R \setminus J_R$ is an invertible element.

(2) Obviously, $\mathrm{l.ann}(b) \subseteq \mathrm{l.ann}(a)$. The inclusion is strict by (1). □

In the sequel, R is a chain ring and U is an exceptional simple chain bimodule. First, make an easy observation concerning exceptional chain bimodules.

Lemma 1.3. *Let M be a uniserial R-module with a countable set $x_0, x_1, x_2, \dots$ of generators. Suppose that every cyclic submodule x_nR is isomorphic to a right submodule of U. Then M is isomorphic to a right submodule of U.*

Proof: Without loss of generality $x_0R \subseteq x_1R \subseteq x_2R \subseteq \dots$. It is sufficient to define by induction a sequence of monomorphisms $\nu_n\colon x_nR \to U$ such that $\nu_{n+1}(x_n) = \nu_n(x_n)$ for each $n = 0, 1, 2, \dots$.

First, fix an arbitrary embedding $\nu_0 : x_0R \to U$.

Suppose that ν_n is defined. By hypothesis, there exists an injective homomorphism $\mu\colon x_{n+1}R \to U$. Since $\mu(x_n)$ has the same right annihilator as $\nu_n(x_n)$, by Remark 1.2 (1) there exists an invertible element $u \in R$ such that $u\mu(x_n) = \nu_n(x_n)$. Define a homomorphism $\nu_{n+1}(x) = u\mu(x)$ for every $x \in x_{n+1}R$. Obviously, ν_{n+1} is injective and $\nu_{n+1}(x_n) = \nu_n(x_n)$. □

Lemma 1.4. *Suppose that U_R is not cyclic and that there exists an embedding of U_R into xR for a suitable element $x \in U$. Then there exists an ω_1-generated uniserial right module.*

Proof: It suffices to construct a directed system of cyclic modules $\{m_\alpha R; (\phi_{\beta,\alpha},\ \beta \leq \alpha) |\ \alpha < \omega_1\}$, indexed by all countable ordinals, satisfying the property that the homomorphisms $\phi_{\beta,\alpha} : m_\beta R \to m_\alpha R$ are injective and not surjective whenever $\beta < \alpha < \omega_1$ and such that all the $m_\alpha R$ are isomorphic to proper right submodules of U_R.

First, fix an arbitrary non-zero element $m_0 \in U_R$.

Let $0 \neq \alpha < \omega_1$ and suppose that we have already defined all the $m_\beta R$ for $\beta \leq \alpha$ and all $\phi_{\gamma,\beta}$ for $\gamma < \beta \leq \alpha$ with the required properties. Since there is a monomorphism $\psi : m_\alpha R \to U_R$ and $\operatorname{im}(\psi) \neq U_R$, we can fix an element $m_{\alpha+1} \in U_R$ such that $\operatorname{im}(\psi) \subset m_{\alpha+1}R$. Now, put $\phi_{\beta,\alpha+1} = \psi\phi_{\beta,\alpha}$.

Let α be a countable limit ordinal. Let $(M,\ \{\phi_\gamma;\ \gamma < \alpha\})$ be a direct limit of the directed system $\{(m_\beta R;\ \beta < \alpha),\ (\phi_{\gamma,\beta};\ \gamma \leq \beta < \alpha)\}$. Clearly, the homomorphisms ϕ_γ are injective for every $\gamma < \alpha$. Since α is of countable cofinality, it is possible to express M as the union of a countable strictly increasing chain of modules $\phi_{\alpha_n}(m_{\alpha_n}R)$, where $\sup\{\alpha_n, n < \omega\} = \alpha$. Since the submodules $\phi_{\alpha_n}(m_{\alpha_n})R$ are embeddable into U, the hypothesis of Lemma 1.3 is satisfied. Therefore there exists an embedding $\psi\colon M \to U_R$. Moreover, $\psi(M)$ is embeddable into a cyclic right submodule $xR \subseteq U_R$ by the hypothesis. Thus the composition of these two monomorphisms gives us the monomorphism $\rho\colon M \to xR$. Now put $m_\alpha = x$ and $\phi_{\beta,\alpha} = \rho\phi_\beta$.

Since the direct limit of the directed system we have just constructed is a union of a strictly increasing chain of submodules which has the length ω_1, we get the required example of an ω_1-generated uniserial module. $\square$

Lemma 1.5. *Let U_R be an infinitely countably generated right R-module. Then every infinitely countably generated right submodule of U is isomorphic to U_R.*

Proof: Let $I = \bigcup_{n<\omega} x_n R$ be an infinitely generated right submodule which is not equal to U_R. We may suppose that $0 \subset x_0R \subset x_1R \subset \dots$. Fix an element $z_0 \in U_R$ such that $I \subseteq z_0R$. Since U_R is countably generated we can express U_R as a union of a strictly increasing chain of cyclic right submodules z_nR, $n < \omega$.

We now show by induction on n that there exists a sequence $y_1, y_2, \dots$ of elements of $J = J(R)$ such that $z_nR \subseteq y_nx_nR$ and $y_{n+1}x_n = y_nx_n$ for every n. As the left socle $\operatorname{Soc}({}_RU)$ of a bimodule is a subbimodule, we must have either $\operatorname{Soc}({}_RU) = 0$ or $\operatorname{Soc}({}_RU) = {}_RU$. But ${}_RU$ is a non-artinian chain bimodule, so that $\operatorname{Soc}({}_RU) = 0$. In particular $Jx_0 \neq 0$. Hence $\bigcup_{y\in J} yx_0R$ is a non-zero subbimodule, so that $\bigcup_{y\in J} yx_0R = U_R$. Therefore there is an element $y_0 \in J$ such that $z_0R \subseteq y_0x_0R$.

Suppose that the element y_n has been defined. If $z_{n+1}R \subseteq y_nx_{n+1}R$, the element $y_{n+1} = y_n$ trivially satisfies our requirements. Otherwise $y_nx_{n+1}R \subset z_{n+1}R$. Put $L = \operatorname{l.ann}(x_n)$. Since $x_nR \subset x_{n+1}R$, it follows from Remark 1.2(2) that $Lx_{n+1} \neq 0$. The fact that $\bigcup_{a\in L} ax_{n+1}R$ is a subbimodule implies the equation $\bigcup_{a\in L} ax_{n+1}R = U_R$. Therefore there is an element $a \in L$ for which $z_{n+1}R \subset ax_{n+1}R$.

Now, put $y_{n+1} = y_n + a$. We need to show that y_{n+1} satisfies our requirements. Since $a \in \mathrm{l.ann}(x_n)$, the equations $y_{n+1}x_n = y_nx_n + ax_n = y_nx_n$ hold true. Finally, we check the strict inclusion $z_{n+1}R \subset y_{n+1}x_{n+1}R$. Assume that $y_{n+1}x_{n+1}R \subseteq z_{n+1}R$. It means that $(y_n + a)x_{n+1} \in z_{n+1}R$. As y_nx_{n+1} is an element of $z_{n+1}R$, it holds true that $ax_{n+1} = y_{n+1}x_{n+1} - y_nx_{n+1} \in z_{n+1}R$. This is a contradiction because $z_{n+1}R \subset ax_{n+1}R$. Hence $z_{n+1} \in y_{n+1}x_{n+1}R$.

As $y_n \in J$, the element $1 - y_n$ is invertible, so the left multiplication by $1 - y_n$ induces a monomorphism $u_n: x_nR \to U_R$. Moreover, $u_{n+1}(x_n) - u_n(x_n) = (y_n - y_{n+1})x_n = 0$. Thus the monomorphisms u_n determines in natural way a homomorphism $u: I \to U_R$, which is injective. In order to complete the proof it suffices to show that u is onto U_R.

Assume that $(1 - y_n)x_n \in x_nR$. Then also $y_nx_n = x_n - (1 - y_n)x_n \in x_nR \subset z_0R \subset z_nR$, a contradiction. Hence $x_n \in (1 - y_n)x_nR$, and so $y_nx_n = x_n - (1 - y_n)x_n \in (1 - y_n)x_nR$. Since $y_nx_nR \subseteq (1 - y_n)x_nR$ for every $n = 1, 2, \dots$, $U = \bigcup_{n<\omega} z_nR \subseteq \bigcup_{n<\omega} y_nx_nR \subseteq \bigcup_{n<\omega}(1 - y_n)x_nR$ is contained in the image of the homomorphism u. □

The claim of the previous lemma is proved in [P2, Lemma 5.8] for countably generated locally coherent uniserial modules over an exceptional coherent chain ring.

Corollary 1.6. *If there exists a non-artinian exceptional simple chain bimodule, there exists an ω_1-generated uniserial module.*

Proof: If an exceptional simple chain bimodule is uncountably generated, the claim is trivial. On the other hand, let U_R be countably generated. Since an exceptional simple bimodule is strictly embeddable into itself by Lemma 1.5, we obtain an example of an ω_1-generated uniserial module applying Lemma 1.4. □

Lemma 1.7. *Let J be a completely prime ideal of a prime chain ring R. Then there exists the right localization of R with respect to J, denote it by R_J, such that the natural homomorphism $R \to R_J$ is injective. We may suppose that $R \subseteq R_J$.*

(1) *Then for every $x \in R_J \setminus R$ there exists $s \in R \setminus J$ such that $x = s^{-1}$.*
(2) *Let M be a uniserial R_J-module. Then M is a uniserial R-module.*

Proof: First, we check the existence of the localization R_J. For this purpose we will apply the existence theorem [C, Theorem 0.5.3]. Take arbitrary elements $r \in R$ and $t \in R \setminus J$. Note that the multiplicative set $R \setminus J$ satisfies the Ore condition, i.e. $r(R \setminus J) \cap tR \neq \emptyset$. Since $rt, tr \in R \setminus J$ whenever $r \in R \setminus J$, it suffices to prove that $rt \neq 0$ if $r \in J \setminus \{0\}$. Suppose that $r \in J$ and $tr = 0$ ($rt = 0$). As R is a chain ring and J is a two-sided ideal, we have $Rt \supseteq J \supseteq rR$ ($tR \supseteq J \supseteq Rr$), and so $rRr = 0$. Since R is a prime ring, $r = 0$. Now, applying [C, Theorem 0.5.3] we get that R_J is a ring and the natural mapping $R \to R_J$

is a ring monomorphism. In the rest of the proof we identify the image of this monomorphism and the ring R.

(1) Fix an arbitrary element $ab^{-1} \in R_J \setminus R$, $a \in R$, $b \in R \setminus J$. As R is a left uniserial module, there exists $s \in R$ such that $sa = b$ or $a = sb$. Assume that $a = sb$. Then $ab^{-1} = sbb^{-1} = s \in R$, a contradiction. Hence $sa = b$ and so $s, a \in R \setminus J$. Finally, $ab^{-1} = a(sa)^{-1} = aa^{-1}s^{-1} = s^{-1}$.

(2) Let $m, n \in M$ and suppose that $mR_J \subseteq nR_J$, i.e. $m = nx$ for suitable $x \in R_J$. If $x \in R$, there is nothing to prove, otherwise $x = s^{-1}$ for a suitable element $s \in R \setminus J$ by (1). Then $ms = ns^{-1}s = n$, hence $nR \subseteq mR$. $\square$

Theorem 1.8. *Over every non-artinian nearly simple chain ring there exists an ω_1-generated uniserial module.*

Proof: If the ring contains an uncountably generated right ideal, the claim is trivial. Otherwise, suppose that all right ideals are countably generated. In order to prove the theorem it is sufficient to find a non-artinian exceptional simple bimodule. Then the assertion follows from Corollary 1.6.

If the ring is not a domain, all elements of the Jacobson radical have nonzero right and left annihilators, because the set of all right (respectively, left) zero-divisors in a chain ring is always a completely prime ideal [BBT, Lemma 2.3(i)]. Hence the Jacobson radical is an exceptional simple bimodule.

If the ring R is a domain, there exists the right ring of fractions of R by Lemma 1.7, denote it by Q. Remark that R is a subbimodule of the R-bimodule Q so that Q/R is R-bimodule. We will show that Q/R is an exceptional simple chain bimodule.

By Lemma 1.7(2) the right and left module Q is uniserial because R is a chain ring. Hence Q and so Q/R is a chain bimodule. Applying Lemma 1.7(1) we see that every element of $Q \setminus R$ is of the form r^{-1} for some $r \in R \setminus \{0\}$. As $j^{-1}j = jj^{-1} = 1 \in R$ for every element $j \in R \setminus \{0\}$, the bimodule Q/R is exceptional.

It remains to prove that Q/R is a simple bimodule. As we have observed, each non-zero cyclic R-submodule of Q strictly containing the submodule R is generated by a suitable element r^{-1} for non-invertible $r \in R \setminus \{0\}$. Fix such an element r^{-1}. Remark that $rR \subset J(R)$. Moreover, $Q = \bigcup_{x \in R\setminus\{0\}} x^{-1}R$ hence it suffices to show that for each $x \in R \setminus \{0\}$ there exists an element $y \in R$ such that $x^{-1}R \subseteq yr^{-1}R$. Note that $Rx \cap (R \setminus rR) \neq \emptyset$. Indeed, the left ideal Rx is not a subset of the right ideal rR because rR contains no non-zero ideal and so no non-zero left ideal. Fix an element $c \in Rx \cap (R \setminus rR)$. As $c \in Rx$, there exists an element $s \in R$ such that $c = sx$. Hence $c^{-1}s = x^{-1}$ and so $x^{-1}R \subseteq c^{-1}R$. Finally, since $rR \subseteq cR$, there exists $y \in R$ such that $cy = r$. Now $yr^{-1} = c^{-1}$ and $x^{-1}R \subseteq yr^{-1}R = c^{-1}R$ which finishes the proof. $\square$

Observe that a nearly simple chain prime ring (in particular, a domain) is not artinian.

Since an artinian nearly simple chain ring R (clearly, it is neither a domain nor a prime ring) has Loewy length equal to 2, every R-module is semiartinian with socle length at most 2. Hence every non-zero uniserial module over an artinian nearly simple chain ring is either simple or it is isomorphic to the module R_R.

Example 1.9. There exist a countable nearly simple chain domain and a countable exceptional nearly simple chain prime ring which is not a domain.

Proof: The both examples are due to Dubrovin, for the first one see [BBT, Section 6.5] or [D1], and for the second see [D2].

The example of a countable non-artinian nearly simple chain domain is the localization of the ring $K[P]$ with respect to suitable Ore system where K is a skew-field and P is a suitable subsemigroup of a semidirect product of the additive group $(Q,+)$ and the multiplicative group $(Q_{>0},.)$. Hence the ring is countable whenever we chose a countable skew-field K.

The example of a countable non-artinian exceptional nearly simple chain prime ring which is not a domain is constructed as a suitable factor of ring R which is contained in a rational closure of a group ring $K[G]$. Since the group G in the construction in [D2] is countable, the ring R is countable iff the skew-field K is countable. $\square$

2. Chain rings

In this section we show that the question "over which chain rings there exists an ω_1-generated uniserial module" leads to checking two extreme cases. First of them, the case of nearly simple chain rings, is studied in the first section. The second situation is close to the case of commutative rings.

Proposition 2.1. *Suppose that a chain ring R properly contains a prime ideal P and an ideal J, $P \subset J$, such that J/P is a simple bimodule. Then there exists an ω_1-generated uniserial right module.*

Proof: Without loss of generality suppose that $P=0$, so R is a prime ring. Note that J is an idempotent because $J^2 \not\subseteq P$ and J/P is a simple bimodule. Hence, by [BBT, Theorem 1.15(1)] J is completely prime. Now, the right localization R_J of the ring R with respect to the prime ideal J (which exists by Lemma 1.7) is a nearly simple chain ring. Since R_J is prime, the socle of the ring R_J vanishes. Hence the Jacobson radical J_J is not artinian. It follows from Theorem 1.8 that there exists an ω_1-generated uniserial R_J-module M. Applying Lemma 1.7(2), M has a structure of a uniserial R-module. Finally, remark that $\mathrm{gen}_R(M) \geq \mathrm{gen}_{R_J}(M) = \omega_1$. $\square$

Recall an another meaning of the term exceptional as it used in [BBT]. A prime ideal is *exceptional* if it is not completely prime.

Corollary 2.2. *If a chain ring R contains an exceptional prime ideal, there exists an ω_1-generated uniserial right module.*

PROOF: By [BBT, Theorem 6.2] there exists a completely prime ideal above an exceptional prime ideal which satisfies the hypothesis of Proposition 2.1. □

Lemma 2.3. *Let R be a chain ring. If $R/\operatorname{rad}(R)$ contains an uncountable strictly decreasing chain of left ideals, there exists an ω_1-generated uniserial module.*

PROOF: If there is an exceptional prime ideal, the existence of an ω_1-generated uniserial module follows immediately from Corollary 2.2. Hence let us suppose that all prime ideals are completely prime. In particular, $\operatorname{rad}(R)$ is a completely prime ideal. We may assume that $\operatorname{rad}(R) = 0$.

Since R contains an uncountable strictly decreasing chain of left ideals, it contains an uncountable strictly decreasing chain of left principle ideals $(Rs_\alpha | \alpha < \omega_1)$ as well. Thus the right ring of fractions $Q(R)$ (which exists by Lemma 1.7) contains as right R-modules an uncountable strictly increasing chain of cyclic submodules $(s_\alpha^{-1}R)$. The union $\bigcup_{\alpha<\omega_1} s_\alpha^{-1}R$ gives us an ω_1-generated uniserial submodule. □

Recall that a module M is said to be *dually slender* provided the covariant functor $\hom_R(M, -)$ commutes with direct sums. Several characterizations of this notion are known. The most important and useful is the following one: The module is dually slender if and only if it is not a union of a countable infinite strictly increasing chain of submodules (for a proof see [T, Lemma 1.2]). Note that the class of all dually slender modules contains all finitely generated modules and it is closed under taking factors and taking finite sums.

A ring is called *right steady* if the class of all dually slender modules coincides with the class of all finitely generated modules. Recall that a factor of a right steady ring is right steady as well.

Theorem 2.4. *For a chain ring R the following conditions are equivalent:*

1. *R is right steady;*
2. *there exists no ω_1-generated uniserial right module;*
3. *$R/\operatorname{rad}(R)$ contains no uncountable strictly decreasing chain of ideals, R contains no uncountably generated right ideal and for every ideal I and for every prime ideal $P \subsetneq I$ there exists an ideal K such that $P \subset K \subset I$.*

PROOF: The implication (1) → (2) is in obvious.

(2) → (3) Let us suppose that the negation of (3) holds true and we will find an ω_1-generated uniserial module.

If there exists an uncountable strictly decreasing chain of ideals, there exists an ω_1-generated uniserial module by Lemma 2.3. Since R is a chain ring, an occurrence of an ω_1-generated right ideal gives us an example of an ω_1-generated uniserial module. Finally, suppose that $R/\operatorname{rad}(R)$ contains no uncountable strictly decreasing chain of ideals and that R contains no uncountably generated right ideal. Then the negation of the third item of (3) implies the hypothesis of Proposition 2.1 is satisfied. Applying the proposition we get a required ω_1-generated uniserial module.

(3) $\rightarrow$ (1) Assume that (3) holds and R is not right steady. Fix an infinitely generated dually slender module M. Since the ring R is local, by [Z, Corollary 1.5] it is possible to chose M containing no maximal submodule. Then every non-zero factor of M is an infinitely generated module. As R contains only countably generated right ideals, we may apply [ZT, Lemma 11]. Hence there exists a prime ideal P such that M/MP is an infinitely generated module and $M/MI = 0$, i.e. $M = MI$ for every ideal $P \subset I$. As no simple bimodule occurs above the prime ideal P, P is an intersection of a strictly decreasing chain of ideals. Thus by [BBT, Theorem 1.15(1)] P is a completely prime ideal. By hypothesis every strictly decreasing chain of ideals is countable. In particular, there exists a decreasing sequence of ideals $I_1 \supset I_2 \supset \cdots \supset I_n \ldots$ such that $P = \bigcap_{n<\omega} I_n$. Hence the right ring of fractions $Q = Q(R/P)$ (which exists by Lemma 1.7) is countably generated as a right R/P-module.

Define modules $M_n = \{m \in M \mid mI_n \subseteq MP\}$ for every integer n. Note that $M_n \subset M$ for every n. Hence $N = M/\bigcup_{n<\omega} M_n$ is nonzero so it is an infinitely generated dually slender R/P-module. We prove that N is a torsion-free R/P-module.

Let $m \in M$ and let $mx \in \bigcup_{n<\omega} M_n$ for some $x \in R \setminus P$. Then $mxI_n \subseteq MP$ for a suitable n. As P is prime, there exists $k \geq n$ such that $I_k \subseteq xI_n$. Thus $mI_k \subseteq MP$ and $m \in M_k$. It implies N is a torsion-free R/P-module. So that the module N is embeddable into the module $N \otimes_{R/P} Q \cong Q^{(\kappa)}$ (recall Q is the right ring of fractions of R/P). Since N is dually slender, N embeds into a direct sum of finitely many copies of Q. Hence we have an infinitely generated dually slender submodule of a countably generated module. By [ZT, Lemma 5] a suitable cyclic R-module contains an infinitely generated dually slender submodule which is an ω_1-generated. It is a contradiction because each submodule of any cyclic R-module is countably generated. $\square$

The condition (3) immediately implies the following fact.

Corollary 2.5. *Let all right and left ideals of a chain ring R be countably generated. Then R is right steady if and only if R is left steady.*

Note that each prime ideal over a chain ring with the Krull dimension is completely prime by [F, 7.17 and 7.38]. In particular, the prime radical $\operatorname{rad}(R)$, i.e. the intersection of all prime ideals of R, is completely prime.

Corollary 2.6. *Let R be a chain ring with the Krull dimension. The following conditions are equivalent:*

(1) *R is right steady;*
(2) *there exists no ω_1-generated uniserial right module;*
(3) *the right ring of fractions $Q(R/\operatorname{rad}(R))_R$ contains no ω_1-generated right R-submodule;*
(4) *all right ideals of R are countably generated and R is of a countable Krull dimension.*

Proof: As there is no idempotent prime ideal in R, every prime factor of R contains no non-trivial simple bimodule. Moreover, a chain ring with the Krull dimension possesses an uncountable strictly decreasing chain of ideals if and only if it has an uncountable Krull dimension. Finally, $R/\operatorname{rad}(R)$ is right steady if and only if R is right steady because $\operatorname{rad}(R)$ is a nilpotent ideal [ZT, Lemma 3]. Now the claim is a consequence of Theorem 2.4. □

Example 2.7. There exists a non-steady commutative valuation domain with the Krull dimension such that all ideals are countably generated.

Proof: Take the inverse natural order on the ordinal ω_1 and define on the group $\mathbb{Z}^{(\omega_1)}$ the corresponding lexicographic order $\preceq$ where $\mathbb{Z}$ is equipped with the natural order. By the Krull theorem (see for example [FS, Theorem I.3.4]) there exists a valuation domain R with the value group order-isomorphic to $(\mathbb{Z}^{(\omega_1)}, \preceq)$. It is not hard to prove that R has the Krull dimension which is at least equal ω_1 so R is not steady. Moreover, every filter on $(\mathbb{Z}^{(\omega_1)}, \preceq)$ is countably generated, hence every ideal of the ring R is countably generated. □

3. An application on serial rings

We start this section with two general lemmas which show the correspondence between classes of dually slender modules over rings R and eRe for an idempotent $e \in R$.

Lemma 3.1. *Let R be an arbitrary ring and let $e \in R$ be an idempotent. Suppose M is a right R-module such that $MeR = M$.*

(1) *Me is an infinitely generated eRe-module if M is an infinitely generated R-module.*
(2) *Me is a dually slender eRe-module if M is a dually slender R-module.*

Proof: Note that Me has a natural structure of a right eRe-module.

(1) Suppose Me is a finitely generated eRe-module, hence $Me = \sum_{i\leq n} m_i eRe$. Since $M = MeR$, we get $M = \sum_{i\leq n} m_i eReR = \sum_{i\leq n} m_i eR$, i.e. M is a finitely generated R-module.

(2) Suppose $Me = \bigcup_{i<\omega} N_i e$ where $N_i e$ is an increasing chain of eRe-submodules. Then $M = MeR = \bigcup_{i<\omega} N_i eR$. As M is dually slender, there exists $n < \omega$

such that $M = N_i eR$. Thus $Me = N_i eRe = N_i e$ which proves Me is a dually slender eRe-module. $\square$

Lemma 3.2. *Let R be an arbitrary ring and let $e \in R$ be an idempotent. Then the functor $- \otimes_{eRe} eR$ maps*

(1) *each infinitely generated eRe-module onto an infinitely generated R-module,*
(2) *each dually slender eRe-module onto a dually slender R-module.*

Proof: We will write $- \otimes -$ instead $- \otimes_{eRe} -$ in the whole proof.

(1) Suppose P is an infinitely generated eRe-module, i.e. $P = \bigcup_{\alpha<\kappa} P_\alpha$ for a suitable infinite cardinal κ and a strictly increasing chain of submodules P_α, $\alpha < \kappa$. Denote by π_α the natural projection P onto P/P_α and put $Q_\alpha = \ker(\pi_\alpha \otimes eR)$ $(= \{\sum_{j\le m} p_j \otimes er_j |\ p_j \in P_\alpha, r_j \in R,\ j \le m < \omega\},\ \alpha < \kappa)$ for every $\alpha < \kappa$. Remark that $(Q_\alpha |\ \alpha < \kappa)$ forms an increasing chain of R-submodules of $P \otimes eR$. As the tensor product commutes with direct limits, $P \otimes eR = \bigcup_{\alpha<\kappa} Q_\alpha$.

It remains to prove that $Q_\alpha \neq P$ for each $\alpha < \kappa$. Fix one $\alpha < \kappa$. Note that $(P/P_\alpha) \otimes eR \neq 0$ because $(P/P_\alpha) \otimes eR \cong_{\mathbb{Z}} ((P/P_\alpha) \otimes eRe) \oplus ((P/P_\alpha) \otimes eR(1-e))$ and $(P/P_\alpha) \otimes eRe \cong P/P_\alpha \neq 0$. Since the tensor product is a right exact functor, $(P \otimes eR)/Q_\alpha \cong (P/P_\alpha) \otimes eR \neq 0$. Hence $Q_\alpha \neq P \otimes eR$ and $P \otimes eR$ is an infinitely generated R-module.

(2) Let $P \otimes eR = \bigcup_{n<\omega} N_n$ for an increasing chain of R-submodules N_i, $i < \omega$. Define eRe-submodules $P_n = \{m \in P \mid\ m \otimes e \in N_n\}$ for each $n < \omega$. Note that the sequence $(P_n |\ n < \omega)$ forms an increasing chain of eRe-submodules of P such that $P = \bigcup_{n<\omega} P_n$. Moreover, it is easy to see that $\{p \otimes e |\ p \in P_n\}R \subseteq N_n$, for each $n < \omega$. As P is a dually slender eRe-module, there exists $n < \omega$ such that $P_n = P$, hence $P_n \otimes eR = N_n = P \otimes eR$. Thus $P \otimes eR$ is a dually slender R-module. $\square$

Proposition 3.3. *Let R be an arbitrary ring and let $\{e_i |\ 1 \le i \le n\}$ be a set of orthogonal idempotents satisfying $\sum_{i\le n} e_i = 1$. Then R is right steady if and only if $e_i Re_i$ is right steady for every $i \le n$.*

Proof: Suppose that R is not right steady and fix an infinitely generated dually slender R-module M. Define by induction a sequence of R-modules M_i: $M_0 = M$, $M_{i+1} = M_i / M_i e_i R$. Clearly, $M_n = 0$. Take a minimal $i < n$ such that M_{i+1} is finitely generated. Then there exists a finitely generated submodule $F \subseteq M_i$ such that $M_i e_i R + F = M_i$. Hence M_i/F is an infinitely generated dually slender module and $(M_i/F)e_i R = M_i/F$. Now we may apply Lemma 3.1 which proves that $(M_i/F)e_i$ is an infinitely generated dually slender $e_i Re_i$-module, hence $e_i Re_i$ is not right steady.

On the other hand, if $e_i Re_i$ is not right steady for at least one i and N is an infinitely generated dually slender $e_i Re_i$-module, then $N \otimes_{e_i Re_i} e_i R$ is a dually slender R-module by Lemma 3.2. Thus R is not right steady. $\square$

Proposition 3.3 allows us to construct (non-) steady subrings of full matrix rings over a (non-) steady ring:

Example 3.4. Let R be a ring and let I, J be ideals of R. Then the subring $\begin{pmatrix} R & I \\ J & R \end{pmatrix}$ of the full matrix ring is right steady iff R is right steady.

As a consequence of Proposition 3.3 we reduce the question of steadiness of a serial ring R to the question of steadiness of chain rings e_iRe_i for the complete set of orthogonal idempotents $\{e_i,\ i \leq n\}$.

Theorem 3.5. *The following conditions are equivalent for a serial ring R with a complete set of orthogonal idempotents $\{e_i,\ i \leq n\}$:*

(1) *R is right steady;*
(2) *e_iRe_i is right steady for every $i \leq n$;*
(3) *there exists no ω_1-generated uniserial right R-module.*

Proof: The equivalence of (1) and (2) follows immediately by Proposition 3.3 and the implication (1) → (3) holds true by definition.

It remains to show (3) → (2). Proving indirectly suppose that e_iRe_i is not right steady for one index $i \leq n$. Applying Theorem 2.4 we get an ω_1-generated uniserial right e_iRe_i-module $N = \bigcup_{\alpha<\omega_1} p_\alpha e_iRe_i$. Since the functor $-\otimes_{e_iRe_i} e_iR$ (denote it by $-\otimes e_iR$) commutes with direct limits, the right R-module $M = N \otimes e_iR$ is the direct limit of a linear directed system of modules $(p_\alpha \otimes e_i)R$, $\alpha < \omega_1$. As every R-module $(p_\alpha \otimes e_i)R$ is a homomorphic image of the uniserial R-module e_iR, M is a union of an increasing chain of uniserial R-modules, hence it is a uniserial R-module as well. By the construction M is at most ω_1-generated. Finally, M is infinitely generated dually slender by Lemma 3.2, so it is precisely an ω_1-generated uniserial right R-module. $\square$

Example 3.6. Let R be a chain ring and I be the maximal ideal of R. Then the ring $S = \begin{pmatrix} R & I \\ R & R \end{pmatrix}$ is right (left) steady iff R is right (left) steady. Denote $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then every right ideal contained in e_1S is either of the form $\begin{pmatrix} iR & iR \\ 0 & 0 \end{pmatrix}$ for some $i \in I$ or of the form $\begin{pmatrix} rR & rI \\ 0 & 0 \end{pmatrix}$ for some $r \in R$. Similarly, every right ideal contained in e_2S is either of the form $\begin{pmatrix} 0 & 0 \\ rR & rR \end{pmatrix}$ or of the form $\begin{pmatrix} 0 & 0 \\ rR & rI \end{pmatrix}$ for some $r \in R$. Since rI is a maximal submodule of rR for every non-zero element $r \in R$, the right ideals e_1S and e_2S are uniserial. Applying the symmetric argument we get Se_1 and Se_2 are uniserial left ideals, hence S is a non-chain serial ring.

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