Integro-differential-difference equations associated with the Dunkl operator and entire functions

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Abstract. In this work we consider the Dunkl operator on the complex plane, defined by

$$\mathcal{D}_k f(z) = \frac{d}{dz} f(z) + k \frac{f(z) - f(-z)}{z}, \quad k \geq 0.$$

We define a convolution product associated with $\mathcal{D}_k$ denoted $*_k$ and we study the integro-differential-difference equations of the type $\mu *_k f = \sum_{n=0}^{\infty} a_{n,k} \mathcal{D}_k^n f$, where $(a_{n,k})$ is a sequence of complex numbers and μ is a measure over the real line. We show that many of these equations provide representations for particular classes of entire functions of exponential type.

Keywords: Dunkl operator, Fourier-Dunkl transform, entire function of exponential type, integro-differential-difference equation

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Introduction

In this paper we consider the first-order differential-difference operator on $\mathbb{C}$

$$\mathcal{D}_k f(z) = \frac{d}{dz} f(z) + k \frac{f(z) - f(-z)}{z}, \quad z \in \mathbb{C}, \ f \in \mathcal{A}(\mathbb{C})$$

($\mathcal{A}(\mathbb{C})$ is the space of entire functions), which is known as the Dunkl operator of index k, $k \geq 0$. It was introduced by C.F. Dunkl (see [4], [5]) and has found a wide area of applications in mathematics and mathematical physics.

It has been shown that there exists a unique intertwining operator V_k between $\mathcal{D}_k$ and $\mathcal{D} = \frac{d}{dz}$ which satisfies

$$V_k \mathcal{D} = \mathcal{D}_k V_k, \quad V_k f(0) = f(0), \quad \text{for all} \quad f \in \mathcal{A}(\mathbb{C}).$$

By using the method of generalized Taylor series, we associate with $\mathcal{D}_k$ the translation operators T_z^k, $z \in \mathbb{C}$, defined on $\mathcal{A}(\mathbb{C})$ by

$$\forall\, \omega \in \mathbb{C}, \ T_z^k f(\omega) = \sum_{n=0}^{\infty} b_n(\omega) \mathcal{D}_k^n f(z), \tag{1}$$

where $b_n(\omega) = V_k(\frac{\omega^n}{n!})$. For an appropriate measure μ over the real line and an entire function f, we define the convolution product of μ and f associated with $\mathcal{D}_k$, denoted $\mu *_k f$ and given by

$$\forall\, z \in \mathbb{C}\,,\ \mu *_k f(z) = \int_{\mathbb{R}} T^k_{-y} f(z)\, d\mu(y).$$

In this work, we are interested in the study of the following integro-differential-difference equations

$$\mu *_k f(z) = \sum_{n\geq 0} a_{n,k} \mathcal{D}^n_k f(z), \tag{2}$$

where $(a_{n,k})_{n\geq 0}$ is a sequence of complex numbers.
These equations characterize a class of entire functions of exponential type which intervenes in classical complex analysis and have many applications in other fields (for more details, one can see [3]). In fact this study shows that when the measure μ satisfies $\int_{\mathbb{R}} e^{\sigma|x|}\, d|\mu| < \infty$, where σ is a positive number, then every entire function of exponential type less than σ, is a solution of such equations and conversely if f is a C^∞-function on $\mathbb{R}$ satisfying the equation (2) and if $\sum_{n\geq 0} a_{n,k} z^n$ is analytic inside the disk $|z| \leq a$, $a \leq \sigma$, then f is the restriction to $\mathbb{R}$ of an entire function of exponential type at most a. After, we develop a method which permits us to construct solutions of these equations which are expressed in terms of normalized spherical Bessel functions of index α

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{(\frac{z}{2})^{2n}}{\Gamma(n+\alpha+1)},\ z \in \mathbb{C}.$$

Next, we suppose that $k > 0$. In this case, the restriction on $\mathbb{R}$ of the translation operators associated with the Dunkl operator given by formula (1) possess an integral representation which is available for a continuous function on $\mathbb{R}$, so that we can consider equations of the type

$$\mu *_k f = \sum_{n=0}^{N} a_{n,k} \mathcal{D}^n_k f,\ a_{N,k} \neq 0,\ N \in \mathbb{N}, \tag{3}$$

when f is a C^N-function on $\mathbb{R}$ and μ is an appropriate measure. We establish, under some assumptions, that every C^N-function on $\mathbb{R}$ satisfying equation (3) is a C^∞-function on $\mathbb{R}$. In particular, if when all but one of the $a_{n,k}$ are zero, $0 \leq n \leq N$, then f is the restriction on $\mathbb{R}$ of an entire function of exponential type.

We point out that the notion of integro-differential equations was analyzed in details by M.H. Mugler [9] in the classical case (which corresponds to $k = 0$). Later, N. Ben Salem and W. Masmoudi have studied the integro-differential equations associated with the Bessel differential operator (see [2]).

The paper is arranged as follows. The first section of this paper is devoted to the study and recall of some results of harmonic analysis associated with the Dunkl operator $\mathcal{D}_k$. Especially we define the translation operators and convolution product associated with $\mathcal{D}_k$ of an appropriate measure and an entire function, we define also the Laplace-Dunkl transform of a measure over the real line and we establish some properties related with these objects.

In the second section, we deal with the integro-differential difference equations of the type $\mu *_k f = \sum_{n\geq 0} a_{n,k}\mathcal{D}_k^n f$. We give a class of functions which are solutions of that equations.

In the third section, we assume that $k > 0$. We study equations of type $\mu *_k f = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f$, where f is a C^N-function on $\mathbb{R}$.

In the last section we establish a Paley-Wiener type theorem associated with $\mathcal{D}_k$ and give some applications. For instance, we proceed to develop conditions on the measure μ such equation of the form $\mu *_k f = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f$ characterize the class of entire functions of exponential type a which are square integrable with respect to $|x|^{2k}dx$ and bounded on the real line. Next, we continue by considering the equations characterizing entire functions of exponential type which have polynomial growth on the real line. The section closes by considering an equivalent condition characterizing the last equations in terms of the moments of the measure μ.

1. Harmonic analysis associated with the Dunkl operator

We consider the following spaces:
- $\mathcal{E}(\mathbb{R})$ the space of C^∞-functions, endowed with the usual topology of uniform convergence of the functions and their derivatives of all order on compact subsets of $\mathbb{R}$;
- $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support;
- $\mathcal{A}(\mathbb{C})$ is the space of entire functions on $\mathbb{C}$ provided with the topology of uniform convergence on every compact of $\mathbb{C}$;
- $\mathcal{A}'(\mathbb{C})$ is the topological dual of $\mathcal{A}(\mathbb{C})$;
- $\mathrm{Exp}(\mathbb{C})$ is the space of entire functions of exponential type, we have

$$\mathrm{Exp}(\mathbb{C}) = \bigcup_{a>0} \mathrm{Exp}_a(\mathbb{C}),$$

where

$$\mathrm{Exp}_a(\mathbb{C}) = \left\{ f \in \mathcal{A}(\mathbb{C}),\ N_a(f) = \sup_{\lambda\in\mathbb{C}} |f(\lambda)|e^{-a|\lambda|} < +\infty \right\}.$$

We provide $\mathrm{Exp}_a(\mathbb{C})$ with the topology defined by the norm $N_a(f)$. For this topology $\mathrm{Exp}_a(\mathbb{C})$ is a Banach space. $\mathrm{Exp}(\mathbb{C})$ is endowed with the inductive limit topology.

The Dunkl operator $\mathcal{D}_k$ associated with the parameter $k \geq 0$, is defined on $\mathbb{C}$ by

$$\mathcal{D}_k(f)(z) = \frac{d}{dz}f(z) + k\frac{f(z) - f(-z)}{z}, \ f \in \mathcal{A}(\mathbb{C}).$$

For $k = 0$, $\mathcal{D}_0$ reduces to the usual derivative which will be denoted by $\mathcal{D}$. It is well known that there exists a unique isomorphism V_k of $\mathcal{A}(\mathbb{C})$ such that

$$V_k\mathcal{D}f = \mathcal{D}_kV_kf, \quad V_kf(0) = f(0). \tag{4}$$

The operator V_k is called the Dunkl intertwining operator of index k between $\mathcal{D}_k$ and $\mathcal{D} = \frac{d}{dz}$ on the space $\mathcal{A}(\mathbb{C})$, (see [1]).
For $k > 0$, V_k has the following representation (see [5, Theorem 5.1])

$$V_kf(z) = \frac{2^{-2k}\Gamma(2k+1)}{\Gamma(k)\Gamma(k+1)}\int_{-1}^{1} f(zt)(1-t^2)^{k-1}(1+t)\,dt, \ f \in \mathcal{A}(\mathbb{C}). \tag{5}$$

For $k \geq 0$, and $\lambda, z \in \mathbb{C}$, the equation

$$\begin{cases} \mathcal{D}_ku(z) = \lambda u(z), \\ \quad u(0) = 1, \end{cases}$$

has a unique solution $\phi^k_{\lambda,0}$ given by

$$\phi^k_{\lambda,0}(z) = j_{k-\frac{1}{2}}(i\lambda z) + \frac{\lambda z}{2k+1}j_{k+\frac{1}{2}}(i\lambda z),$$

where j_α is the normalized spherical Bessel function defined for $\alpha \geq -\frac{1}{2}$, by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{(\frac{z}{2})^{2n}}{\Gamma(n+\alpha+1)}, \ z \in \mathbb{C}.$$

We remark that $\phi^k_{\lambda,0}(z) = V_k(e^{\lambda\cdot})(z)$. Formula (5) and the last result imply that

$$|\phi^k_{\lambda,0}(z)| \leq e^{|\lambda||z|}, \ |\phi^k_{\lambda,0}(x)| \leq e^{|x||\mathcal{R}e\lambda|}, \ |\phi^k_{-iy,0}(x)| \leq 1, \tag{6}$$

for all $x, y \in \mathbb{R}$ and $\lambda, z \in \mathbb{C}$.
The function $(\lambda, z) \longmapsto \phi^k_{\lambda,0}(z)$ (called Dunkl kernel) is analytic on $\mathbb{C}\times\mathbb{C}$. Therefore, there exist unique analytic functions b_n, $n \in \mathbb{N}$, on $\mathbb{C}$ such that

$$\phi^k_{\lambda,0}(z)=\sum_{n=0}^{\infty} b_n(z)\lambda^n,\ \lambda, z\in\mathbb{C},\ \text{where } b_n(z)=V_k(\frac{\omega^n}{n!})(z)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\phi^k_{\lambda,0}(z)\,|_{\lambda=0},$$

namely

$$b_{2n}(z)=\frac{1}{(k+\frac{1}{2})_n\ n!}\left(\frac{z}{2}\right)^{2n},\quad b_{2n+1}(z)=\frac{1}{(k+\frac{1}{2})_{n+1}\ n!}\left(\frac{z}{2}\right)^{2n+1},\quad \forall\, n\in\mathbb{N}.$$

We remark that for all $z\in\mathbb{C}$ and for all $n\in\mathbb{N}$, we have

$$\mathcal{D}_k b_{n+1}=b_n \ \text{ and } \ |b_n(z)|\le\frac{|z|^n}{n!}\,. \tag{7}$$

In the same context, we denote by

$$\phi^k_{\lambda_0,n}(x)=V_k(x^n e^{\lambda_0 x})=\frac{d^n}{dt^n}\phi^k_{t,0}(x)_{|t=\lambda_0},\quad n\in\mathbb{N}\ \text{ and }\ \lambda_0\in\mathbb{C}. \tag{8}$$

Definition 1.1. The *translation operators associated with the Dunkl operator*, denoted by T^k_z, $z\in\mathbb{C}$, are defined on $\mathcal{A}(\mathbb{C})$ by

$$\forall\,\omega\in\mathbb{C},\ T^k_z f(\omega)=\sum_{n=0}^{\infty} b_n(\omega)\mathcal{D}^n_k f(z).$$

We next collect some properties of translation operators.

Proposition 1.2. *The operators T^k_z satisfy the following properties.*

(i) *For every $z\in\mathbb{C}$, the operator T^k_z is linear and continuous map from $\mathcal{A}(\mathbb{C})$ into itself and*

$$T^k_z f(\omega)=V_{k,z}V_{k,\omega}\left[V^{-1}_k(f)(z+\omega)\right],\ \omega\in\mathbb{C}.$$

(*We use the notation $V_{k,z}$ when we wish to emphasize the functional dependence on the variable z*).

(ii) *For all function f in $\mathcal{A}(\mathbb{C})$ and for every $z\in\mathbb{C}$, $z'\in\mathbb{C}$ and $\omega\in\mathbb{C}$, we have*

$$T^k_0=\textit{identity},\quad T^k_z f(\omega)=T^k_\omega f(z),\quad \mathcal{D}_kT^k_z=T^k_z\mathcal{D}_k\ \text{ and }\ T^k_zT^k_{z'}=T^k_{z'}T^k_z.$$

(iii) *The function $(z,\omega)\longrightarrow T^k_z f(\omega)$ is the unique solution of the following Cauchy problem*

$$\begin{cases}\mathcal{D}_{k,z}u(z,\omega)=\mathcal{D}_{k,\omega}u(z,\omega),\\ \qquad u(z,0)=f(z).\end{cases}$$

(iv) *For all $z\in\mathbb{C}$, $\omega\in\mathbb{C}$ and $\lambda\in\mathbb{C}$, we have*

$$T^k_z\phi^k_{\lambda,0}(\omega)=\phi^k_{\lambda,0}(z)\phi^k_{\lambda,0}(\omega)\qquad (\textit{product formula}).$$

Remark. For $k > 0$, it was pointed out in [10], [11] that the translation operators T_x^k, $x \in \mathbb{R}$, may be represented as:

$$\forall\, y \in \mathbb{R}, \quad T_x^k f(y) = \int_{\mathbb{R}} f(z)\, d\mu_{x,y}^k(z), \quad f \in C(\mathbb{R}), \tag{9}$$

($C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$), $\mu_{x,y}^k$ is a real bounded measure on $\mathbb{R}$ with support in $[-|x| - |y|\,,\, -||x| - |y||] \cup [||x| - |y||\,,\, |x| + |y|]$, for $x, y \neq 0$, $\mu_{x,y}^k(\mathbb{R}) = 1$ and $\|\mu_{x,y}^k\| \leq 4$, for all $x, y \in \mathbb{R}$.

Let us now recall the following generalized Taylor formula with integral remainder (see [8]), which will be used frequently.

Theorem 1.3. *Let f be a function of class C^{n+1} on $\mathbb{R}$, $n \in \mathbb{N}$. Then we have the following generalized Taylor formula with integral remainder*

$$f(x) = \sum_{p=0}^{n} b_p(x) \mathcal{D}_k^p f(0) + \int_{-|x|}^{|x|} W_n(x, y) \mathcal{D}_k^{n+1} f(y) |y|^{2k}\, dy,$$

where $\{W_n\}$, $n = 0, 1, 2 \ldots$, is a sequence of functions constructed inductively from the function $|y|^{2k}$ and satisfying

$$\int_{-|x|}^{|x|} |W_n(x, y)| |y|^{2k}\, dy \leq b_{n+1}(|x|) + |x| b_n(|x|).$$

Definition 1.4. (i) The *Borel-Dunkl transform* of an analytic functional $S \in \mathcal{A}'(\mathbb{C})$ is defined by

$$\mathcal{F}_k(S)(\lambda) = \langle\, S,\, \phi_{\lambda,0}^k(.)\, \rangle, \ \ \lambda \in \mathbb{C}.$$

(ii) The *Fourier-Dunkl transform* of a distribution μ in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\mathcal{F}_k(\mu)(\lambda) = \langle \mu\,,\, \phi_{-i\lambda,0}^k(.) \rangle.$$

(iii) The *k-convolution* of two distributions $\mu, \nu \in \mathcal{E}'(\mathbb{R})$ is given by

$$\langle \mu *_k \nu\,,\, f \rangle = \langle \mu_x\,,\, \langle \nu_y\,,\, T_x^k f(y) \rangle \rangle, \ \ f \in \mathcal{E}(\mathbb{R}).$$

Next, let us recall the following Paley-Wiener type theorem associated with the operator $\mathcal{D}_k$ (for some details see [1]).

Theorem 1.5. *The Borel-Dunkl transform $\mathcal{F}_k$ is a topological isomorphism from $\mathcal{A}'(\mathbb{C})$ onto* $\mathrm{Exp}(\mathbb{C})$.

Theorem 1.6 (Pólya representation). *If f is an entire function of exponential type a, $a>0$, then f has the following integral representation*

$$f(z)=\frac{1}{2i\pi}\int_{|\omega|=a+\epsilon}\phi^k_{z,0}(\omega)F(\omega)\,d\omega,$$

where $\epsilon>0$ and F is an analytic function outside the disk centered at the origin and with radius a.

Proof: From the Paley-Wiener Theorem 1.5, there exists an analytic functional $S\in\mathcal{A}'(\mathbb{C})$ such that

$$\forall\, z\in\mathbb{C},\quad f(z)=\langle\, S,\,\phi^k_{z,0}(.)\,\rangle.$$

Since the analytic functional S is given by a complex measure μ with support in the disk centered at the origin and with radius a, (see [7]), we have

$$\forall\, z\in\mathbb{C},\quad f(z)=\int_{\mathbb{C}}\phi^k_{z,0}(\omega)\,d\mu(\omega).$$

On the other hand, by using the Cauchy integral formula, we can write for all $z\in\mathbb{C}$

$$\phi^k_{z,0}(\omega)=\frac{1}{2i\pi}\int_{|\xi|=a+\epsilon}\frac{\phi^k_{z,0}(\xi)}{\xi-\omega}\,d\xi,\quad \epsilon>0.$$

From Fubini's Theorem we deduce that

$$f(z)=\frac{1}{2i\pi}\int_{|\xi|=a+\epsilon}\phi^k_{z,0}(\xi)F(\xi)\,d\xi,$$

where $F(\xi)=\int_{\mathbb{C}}\frac{d\mu(\omega)}{\xi-\omega}$, ($F$ is called the Borel Transform of the measure μ). $\square$

Proposition 1.7. *Let f be an entire function of exponential type a, $a>0$. Then*
 (i) *for every $n\in\mathbb{N}$, the function $\mathcal{D}^n_k f$ is entire and of exponential type a;*
 (ii) *for every $z,\omega\in\mathbb{C}$ and $\epsilon>0$*

$$|T^k_z f(\omega)|\le C_\epsilon e^{(a+\epsilon)(|z|+|\omega|)},$$

where C_ϵ is a positive constant.

Proof: (i) It is clear that for $n\in\mathbb{N}$, the function $\mathcal{D}^n_k f$ is entire. Let us show that $\mathcal{D}^n_k f$ is of exponential type a. From the Pólya representation Theorem 1.6, we deduce

$$\mathcal{D}^n_k f(z)=\frac{1}{2i\pi}\int_{|\omega|=a+\epsilon}\omega^n\phi^k_{z,0}(\omega)F(\omega)\,d\omega.$$

Now using the property of the function $\phi^k_{z,0}$, relation (6), we obtain

$$|\mathcal{D}_k^n f(z)| \leq (a+\epsilon)^{n+1} M_\epsilon e^{(a+\epsilon)|z|}\,, \tag{10}$$

where $M_\epsilon = \sup\{|F(\omega)|;\ |\omega| = a+\epsilon\}$. Since ϵ is arbitrary, we conclude that $\mathcal{D}_k^n f$ is of exponential type a.

(ii) We have for $f \in \mathrm{Exp}_a(\mathbb{C})$, $T_z^k f(\omega) = \sum_{n=0}^{\infty} b_n(\omega)\mathcal{D}_k^n f(z)$. A combination of (7) and (10) gives the result. □

Notation. For $\sigma > 0$, let $M_\sigma(\mathbb{R})$ be the space of Radon measures on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} e^{\sigma|x|}\, d|\mu|(x) < \infty.$$

Definition 1.8. Let f be an entire function of exponential type $a > 0$ and $\mu \in M_\sigma(\mathbb{R})$ with $\sigma > a$. The *convolution product associated with* $\mathcal{D}_k$ of the function f and the measure μ is the function denoted $\mu *_k f$, defined by

$$\forall\, z \in \mathbb{C}, \quad \mu *_k f(z) = \int_{\mathbb{R}} T_{-y}^k f(z)\, d\mu(y).$$

Proposition 1.9. *Let f be an entire function of exponential type $a > 0$ and $\mu \in M_\sigma(\mathbb{R})$ with $\sigma > a$. Then $\mu *_k f \in \mathrm{Exp}_a(\mathbb{C})$.*

PROOF: We have for all $z \in \mathbb{C}$

$$\mu *_k f(z) = \int_{\mathbb{R}} T_{-y}^k f(z)\, d\mu(y).$$

With the hypotheses on the measure μ, we deduce easily that $z \longmapsto \mu *_k f(z)$ is entire. Now, let $\epsilon \in \mathbb{R}$, $0 < \epsilon < \sigma - a$, by using Proposition 1.7(ii) we have

$$\forall\, z \in \mathbb{C}, \quad |\mu *_k f(z)| \leq C'_\epsilon e^{(a+\epsilon)|z|},$$

where C'_ϵ is a positive constant. Since ϵ is arbitrary, we deduce that $\mu *_k f$ is of exponential type a. □

2. Integro-differential-difference equations associated with the Dunkl operator for the class of entire functions of exponential type

Definition 2.1. Let μ be a measure in $M_\sigma(\mathbb{R})$. The *Laplace-Dunkl transform* of the measure μ is the function denoted $\mathcal{L}_k(\mu)$, defined by

$$\mathcal{L}_k(\mu)(z) = \int_{\mathbb{R}} \phi^k_{-z,0}(y)\, d\mu(y).$$

We remark that for $\mu \in M_\sigma(\mathbb{R})$, the function $\mathcal{L}_k(\mu)$ is analytic in the strip $|\mathcal{R}ez| \leq \sigma$.

Theorem 2.2. *Let μ be a measure in $M_\sigma(\mathbb{R})$. If the equation*

$$\forall\, z \in \mathbb{C}\,, \quad \mu *_k f(z) = \sum_{n=0}^{\infty} a_{n,k} \mathcal{D}_k^n f(z), \quad a_{n,k} \in \mathbb{C} \tag{11}$$

is satisfied by any function f in $\mathrm{Exp}_a(\mathbb{C})$, $0 < a < \sigma$, where $\sum_{n\geq 0} a_{n,k} z^n$ is analytic in the closed disk centered at the origin and with radius a, then

$$a_{n,k} = \frac{1}{n!}\frac{d^n}{dz^n}\mathcal{L}_k(\mu)(z)_{|z=0}.$$

Conversely, if the sequence $(a_{n,k})_{n\geq 0}$, is related to the measure in this fashion, then (11) holds for each class of entire function of exponential type a with $0 < a < \sigma$.

PROOF: Let $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| \leq a$. Since the function $z \longmapsto \phi^k_{\lambda_0,0}(z)$ is of exponential type $|\lambda_0|$, we have

$$\mu *_k \phi^k_{\lambda_0,0}(z) = \phi^k_{\lambda_0,0}(z) \int_{\mathbb{R}} \phi^k_{-\lambda_0,0}(y)\, d\mu(y) = \phi^k_{\lambda_0,0}(z)\mathcal{L}_k(\mu)(\lambda_0).$$

On the other hand, we have

$$\sum_{n\geq 0} a_{n,k}\mathcal{D}_k^n \phi^k_{\lambda_0,0}(z) = \phi^k_{\lambda_0,0}(z) \sum_{n\geq 0} a_{n,k}\lambda_0^n\,.$$

So we deduce that

$$\Big(\mathcal{L}_k(\mu)(\lambda_0) - \sum_{n\geq 0} a_{n,k}\lambda_0^n\Big)\phi^k_{\lambda_0,0}(z) = 0.$$

Taking $z = 0$, we obtain $\mathcal{L}_k(\mu)(\lambda_0) = \sum_{n\geq 0} a_{n,k}\lambda_0^n$. This holds for every λ_0 such that $|\lambda_0| \leq a$.
So $a_{n,k} = \frac{1}{n!}\frac{d^n}{dz^n}\mathcal{L}_k(\mu)(z)_{|z=0}$. Conversely, let $f \in \mathrm{Exp}_a(\mathbb{C})$, we have

$$\mu *_k f(z) = \int_{\mathbb{R}} \Big(\sum_{n\geq 0} b_n(-y)\mathcal{D}_k^n f(z)\Big)\, d\mu(y) = \sum_{n\geq 0}\Big(\int_{\mathbb{R}} b_n(-y)d\mu(y)\Big)\mathcal{D}_k^n f(z).$$

The last identity is justified by the fact that

$$\int_{\mathbb{R}} \sum_{n\geq 0} |b_n(-y)||\mathcal{D}_k^n f(z)|\, d|\mu|(y) < +\infty$$

which is a consequence of Proposition 1.7(i) and the relation (7). We conclude by observing that

$$\int_{\mathbb{R}} b_n(-y)\, d\mu(y) = \frac{1}{n!}\int_{\mathbb{R}} \frac{d^n}{d\lambda^n}\phi^k_{-\lambda,0}(y)_{|\lambda=0}\, d\mu(y) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{L}_k(\mu)(\lambda)_{|\lambda=0}.$$

□

Theorem 2.3. *Let $\mu \in M_\sigma(\mathbb{R})$ and let f be a C^∞-function on $\mathbb{R}$ satisfying*

$$\forall\, x \in \mathbb{R},\ \mu *_k f(x) = \sum_{n\geq 0} a_{n,k}\mathcal{D}_k^n f(x),$$

where $(a_{n,k})_{n\geq 0}$ is a sequence of complex numbers such that the series $\sum_{n\geq 0} a_{n,k}z^n$ is analytic inside the closed disk $|z| \leq a$, $0 < a < \sigma$. Then f is the restriction on $\mathbb{R}$ of an entire function of exponential type at most a.

Proof: Let x be fixed in $\mathbb{R}$. From the convergence of the series $\sum_{n\geq 0} a_{n,k}\mathcal{D}_k^n f(x)$, we deduce that there exists $N_1(x) \in \mathbb{N}$ such that $|a_{n,k}\mathcal{D}_k^n f(x)| \leq 1$, for all $n \geq N_1(x)$. On the other hand, since the series $\sum_{n\geq 0} a_{n,k}z^n$ is analytic in the disk $|z| \leq a$, we have $\limsup_{n\longrightarrow+\infty} |a_{n,k}|^{\frac{1}{n}} \leq \frac{1}{a}$. Thus for every ϵ, $0 < \epsilon < \frac{1}{a}$, there exists $N_2 \in \mathbb{N}$ such that, $|a_{n,k}|^{\frac{1}{n}} > (\frac{1}{a} - \epsilon)$, for all $n \geq N_2$. Hence, for $n \geq \max(N_1(x), N_2)$, we have $|\mathcal{D}_k^n f(x)| \leq (\frac{1}{a} - \epsilon)^{-n}$. By applying the Delsarte Taylor formula with integral remainder given in Theorem 1.3, to the function f and the relation (7), we obtain

$$f(x) = \sum_{n=0}^{N} b_n(x)\mathcal{D}_k^n f(0) + R_N(x),$$

where

$$\begin{aligned}|R_N(x)| &\leq \sup_{0\leq|t|\leq|x|} |\mathcal{D}_k^{N+1} f(t)|\,(b_{N+1}(|x|) + |x| b_N(|x|))\\ &\leq \frac{|x|^{N+1}}{(N+1)!}(2+N) \sup_{0\leq|t|\leq|x|} |\mathcal{D}_k^{N+1} f(t)|.\end{aligned}$$

For each t, $0 \leq |t| \leq |x|$, the above analysis shows that there exists $N_t \in \mathbb{N}$ such that

$$|\mathcal{D}_k^n f(t)| \leq (\frac{1}{a} - \epsilon)^{-n}, \quad \text{for } \ n \geq N_t.$$

But $[-|x|\,,\,|x|]$ is compact, so there is some N' independent of t such that

$$\sup_{0\leq|t|\leq|x|} |\mathcal{D}_k^n f(t)| \leq (\frac{1}{a} - \epsilon)^{-n}, \quad \text{for } \ n \geq N'.$$

Hence for $N \geq N'$, we have $|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!}(2+N)(\frac{1}{a}-\epsilon)^{-(N+1)}$, which implies that $R_N(x)$ tends to zero as N tends to $+\infty$. Consequently the function f may be expanded in a generalized Taylor series $f(x) = \sum_{n\geq 0} b_n(x)\mathcal{D}_k^n f(0)$. Hence f is the restriction on $\mathbb{R}$ of the entire function g given by $g(z) = \sum_{n\geq 0} b_n(z)\mathcal{D}_k^n f(0)$. We deduce from (7) and (10) that g is of exponential type at most a. □

Example. Let μ be the measure defined by

$$d\mu(y) = \frac{a}{2}e^{-a|y|}|y|^{2k}dy, \quad a > 0, \quad k \geq 0.$$

Consider the equation

$$\mu *_k f = \sum_{n=0}^{\infty} a_{n,k}\mathcal{D}_k^{2n} f, \tag{12}$$

where $a_{n,k}$ is given by

$$a_{n,k} = \frac{2^{2k}\Gamma(k+\frac{1}{2})\Gamma(k+n+1)}{n!\Gamma(\frac{1}{2})a^{2(k+n)}}.$$

By computation, we have

$$\mathcal{L}_k(\mu)(z) = \frac{2^{2k}a^2\Gamma(k+\frac{1}{2})\Gamma(k+1)}{\Gamma(\frac{1}{2})(a^2-z^2)^{k+1}} = \sum_{n=0}^{\infty} a_{n,k}z^{2n}, \quad \text{for} \quad |z| < a.$$

Due to Theorem 2.2, Theorem 2.3 shows that equation (12) characterizes entire functions of exponential type less than a.
This example shows that the relation between exponential type and domain of analyticity is sharp since $\mathcal{L}_k(\mu)$ has a singularity at $\pm a$.

We proceed now to develop a method which permits us to construct solutions of (11), expressed in terms of functions given by (8).

Proposition 2.4. *Let $\mu \in M_\sigma(\mathbb{R})$ and λ_0 a zero of multiplicity N of the function*

$$g(z) = \sum_{n=0}^{s} c_{n,k}z^n - \mathcal{L}_k(\mu)(z),$$

where $(c_{n,k})_{0\leq n\leq s}$ is a finite sequence in $\mathbb{C}$. The function defined by $f(x) = \sum_{m=0}^{N-1} a_m\phi^k_{\lambda_0,m}(x)$ is a solution of the equation $\mu *_k f(x) = \sum_{n=0}^{s} c_{n,k}\mathcal{D}_k^n f(x)$, where $(a_m)_{0\leq m\leq N-1}$ is a finite sequence in $\mathbb{C}$.

Proof: We have, for all $x \in \mathbb{R}$

$$\mu *_k f(x) = \sum_{m=0}^{N-1} a_m \sum_{j=0}^{m} \binom{m}{j} (\mathcal{L}_k(\mu))^{(j)}(\lambda_0)\phi^k_{\lambda_0,m-j}(x).$$

Now we use the fact that λ_0 is a zero of multiplicity N of the function g, so we have

$$(\mathcal{L}_k(\mu))^{(j)}(\lambda_0) = \begin{cases} \sum_{n=j}^{s} j!\binom{n}{j} c_{n,k}\lambda_0^{n-j}, & \text{for } \ 0 \le j \le s, \\ 0, & \text{if } \ s < j \le N. \end{cases}$$

Hence

$$\mu *_k f(x) = \sum_{n=0}^{s} c_{n,k} V_{k,y}\left[\sum_{j=0}^{n}\sum_{m=j}^{N-1} a_m \binom{m}{j} j! \binom{n}{j} y^{m-j} e^{\lambda_0 y}\lambda_0^{n-j}\right](x).$$

Using the following relation which is obtained by the generalized product rule,

$$\mathcal{D}^n\left\{\sum_{m=0}^{N-1} a_m x^m e^{\lambda_0 x}\right\} = e^{\lambda_0 x}\sum_{j=0}^{n}\sum_{m=j}^{N-1} a_m \binom{m}{j} j! \binom{n}{j} x^{m-j}\lambda_0^{n-j}$$

and (4), we deduce the result. □

In [1], we have called the functions $\phi^k_{\lambda,m}$ k-exponential-monomials which can be expressed in terms of normalized spherical Bessel functions, namely

$$\phi^k_{\lambda,m}(x) = \begin{cases} x^{2n}\sum_{s=0}^{n} f_{2n,s}\phi^{k+s}_{\lambda,0}(x), & \text{if } m = 2n, \\ x^{2n+1}\sum_{s=0}^{n} f_{2n,s}\left[\phi^{k+s}_{\lambda,0}(x) - \frac{k+s}{k+s+\frac{1}{2}} j_{k+s+\frac{1}{2}}(i\lambda x)\right], & \text{if } m = 2n+1, \end{cases}$$

where $f_{2n,s}$ are given by

$$f_{2n,s} = (-1)^s\binom{n}{s}\frac{(k)_s}{(k+\frac{1}{2})_s}, \quad 0 \le s \le n.$$

We can extend the previous proposition to infinite case.

Proposition 2.5. *Let $\mu \in M_\sigma(\mathbb{R})$ and λ_0 a zero of multiplicity N of the function*

$$g(z) = \sum_{n=0}^{\infty} c_{n,k} z^n - \mathcal{L}_k(\mu)(z).$$

Suppose that the series $\sum_{n\ge 0} c_{n,k} z^n$ is analytic in the disk $|z| \le a < \sigma$ and $|\lambda_0| < a$. Then every function of the form $f(x) = \sum_{m=0}^{N-1} a_m \phi^k_{\lambda_0,m}(x)$ is a solution of the integro-differential-difference equation

$$\mu *_k f = \sum_{n=0}^{\infty} c_{n,k}\mathcal{D}^n_k f.$$

Proof: We proceed as in the previous proof, we remark that we can change the order of summation by using the uniform convergence of series. □

Lemma 2.6. *If ψ is analytic on a neighborhood of a contour γ in $\mathbb{C}$ and*

$$\int_\gamma \phi^k_{z,0}(\omega)\psi(\omega)\,d\omega = 0,$$

then ψ is analytic inside γ.

Proof: It is obtained in the same way as for Lemma 6.10.6, p. 110 in [3]. □

In the following proposition, we show that every solution of equation (11) which is entire of exponential type is a sum of k-exponential-monomials functions.

Proposition 2.7. *Let f be an entire function of exponential type a, $a > 0$, $\sum_{n\geq 0} a_{n,k} z^n$ an analytic function in a closed disk $|z| \leq b$, which contains the conjugate indicator diagram of f and $\mu \in M_\sigma(\mathbb{R})$, with $\sigma \geq b$. If moreover f satisfies the equation*

$$\mu *_k f = \sum_{n=0}^{\infty} a_{n,k}\mathcal{D}^n_k f.$$

Then f is of the following form

$$f(z) = \sum_{s=0}^{m}\sum_{n=0}^{l_s-1} \beta_{n,s}\phi^k_{\lambda_s,n}(z),\ \beta_{n,s} \in \mathbb{C},$$

where λ_s, $0 \leq s \leq m$, are the zeros of multiplicity l_s of the function $g(z) = \mathcal{L}_k(\mu)(z) - \sum_{n=0}^{\infty} a_{n,k} z^n$, which are contained in the conjugate indicator diagram of f, (m is possibly infinite).

Proof: From Pólya representation Theorem 1.6, we have

$$f(z) = \frac{1}{2i\pi}\int_{|\omega|=a+\epsilon} \phi^k_{z,0}(\omega)F(\omega)\,d\omega, \quad z \in \mathbb{C},$$

where F is analytic outside the disk $|z| \leq a$ and $\epsilon > 0$. Hence we have

$$\forall\, z \in \mathbb{C},\ \mu *_k f(z) = \frac{1}{2i\pi}\int_{|\omega|=a+\epsilon} \phi^k_{z,0}(\omega)\mathcal{L}_k(\mu)(\omega)F(\omega)\,d\omega.$$

On the other hand, we have

$$\sum_{n=0}^{\infty} a_{n,k}\mathcal{D}^n_k f(z) = \frac{1}{2i\pi}\int_{|\omega|=a+\epsilon}\left(\sum_{n=0}^{\infty} a_{n,k}\omega^n\right)\phi^k_{z,0}(\omega)F(\omega)\,d\omega.$$

Since f is a solution of the equation (11), we must have

$$\int_{|\omega|=a+\epsilon} \phi_{z,0}^k(\omega) \left[\sum_{n=0}^{\infty} a_{n,k}\omega^n - \mathcal{L}_k(\mu)(\omega)\right] F(\omega)\, d\omega = 0.$$

By using Lemma 2.6, we deduce that the function

$$\omega \longmapsto \left[\sum_{n=0}^{\infty} a_{n,k}\omega^n - \mathcal{L}_k(\mu)(\omega)\right] F(\omega)$$

is analytic inside the disk $|z| \leq a + \epsilon$. Hence the function F has at most poles at the zeros of the function $\omega \longmapsto g(\omega) = \mathcal{L}_k(\mu)(\omega) - \sum_{n=0}^{\infty} a_{n,k}\omega^n$, contained in the conjugate indicator diagram of f. Now by using the Pólya representation and the residue theorem, we deduce that

$$f(z) = \sum_{s=0}^{m} \operatorname{Res}(\phi_{z,0}^k(\omega)F(\omega), \lambda_s) = \sum_{s=0}^{m}\sum_{n=0}^{l_s-1} \beta_{n,s}\phi_{\lambda_s,n}^k(z),$$

where $\beta_{n,s} = \frac{1}{(l_s-1)!}\binom{l_s-1}{n}\frac{d^{l_s-1-n}}{d\omega^{l_s-1-n}}\Big[(\omega-\lambda_s)^{l_s}F(\omega)\Big]_{|\omega=\lambda_s}$. □

3. Integro-differential equations associated with the Dunkl operator on the space of C^n-functions on $\mathbb{R}$

In the following we suppose that $k > 0$. Then the translation operators T_x^k, $x \in \mathbb{R}$ associated with the Dunkl operator, are given for a continuous function on $\mathbb{R}$ by formula (9).

Definition 3.1. Let f be a continuous function on $\mathbb{R}$. We say that the nonnegative function $\psi \in C(\mathbb{R})$ is a *bounding function* of f, if we have

(i) $\forall\, x \in \mathbb{R}$, $|f(x)| \leq \psi(x)$,

(ii) there exists a constant $A = A(\psi, k)$ such that

$$\forall\, x, y \in \mathbb{R}, \int_{\mathbb{R}} \psi(z)\, d|\mu_{x,y}^k|(z) \leq A\psi(x)\psi(y).$$

The smallest constant satisfying the latter inequality will be called the supporting constant.

Example. $\psi(x) = e^{a|x|}$, $a > 0$, we have

$$\forall\, x, y \in \mathbb{R} \int_{\mathbb{R}} e^{a|z|}\, d|\mu_{x,y}^k|(z) \leq 4e^{a(|x|+|y|)}$$

which can be seen by using the properties of the measure $\mu_{x,y}^k$.

Lemma 3.2. *Let f be a function of class C^m on $\mathbb{R}$, $m \in \mathbb{N}$, such that $\mathcal{D}_k^m f$ is of class C^n on $\mathbb{R}$, $n \in \mathbb{N}$. Then f is of class C^{m+n} on $\mathbb{R}$.*

Proof: See [8]. $\square$

Lemma 3.3. *Let $f \in C^1(\mathbb{R})$ and μ be a measure on $\mathbb{R}$. If ψ is a bounding function of f and $\mathcal{D}_k f$, satisfying $\int_{\mathbb{R}} \psi(-y)\, d|\mu|(y) < +\infty$, then $\mu *_k f \in C^1(\mathbb{R})$ and we have*

$$\mathcal{D}_k(\mu *_k f) = \mu *_k \mathcal{D}_k f.$$

Proof: Using the bounding function, by differentiation under the integral, we can see that $\mu *_k f \in C^1(\mathbb{R})$. On the other hand, we have by Theorem 1.3

$$\begin{aligned}\mathcal{D}_k(\mu *_k f)(x) &= \lim_{a \longrightarrow 0} \frac{T_x^k(\mu *_k f)(a) - \mu *_k f(x)}{b_1(a)} \\ &= \lim_{a \longrightarrow 0} \int_{\mathbb{R}} \frac{T_x^k(T_{-y}^k f)(a) - T_{-y}^k f(x)}{b_1(a)}\, d\mu(y),\end{aligned}$$

where $T_x^k(T_{-y}^k f)(a) - T_{-y}^k f(x) = \int_{-|a|}^{|a|} W_0(a,t) T_x^k T_{-y}^k(\mathcal{D}_k f)(t)|t|^{2k}\, dt$. Since

$$\left| \int_{-|a|}^{|a|} W_0(a,t) T_x^k T_{-y}^k(\mathcal{D}_k f)(t)|t|^{2k}\, dt \right| \leq A^2 \sup_{|t| \leq |a|} \psi(t)\psi(-y)\psi(x)(b_1(|a|) + |a|),$$

we have for $0 < a < 1$

$$\left| \frac{T_x^k(T_{-y}^k f)(a) - T_{-y}^k f(x)}{b_1(a)} \right| \leq A^2 \left(\frac{2\Gamma(k+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} + 1 \right) \sup_{|t| \leq 1} \psi(t)\psi(-y)\psi(x).$$

As $y \longmapsto \psi(-y) \in L^1(d|\mu|)$, the dominated convergence theorem and the following formula

$$\lim_{a \longmapsto 0} \frac{T_a^k(T_{-y}^k f)(x) - T_{-y}^k f(x)}{b_1(a)} = \mathcal{D}_k(T_{-y}^k f)(x) = T_{-y}^k(\mathcal{D}_k f)(x)$$

yield $\mathcal{D}_k(\mu *_k f)(x) = (\mu *_k \mathcal{D}_k f)(x)$. $\square$

Lemma 3.4. *Let f be in $C^\infty(\mathbb{R})$. Suppose that there exist a positive constant B and a nonnegative continuous function ψ on $\mathbb{R}$ such that*

$$\forall\, x \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}, \quad |\mathcal{D}_k^n f(x)| \leq B^n \psi(x).$$

Then f is the restriction to $\mathbb{R}$ of an entire function of exponential type B.

PROOF: From Theorem 1.3, we have for all $x \in \mathbb{R}$,

$$f(x) = \sum_{s=0}^{n-1} b_s(x)\mathcal{D}_k^s f(0) + R_n(x)$$

with

$$|R_n(x)| \leq \sup_{|t|\leq|x|} |\mathcal{D}_k^n f(t)|(b_n(|x|) + |x|b_{n-1}(|x|)) \leq B^n \sup_{|t|\leq|x|} \psi(t)\frac{n+1}{n!}|x|^n.$$

The latter term goes at zero as n tends to $+\infty$. Hence for $x \in \mathbb{R}$, $f(x)$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty} b_n(x)\mathcal{D}_k^n f(0),\ x \in \mathbb{R}.$$

Put $g(z) = \sum_{n=0}^{\infty} b_n(z)\mathcal{D}_k^n f(0)$. This series defines an entire function and we have, $|g(z)| \leq \psi(0)e^{B|z|}$. Then g is entire of exponential type at most B. □

Proposition 3.5. *Let $f \in C(\mathbb{R})$, ψ a bounding function of f and μ a measure on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \psi(-t)\, d|\mu|(t) = M < +\infty.$$

If f is a solution of the equation

$$\mathcal{D}_k f = \mu *_k f,$$

then f is the restriction to $\mathbb{R}$ of an entire function of exponential type AM, where A is the supporting constant.

PROOF: Fix a, and choose a sequence such that $x_n \longrightarrow a$ as $n \longrightarrow +\infty$. Since for $\delta > 0$

$$|T_{-y}^k f(x_n) - T_{-y}^k f(a)| \leq 2A \max_{|x-a|\leq\delta} \psi(x)\, \psi(-y)$$

and the latter terms is in $L^1(d|\mu|)$, the dominated convergence theorem yields

$$\lim_{n\longrightarrow+\infty} (\mathcal{D}_k f(x_n) - \mathcal{D}_k f(a)) = 0.$$

Hence $\mathcal{D}_k f \in C(\mathbb{R})$, so that Lemma 3.2 implies $f \in C^1(\mathbb{R})$. From the following inequality

$$|\mathcal{D}_k f(x)| \leq \int_{\mathbb{R}} |T_{-y}^k f(x)|\, d|\mu|(y) \leq AM\psi(x),$$

we see that $\mathcal{D}_k f$ has a bounding function $AM\psi$. By Lemma 3.3 we obtain

$$\mathcal{D}_k^2 f(x) = \int_{\mathbb{R}} T_{-y}^k(\mathcal{D}_k f)(x)\, d\mu(y), \quad \text{so} \quad |\mathcal{D}_k^2 f(x)| \leq (AM)^2\psi(x).$$

An induction argument shows that $f \in C^\infty(\mathbb{R})$ and we have for $n \geq 1$

$$\mathcal{D}_k^n f(x) = \mu *_k \mathcal{D}_k^{n-1} f(x), \quad \text{with} \quad |\mathcal{D}_k^n f(x)| \leq (AM)^n \psi(x), \quad \text{for} \quad n \geq 0.$$

The result follows from Lemma 3.4. □

Example. Let μ be the measure defined by

$$\mu = \sum_{s\in\mathbf{Z}} \tau \frac{4(-1)^s}{\pi^2(2s+1)^2}\delta_{-\frac{2s+1}{2\tau}\pi}$$

where δ_a denotes the Dirac point mass measure at a. If f is bounded on the real axis and satisfies the equation

$$\mathcal{D}_k f(x) = \mu *_k f(x) = \frac{4\tau}{\pi^2}\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^2} T^k_{\frac{2n+1}{2\tau}\pi} f(x),$$

then f is the restriction on $\mathbb{R}$ of an entire function of exponential type $|\mu|(\mathbb{R}) = \tau$.

Lemma 3.6. *Let $f \in C^n(\mathbb{R})$, $n \geq 1$, satisfying*

$$\forall\, x \in \mathbb{R}, \quad |\mathcal{D}_k^n f(x)| \leq A e^{\tau|x|},$$

where $A = A_k$ is a positive constant and $\tau > 0$. Then we have

$$|\mathcal{D}_k^{n-s} f(x)| \leq \tau^{-s}\left(2^s A + 2^{s-1}\tau C_{n-1} + 2^{s-2}\tau^2 C_{n-2} + \cdots + \tau^s C_{n-s}\right) e^{\tau|x|},$$

where $C_{n-s} = |\mathcal{D}_k^{n-s} f(0)|$, $0 < s \leq n$.

Proof: From the Delsarte Taylor formula with integral remainder, Theorem 1.3, we deduce that

$$\begin{aligned} |\mathcal{D}_k^{n-1} f(x)| &\leq |\mathcal{D}_k^{n-1} f(0)| + 2A\int_0^{|x|} e^{\tau y}\, dy \\ &\leq \frac{1}{\tau}(2A + \tau C_{n-1})e^{\tau|x|}. \end{aligned}$$

We complete by induction. □

From Proposition 2.7, we deduce that an entire function of exponential type which satisfies an equation of the form in the following proposition is a sum of k-exponential-monomials functions. The following proposition makes it clear why this hypothesis on the analyticity of the function is chosen, since a solution which is of exponential growth on the real line is shown to be entire of exponential type.

Proposition 3.7. *Let $f \in C^n(R)$ satisfy*

(i) $|f(x)| \le Me^{\tau|x|}$,

(ii) $\mathcal{D}_k^n f = \mu *_k f$, *for* $n \ge 2$,

where μ is a measure on $\mathbb{R}$ such that $B = \int_{\mathbb{R}} e^{\tau|t|}\, d|\mu|(t) < +\infty$. Then f is the restriction to $\mathbb{R}$ of an entire function of exponential type at most $(4B)^{\frac{1}{n}}$.

Proof: We have

$$|\mathcal{D}_k^n f(x)| \le \int_{\mathbb{R}} |T_{-t}^k f(x)|\, d|\mu|(t) \le 4MBe^{\tau|x|}.$$

By Lemma 3.6, we deduce that

$$|\mathcal{D}_k^{n-s} f(x)| \\ \le \tau^{-s}\left(2^s \times 4MB + 2^{s-1}\tau C_{n-1} + 2^{s-2}\tau^2 C_{n-2} + \cdots + \tau^s C_{n-s}\right) e^{\tau|x|},$$

for $0 < s \le n$. On the other hand, by using Lemma 3.3, we obtain

$$\forall\, x \in \mathbb{R}, \quad \forall\, s \in \mathbb{N}, \quad 0 \le s \le n, \quad \mathcal{D}_k^{n+s} f(x) = \mu *_k \mathcal{D}_k^s f(x).$$

So that, $|\mathcal{D}_k^{n+s} f(x)| \le 4B\tilde{C}_s e^{\tau|x|}$, for $0 < s < n$, and $|\mathcal{D}_k^{2n} f(x)| \le (4B)^2 Me^{\tau|x|}$, where $\tilde{C}_s = \tau^{s-n}\left(2^{n-s} \times 4MB + 2^{n-s-1}\tau C_{n-1} + \cdots + \tau^{n-s} C_s\right)$. Repeating this process we obtain for $m \in \mathbb{N}$ and $0 < s < n$,

$$|\mathcal{D}_k^{nm+s} f(x)| \le (4B)^m \tilde{C}_s e^{\tau|x|} \quad \text{and} \quad |\mathcal{D}_k^{nm} f(x)| \le (4B)^m Me^{\tau|x|}.$$

Hence we deduce by Lemma 3.3 that $f \in C^\infty(\mathbb{R})$. Furthermore, we have

$$\limsup_{j \longrightarrow +\infty} |\mathcal{D}_k^j f(x)|^{\frac{1}{j}} \le \lim_{m \longrightarrow +\infty} \left((4B)^m \tilde{C}_s\right)^{\frac{1}{nm+s}} = (4B)^{\frac{1}{n}}$$

for $0 < s < n$ fixed. It then follows from Lemma 3.4 that f is entire of exponential type at most $(4B)^{\frac{1}{n}}$. $\square$

Lemma 3.8. *Let $f \in C^N(\mathbb{R})$ satisfy*

(i) $|f(x)| \le Me^{\tau|x|}$,

(ii) $|\sum_{n=0}^N a_{n,k} \mathcal{D}_k^n f(x)| \le Be^{\tau|x|}$, $a_{N,k} \ne 0$.

Then $|\mathcal{D}_k^s f(x)| \le M_{s,k} e^{\tau|x|}$, $0 < s \le N$, where $M_{s,k}$ is a constant depending upon the function, k and s.

Proof: Proceed by induction on N. If $|a_{1,k}\mathcal{D}_k f(x) + a_{0,k} f(x)| \le Be^{\tau|x|}$, then

$$|\mathcal{D}_k f(x)| \le \frac{1}{|a_{1,k}|}(B + |a_{0,k}|M)e^{\tau|x|}.$$

This yields the statement for $N = 1$. Now if $|\sum_{n=0}^{N+1} a_{n,k}\mathcal{D}_k^n f(x)| \leq Be^{\tau|x|}$, then the generalized Taylor formula with integral remainder gives

$$\begin{aligned}\left|\sum_{n=0}^{N} a_{n+1,k}\mathcal{D}_k^n f(x)\right| &\leq \left|\sum_{n=0}^{N} a_{n+1,k}\mathcal{D}_k^n f(0)\right| \\ &\quad + \int_{-|x|}^{|x|} |W_0(x,y)|\left|\sum_{n=1}^{N+1} a_{n,k}\mathcal{D}_k^n f(y)\right||y|^{2k}\,dy \\ &\leq \left[P_0 + (|a_{0,k}|M + B)2\tau^{-1}\right] e^{\tau|x|},\end{aligned}$$

where $P_0 = |\sum_{n=0}^{N} a_{n+1,k}\mathcal{D}_k^n f(0)|$. By our induction hypothesis, $|\mathcal{D}_k^s f(x)| \leq M_{s,k}e^{\tau|x|}$, for $0 < s \leq N$. Further

$$|\mathcal{D}_k^{N+1} f(x)| \leq \frac{1}{|a_{N+1,k}|}\left[\sum_{n=0}^{N} |a_{n,k}|M_{n,k} + B\right] e^{\tau|x|}.$$

This concludes the proof of Lemma 3.8. □

Proposition 3.9. *Let μ be a measure on $\mathbb{R}$ such that $B = \int_{\mathbb{R}} e^{\tau|t|}\,d|\mu|(t) < +\infty$, $\tau > 0$, and $f \in C^N(\mathbb{R})$ satisfy*

(i) $|f(x)| \leq Me^{\tau|x|}$, $x \in \mathbb{R}$,

(ii) $\mu *_k f(x) = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f(x)$, $a_{N,k} \neq 0$.

Then f is infinitely differentiable.

PROOF: First

$$\left|\sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f(x)\right| \leq 4MBe^{\tau|x|}.$$

Observe that by Lemma 3.8, $|\mathcal{D}_k^s f(x)| \leq M_{s,k}e^{\tau|x|}$, $0 < s \leq N$. Further, Lemma 3.3 implies that the right hand side of the following equation

$$\mathcal{D}_k^N f(x) = \frac{1}{|a_{N,k}|}\left[\mu *_k f(x) - \sum_{n=0}^{N-1} a_{n,k}\mathcal{D}_k^n f(x)\right],$$

is differentiable and that in fact

$$\mathcal{D}_k^{N+1} f(x) = \frac{1}{|a_{N,k}|}\left[\mu *_k \mathcal{D}_k f(x) - \sum_{n=0}^{N-1} a_{n,k}\mathcal{D}_k^{n+1} f(x)\right].$$

Therefore

$$|\mathcal{D}_k^{N+1} f(x)| \leq \frac{1}{|a_{N,k}|}\left(\sum_{n=1}^{N} |a_{n-1,k}|M_{n,k} + 4M_{1,k}B\right)e^{\tau|x|}.$$

This process can be repeated infinitely, proving the proposition. □

Remark. Proposition 3.7 and 3.9 have shown that the equations $\mu *_k f(x) = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f(x)$ have solutions which are entire functions of exponential type when all but one of the $a_{n,k}$ are zero, $1 \leq n \leq N$, and are at least infinitely differentiable otherwise.

4. Characterizations for certain classes of entire functions of slow growth on the real axis associated with the Dunkl operator

In this section we show a Paley-Wiener type Theorem associated with $\mathcal{D}_k$ and we proceed to develop conditions on the measure (more precisely on the Fourier-Dunkl transform of the measure which will be defined below) such that equations of the form $\mu *_k f = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f$ characterize the class of entire functions of exponential type a which are square integrable with respect to $|x|^{2k}dx$ and bounded on the real line.
Next, we continue by considering the equations characterizing entire functions of exponential type which have polynomial growth on the real line. The section closes by giving some other results by considering these same equations in the same order.

Notations. We denote by

- $L_k^p(\mathbb{R})$, $1 \leq p < \infty$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,k} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2k}\, dx\right)^{\frac{1}{p}} < +\infty.$$

- $L_k^2([-a\,,\,a])$ the subspace of functions in $L_k^2(\mathbb{R})$ vanishing outside $[-a\,,\,a]$, $a > 0$.
- $\mathrm{Exp}_a^B(\mathbb{C})$, the space of entire functions of exponential type a which are bounded on the real line.
- $L_{k,a}^2(\mathbb{R})$ the subspace of $\mathrm{Exp}_a^B(\mathbb{C})$ consisting of functions belonging to $L_k^2(\mathbb{R})$.

- The Fourier-Dunkl transform on $L_k^1(\mathbb{R})$ is defined by

$$\mathcal{F}_k(f)(\xi) = c_k \int_{\mathbb{R}} f(x)\phi_{-i\xi,0}^k(x)|x|^{2k}\, dx,$$

where $c_k = \frac{1}{2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2})}$.
- Let μ be a finite Radon measure on the real line. The Fourier-Dunkl transform of μ is given by

$$\mathcal{F}_k(\mu)(y) = \int_{\mathbb{R}} \phi_{-iy,0}^k(x)\, d\mu(x).$$

Many properties of the Euclidean Fourier transform carry over to Fourier-Dunkl transform. In particular $\mathcal{F}_k(f) \in C_0(\mathbb{R})$ for $f \in L_k^1(\mathbb{R})$ ($C_0(\mathbb{R})$ is the space of

continuous functions on $\mathbb{R}$ such that vanish at infinity), and there holds an L^1-inversion Theorem: If f, $\mathcal{F}_k(f) \in L^1_k(\mathbb{R})$ then $f = \overline{\mathcal{F}_k}\mathcal{F}_k(f) = \mathcal{F}_k\overline{\mathcal{F}_k}(f)$ a.e, where $\overline{\mathcal{F}_k}(f)(\xi) = \mathcal{F}_k(f)(-\xi)$. Moreover, the Fourier-Dunkl transform $\mathcal{F}_k$ is a topological isomorphism from $S(\mathbb{R})$ onto itself ($S(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions on the $\mathbb{R}$), so $\mathcal{F}_k$ can be extended to a Plancherel transform on $L^2_k(\mathbb{R})$. For details see [6].
Let f in $L^p_k(\mathbb{R})$, $1 \le p < \infty$. We define the distribution S_f by

$$\langle S_f\,,\, \varphi\rangle_k = \int_{\mathbb{R}} f(x)\varphi(x)|x|^{2k}\,dx,\ \ \varphi \in S(\mathbb{R}). \tag{13}$$

Let f in $\mathcal{A}(\mathbb{C})$ be such that

$$f(z) = \int_{\mathbb{R}} g(t)\phi^k_{iz,0}(t)|t|^{2k}\,dt, \quad z \in \mathbb{C},$$

with $g \in L^2_k([-a\,,\,a])$, $a > 0$. Then f is an entire function of exponential type a. The following Paley-Wiener type theorem associated with the operator $\mathcal{D}_k$ asserts the converse of this is true, if we know that f restricted to the real axis belongs to $L^2_k(\mathbb{R})$. More precisely, we have

Theorem 4.1. *Suppose $f \in L^2_k(\mathbb{R}) \cap \mathcal{A}(\mathbb{C})$. Then*

$$f(z) = c_k \int_{-a}^{a} g(t)\phi^k_{iz,0}(t)|t|^{2k}\,dt,$$

where $g \in L^2_k([-a\,,\,a])$ if and only if f is of exponential type a.

Proof: Suppose f is of exponential type a and its restriction to the real axis belongs to $L^2_k(\mathbb{R})$. Let g be the Fourier-Dunkl transform of f. Then

$$f(x) = \lim_{T \longrightarrow +\infty} c_k \int_{-T}^{T} g(t)\phi^k_{ix,0}(t)|t|^{2k}\,dt,$$

where the limit is in the topology of $L^2_k(\mathbb{R})$. If $t < -a$, let Γ be the closed curve in the upper half plane which consists of the segment $L_1 = [-T, -\epsilon]$, γ_ϵ, $\epsilon > 0$ is the small semicircle from $-\epsilon$ to ϵ, oriented counterclockwise, $L_2 = [\epsilon, T], L_3 = [T, T + iT], L_4 = [T + iT, -T + iT]$ and $L_5 = [-T + iT, -T]$. We can use a similar argument with Γ in the lower half plane if $t > a$. We obtain the result by proceeding in the same way as [3, Theorem 6.8.1, p. 103] and using that

$$\lim_{\epsilon \longrightarrow 0} \int_{\gamma_\epsilon} f(z)\phi^k_{ix,0}(z)|z|^{2k}\,dz = 0.$$

□

Remarks. (i) If $f \in L^2_{k,a}(\mathbb{R})$, then for all $n \in \mathbb{N}$, $\mathcal{D}_k^n f \in L^2_{k,a}(\mathbb{R})$.
(ii) Let ξ_k be the function defined by $\xi_k(x) = \mathcal{F}_k(\chi_{[-a,a]})$, where $\chi_{[-a,a]}$ is the characteristic function of the interval $[-a, a]$. We have

$$\xi_k(x) = \frac{1}{2^{k-\frac{1}{2}}\Gamma(\frac{1}{2})} a^{2k+1} \sum_{n=0}^{\infty} \frac{(-1)^n (xa)^{2n}}{(2n)!(2n+2k+1)} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+k+\frac{1}{2})}. \tag{14}$$

The function ξ_k belongs to $L^2_{k,a}(\mathbb{R})$ and its Fourier-Dunkl transform equals 1 on the interval $[-a, a]$. In the case $k = 0$, we have $\xi_0(x) = \sqrt{\frac{2}{\pi}} \, \frac{\sin ax}{x}$.

Theorem 4.2. *Let $f \in L^2_k(\mathbb{R}) \cap C^N(\mathbb{R})$ and μ be a finite Radon measure on $\mathbb{R}$ such that*

$$\mathcal{F}_k(\mu)(t) = \begin{cases} \sum_{n=0}^N a_{n,k}(it)^n, & \text{for } |t| \le a \text{ and } a_{n,k} \text{ are complex,} \\ g(t), & \text{for } |t| > a \text{ where } g(t) \ne \sum_{n=0}^N a_{n,k}(it)^n. \end{cases}$$

Then

$$\mu *_k f(x) = \sum_{n=0}^{N} a_{n,k} \mathcal{D}_k^n f(x) \tag{15}$$

if and only if $f \in L^2_{k,a}(\mathbb{R})$. Further, if (15) holds for every $f \in L^2_{k,a}(\mathbb{R})$ but not for every $f \in L^2_{k,b}(\mathbb{R})$, where $b > a$, then $\mathcal{F}_k(\mu)$ has the form above.

Proof: Suppose $f \in L^2_{k,a}(\mathbb{R})$. From the Paley-Wiener type Theorem 4.1 and the assumptions on the measure μ, we have

$$\mathcal{F}_k(f)(t) \left[\mathcal{F}_k(\mu)(t) - \sum_{n=0}^{N} a_{n,k}(it)^n \right] = 0.$$

Then

$$\mathcal{F}_k(\mu *_k f)(t) = \mathcal{F}_k(\sum_{n=0}^{N} a_{n,k} \mathcal{D}_k^n f)(t).$$

So, we see that (15) holds. Conversely, if (15) holds, by the Fourier-Dunkl transform and the assumptions on the measure μ, it is clear that $\mathcal{F}_k(f)(t) = 0$, for $|t| > a$. Taking the inverse Fourier-Dunkl transform, we have

$$f(x) = c_k \int_{-a}^{a} \phi^k_{ix,0}(t) \mathcal{F}_k(f)(t) |t|^{2k} \, dt.$$

Hence $f \in L^2_{k,a}(\mathbb{R})$ by the Paley-Wiener type Theorem 4.1. Finally, if (15) holds for every $f \in L^2_{k,a}(\mathbb{R})$, it certainly holds for the function ξ_k given by relation (14), hence

$$\mathcal{F}_k(\mu)(t) = \sum_{n=0}^{N} a_{n,k}(it)^n \quad \text{for} \quad |t| \leq a.$$

If $\mathcal{F}_k(\mu)$ has this form for the interval $|t| \leq b$, where $b > a$, the previous argument would show that (15) holds for every $f \in L^2_{k,b}(\mathbb{R})$, contradiction to the hypothesis. This finishes the proof. $\square$

Example. Let μ be the measure defined by

$$\mu = -\frac{a^2}{3}\delta_0 + \frac{2a^2}{\pi^2} \sum_{n\in\mathbf{Z}, n\neq 0} \frac{(-1)^{n+1}}{n^2}\delta_{\frac{n\pi}{a}} .$$

By computation, we have $\mathcal{F}_k(\mu)(t) = \frac{1}{2k+1}(it)^2$ for $|t| \leq a$, and for $|t| > a$ the graph repeats. Then

$$\frac{1}{2k+1}\mathcal{D}_k^2 f(x) = -\frac{a^2}{3} f(x) + \frac{2a^2}{\pi^2} \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{(-1)^{n+1}}{n^2} T^k_{-\frac{n\pi}{a}} f(x)$$

for $f \in L^2_k(\mathbb{R}) \cap C^2(\mathbb{R})$ is characteristic of $f \in L^2_{k,a}(\mathbb{R})$.

Theorem 4.3. *Let μ be a measure such that*

$$\mathcal{F}_k(\mu)(t) = \begin{cases} \sum_{n=0}^{N} a_{n,k}(it)^n, & \text{for } |t| \leq a, \\ g(t), & \text{for } |t| > a, \quad \text{where } g(t) \neq \sum_{n=0}^{N} a_{n,k}(it)^n. \end{cases}$$

Further, suppose that $\int_{\mathbb{R}} |t|^J \, d|\mu|(t) < +\infty$, where $J \geq 0$. Let $f \in C^N(\mathbb{R})$ be such that $f(x) = O(|x|^J)$ as $|x| \longrightarrow +\infty$. Then

$$\mu *_k f = \sum_{n=0}^{N} a_{n,k}\mathcal{D}_k^n f \tag{16}$$

if and only if f is entire of exponential type a. Further, if (16) does hold for this class of functions, then $\mathcal{F}_k(\mu)(t) = \sum_{n=0}^{N} a_{n,k}(it)^n$ for $|t| \leq a$.

Proof: Let f be an entire function of exponential type a and S_f the distribution defined by (13), then $\mathcal{F}_k(S_f) \in S'(\mathbb{R})$ whose support is in the interval $[-a\,,\,a]$. The k-convolution $S_f *_k \mu$ is also a distribution and $\mathcal{F}_k(S_f *_k \mu) = \mathcal{F}_k(S_f)\mathcal{F}_k(\mu)$. By the hypothesis on the measure and the properties of Fourier-Dunkl transform, we see that (16) holds for distributions and thus for functions. Conversely, applying Fourier-Dunkl transform to (16) and using the assumptions on the measure μ yield that the support of $\mathcal{F}_k(S_f)$ is in the interval $[-a\,,\,a]$. Hence f is the restriction to $\mathbb{R}$ of the entire function g of exponential type a given by $g(z) = \overline{\mathcal{F}_k}\mathcal{F}_k(S_f)(z)$. Finally, if (16) holds for every $f \in \mathrm{Exp}_a(\mathbb{C})$ such that $f(x) = O(|x|^J)$ as $|x| \longrightarrow +\infty$, for any fixed $J \geq 0$. Since $z \longmapsto \phi^k_{ix,0}(z)$ is of exponential type a, for $|x| \leq a$, the characterization of the measure follows. □

Example. For $\chi \in D(\mathbb{R})$ (the space of C^∞-functions on $\mathbb{R}$ with compact support), satisfying the following conditions: $0 \leq \chi \leq 1$, $\chi(x) = 1$, $\forall\, x \in [-a\,,\,a]$, $a > 0$ and $\operatorname{supp}\chi \subset \,]-a-\epsilon,\ a+\epsilon\,[$, $\epsilon > 0$, we define

$$\varphi_k(x) = c_k \int_{\mathbb{R}} \phi^k_{ix,0}(t)\chi(t)|t|^{2k}\,dt = \overline{\mathcal{F}_k}(\chi)(x).$$

We note that φ_k is an entire function of exponential type which is in $L^2_k(\mathbb{R})$. Since

$$\varphi_k(x) = c_k(-ix)^{-n} \int_{\mathbb{R}} \phi^k_{ix,0}(t)\mathcal{D}^n_k\chi(t)|t|^{2k}\,dt, \quad n \in \mathbb{N},$$

we have

$$|\varphi_k(x)| \leq c_k|x|^{-n} \int_{\mathbb{R}} |\mathcal{D}^n_k\chi(t)||t|^{2k}\,dt,$$

for arbitrary n, so that $\int_{\mathbb{R}} |t|^J|\varphi_k(t)|\,dt < +\infty$, for $J \geq 0$. A similar argument shows that $\int_{\mathbb{R}} |t|^J|\mathcal{D}^n_k\varphi_k(t)|\,dt < +\infty$, for any $n \geq 0$ and $J \geq 0$. $\mathcal{F}_k(\mathcal{D}^n_k\varphi_k)(t) = (it)^n$, for $|t| \leq a$. Then $\mathcal{D}^n_k f(x) = \int_{\mathbb{R}} T^k_{-t}f(x)\mathcal{D}^n_k\varphi_k(t)\,dt$, $n \geq 0$, characterizes entire functions of exponential type a which have polynomial growth on the real line.

Theorem 4.4. *Let $\mu \in M_\sigma(\mathbb{R})$, with $\sigma > 0$. The following equation*

$$\mu *_k f(z) = \sum_{n=0}^{N} a_{n,k}\mathcal{D}^n_k f(z), \quad z \in \mathbb{C}, \tag{17}$$

is satisfied for every entire function of exponential type a, $(a < \sigma)$ if and only if

$$\int_{\mathbb{R}} m_s(-t)d\mu(t) = \begin{cases} s!a_{s,k}, & \textit{if } \ s \geq 0,\ s \leq N, \\ 0, & \textit{if } \ s > N, \end{cases}$$

where $m_s(z) = V_k(z^s)$.

Proof: We have just to show the "only if" part of this theorem. The "if" part is easy to see when we take in particular $f(z) = m_s(z)$, $s \geq 0$, in equation (17).

Let P be any polynomial of degree j. From Theorem 1.3, we deduce that $T_z^k P(t) = \sum_{n=0}^{j} \mathcal{D}_k^n P(z) b_n(t)$. Thus $\mu *_k P(z) = \sum_{n=0}^{N} a_{n,k} \mathcal{D}_k^n P(z)$. Fixing $z \in \mathbb{C}$, the functional

$$F_z^k : g \longmapsto \int_{\mathbb{R}} \left\{ T_{-t}^k g(z) - \sum_{n=0}^{N} \mathcal{D}_k^n g(z) b_n(-t) \right\} d\mu(t)$$

is a continuous linear functional on the Banach space $X = \{g \in L^1(d|\mu|),\ g$ is entire$\}$, it has the property that $F_z^k[P] = 0$, $\forall P \in \mathcal{P}$, the polynomial space on $\mathbb{C}$. Let $f \in \mathrm{Exp}_a(\mathbb{C})$, $a < \sigma$, and $z \in \mathbb{C}$ fixed, thus $T_z^k f \in X$. Since $T_z^k f$ is analytic, Theorem 1.3 implies that $T_z^k f(t) = \sum_{n=0}^{\infty} \mathcal{D}_k^n f(z) b_n(t)$, (see [8]). By the dominated convergence theorem, we have

$$\|P_{j,z} - T_z^k f\|_1 = \int_{\mathbb{R}} |P_{j,z}(t) - T_z^k f(t)|\, d|\mu|(t) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty,$$

where $P_{j,z}(t) = \sum_{n=0}^{j} \mathcal{D}_k^n f(z) b_n(t)$. Consequently $T_z^k f \in \overline{\mathcal{P}}$, the closure of $\mathcal{P}$ in X. We deduce the result by Hahn Banach theorem. □

Remark. Combining the last theorem with Theorem 2.2, we see that for $\mu \in M_\sigma(\mathbb{R})$, $\sigma > 0$, $\mathcal{L}_k(\mu)(z) = \sum_{n=0}^{N} a_{n,k} z^n$ in a disc if and only

$$\int_{\mathbb{R}} m_s(-t) d\mu(t) = \begin{cases} s! a_{s,k} & \text{for } 0 \le s \le N \\ 0 & \text{if } s > N. \end{cases}$$

We will consider the following cases, where μ represents the measure in each case.

1. $f \in L_k^2(\mathbb{R})$ and $\mathcal{F}_k(\mu)$ exists.
2. $f(x) = O(|x|^J)$ for $J \ge 0$ as $|x| \longrightarrow \infty$ and $\mathcal{F}_k(\mu)$ exists.
3. f has arbitrary growth and $\mu \in M_b(\mathbb{R})$ with $b > a$.

Theorem 4.5. *Let f belong to $\mathrm{Exp}_a(\mathbb{C})$ and satisfy both*

$$\mu *_k f(x) = \sum_{n=0}^{N} a_{n,k} \mathcal{D}_k^n f(x) \quad \text{and} \quad \mu_1 *_k f(x) = \sum_{m=0}^{M} b_{m,k} \mathcal{D}_k^m f(x), \tag{18}$$

where for case (1) *and* (2), *the set of common zeros of the functions*

$$g(t) = \mathcal{F}_k(\mu)(t) - \sum_{n=0}^{N} a_{n,k} (it)^n \quad \text{and} \quad g_1(t) = \mathcal{F}_k(\mu_1)(t) - \sum_{m=0}^{M} b_{m,k} (it)^m$$

for $|t| \leq a$, is at most the origin (where roots of multiplicity r and r_1 occur) and for case (3) the set of common zeros of the complex functions

$$G(z) = \mathcal{L}_k(\mu)(z) - \sum_{n=0}^{N} a_{n,k} z^n \quad \text{and} \quad G_1(z) = \mathcal{L}_k(\mu_1)(z) - \sum_{m=0}^{M} b_{m,k} z^m,$$

is at most the origin (where roots of multiplicity r and r_1 occur). Then, case (1), f is identically zero. Case (2) and (3), f is a polynomial of degree $\min(r-1\,,\, r_1-1)$.

Proof: (1) Follows easily from Paley-Wiener type Theorem 4.1 and the assumptions on the measures μ and μ_1. Case (2), applying Fourier-Dunkl transform to (18) and using the assumptions on the measures μ and μ_1 yield that $\operatorname{supp} \mathcal{F}_k(S_f) \subset \{0\}$, then there exists $N_1 \in \mathbb{N}$ such that $\mathcal{F}_k(S_f) = \sum_{n=0}^{N_1} c_n \delta_0^{(n)}$. Since $g\mathcal{F}_k(S_f)$ is zero distribution and $\lambda = 0$ is a zero of g of order r, hence $c_n = 0$ if $n \geq r$. Thus $f(x) = \sum_{n=0}^{r-1} c_{n,k} x^n$, $c_{n,k} \in \mathbb{C}$. The same work for g_1, implies that f is a polynomial of degree at most $\min(r-1\,,\, r_1-1)$. Finally (3), from Pólya representation Theorem 1.6, Lemma 2.6 and the residue theorem, we deduce the assertion. □

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