A factorization of quasiorder hypergroups

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Abstract. The contribution is devoted to the question of the interchange of the construction of a quasiorder hypergroup from a quasiordered set and the factorization.

Keywords: quasiorder hypergroup, congruence on a hypergroup, relational system

Classification: 20N20, 18A40

A concept of hypergroups was formalized by [1], [2], [9], [12], [14] as follows. Let H be a non-void set and "$\circ$" a mapping of $H \times H$ into $\mathscr{P}^*(H)$ (the set of all non-void subsets of H). The pair $(H, \circ)$ is called *a hypergroupoid*. For $A, B \in H$ we denote $A \circ B = \bigcup\{a \circ b; a \in A, b \in B\}$.

A hypergroupoid $(H, \circ)$ is called *a hypergroup* if "$\circ$" is associative, i.e. $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in H$, and the so-called *reproduction axiom*, i.e. $a \circ M = M = M \circ a$ for any $a \in H$, is satisfied.

Let R be a binary relation on a non-void set A. The pair $\mathscr{A} = (A, R)$ is called a *relational system*. A relational system A is called *transitive* if R is transitive and $\mathscr{A}$ is called a *quasiordered set* whenever R is a quasiorder on A, i.e. R is a reflexive and transitive relation.

The following fact is well-known. Let $\mathscr{A} = (A, R)$ be a relational system. Denote $U_R(a) = \{x \in A; \langle a, x\rangle \in R\}$ and, for $M \subseteq A$, $U_R(M) = \{x \in A; \langle a, x\rangle \in R$ for all $a \in M\}$. Let $\mathscr{A} = (A, \leq)$ be a quasiordered set. Define for $a, b \in A$

$$a \circ b = U_{\leq}(a) \cup U_{\leq}(b). \tag{1}$$

Then $(A, \circ)$ is a hypergroup which is called a *quasiorder hypergroup* (see e.g. [9]).

The concept of congruence on a hypergroup $(H, \circ)$ was defined by several authors. It was shown in [9, p. 151] that the definitions are equivalent. Let θ be an equivalence on a set A and $M \subseteq A$. Denote

$$\theta(M) = \{x \in A; \langle a, x\rangle \in \theta \ \text{ for some } \ a \in M\}.$$

Definition 1 ([9])**.** Let $(H, \circ)$ be a hypergroup and θ be an equivalence on H. We call θ a *congruence* on $(H, \circ)$ if for each $a, b, c, d \in H$ we have:

$$\langle a, b\rangle \in \theta \text{ and } \langle c, d\rangle \in \theta \text{ imply } \theta(a \circ c) = \theta(b \circ d).$$

The motivation of our paper is the following: Let $(H, \leq)$ be a quasiordered set and $\mathscr{H} = (H, \circ)$ be a hypergroup, where "$\circ$" is defined by (1) (i.e. it is the *induced quasiorder hypergroup*). From now on, a quasiorder will be denoted by the symbol "$\leq$". Let θ be a congruence on $(H, \circ)$.

I. Does there exist an equivalence ψ on $(H, \leq)$ such that $(H/\psi, \leq/\psi)$ is a quasiordered set and $\mathscr{H}/\theta = (H, \circ)/\theta$ is isomorphic to the quasiorder hypergroup induced by $(H/\psi, \leq/\psi)$?

   It can be visualized by the following diagram:

(D1)
$$\begin{tikzcd}
(H, \leq) \arrow[r, dotted, "\psi\,?"] \arrow[d] & (H/\psi, \leq/\psi) \arrow[d, dotted] \\
\mathscr{H} = (H, \circ) \arrow[r] & \mathscr{H}/\theta = (H, \circ)/\theta
\end{tikzcd}$$

II. Suppose that $\mathscr{H} = (H, \circ)$ be a quasiorder hypergroup induced by a quasiordered set $(H, \leq)$ and let ψ be an equivalence on $(H, \circ)$ such that $(H/\psi, \leq/\psi)$ is a quasiordered set again. Under what conditions on ψ does there exist a congruence θ on $\mathscr{H}$ such that $\mathscr{H}/\theta$ is isomorphic to the quasiorder hypergroup induced by $(H/\psi, \leq/\psi)$?

   It can be visualized by the following diagram:

(D2)
$$\begin{tikzcd}
(H, \leq) \arrow[r, "\psi"] \arrow[d] & (H/\psi, \leq/\psi) \arrow[d, dotted] \\
\mathscr{H} = (H, \circ) \arrow[r, dotted, "\theta\,?"] & \mathscr{H}/\theta = (H, \circ)/\theta
\end{tikzcd}$$

As θ and ψ are equivalences on the same set H, we can easily simplify our problems by considering $\theta = \psi$, i.e. we can ask what conditions must be satisfied by an equivalence on a quasiordered set to be a congruence on the induced quasiorder hypergroup and vice versa. First of all, we need several concepts and properties of relational systems.

Definition 2. Let $\mathscr{A} = (A, R)$ be a relational system and θ be an equivalence on A. For $a \in A$ denote by $[a]_\theta$ the θ-class containing the element a. Define R/θ on A/θ as follows:

$$\langle [a]_\theta, [b]_\theta \rangle \in R/\theta \ \text{ if and only if there exist } \ x \in [a]_\theta, y \in [b]_\theta \ \text{ with } \ \langle x, y\rangle \in R.$$

The system $\mathscr{A}/\theta = (A/\theta, R/\theta)$ is called a *quotient system of* $\mathscr{A}$ *by* θ.

The following statement is almost trivial:

Lemma 1. *Let* $\mathscr{A} = (A, R)$ *be a relational system and* θ *be an equivalence on* A. *If* R *is reflexive or symmetric, then also* R/θ *has the same property.*

Unfortunately, a similar statement fails for transitive relational systems, see the following:

Example 1. Let $A = \{a, b, c, d\}$ and $(A, \leq)$ be a quasiordered set visualized in Figure 1 below.

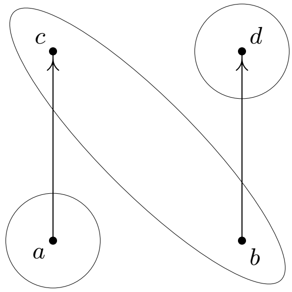


Figure 1

Let θ be an equivalence on A defined by the partition $\{a\}, \{b, c\}, \{d\}$. Then we have on A/θ the following:

$$\begin{aligned} & [a]_\theta \leq/\theta\, [b]_\theta \ \text{ since } \ a \leq c \ \text{ and } \ c \in [b]_\theta, \\ & [b]_\theta \leq/\theta\, [d]_\theta \ \text{ since } \ b \leq d, \\ \text{but } \ & [a]_\theta \leq/\theta\, [d]_\theta \ \text{ does not hold, i.e. } \text{“} \leq/\theta \text{”} \ \text{ is not transitive.} \end{aligned}$$

This example motivates us to introduce the following concept:

Definition 3. Let $\mathscr{A} = (A, R)$ be a relational system and θ be an equivalence on A. We say that θ is *compatible* (with $\mathscr{A}$) if either $\theta = A \times A$ or it satisfies the following condition:

(c) for each $x, y, z \in A$ with $\langle x, y\rangle \in \theta$ and $\langle y, z\rangle \in R$ there exists $q \in A$ such that $\langle x, q\rangle \in R$ and $\langle q, z\rangle \in \theta$.

Theorem 1. *Let $\mathscr{A} = (A, \leq)$ be a quasiordered set and θ be a compatible equivalence on A. Then $\mathscr{A}/\theta = (A/\theta, \leq/\theta)$ is a quasiordered set.*

Proof: By Lemma 1 "$\leq/\theta$" is reflexive on A/θ. Suppose $[a]_\theta \leq/\theta\ [b]_\theta$ and $[b]_\theta \leq/\theta\ [c]_\theta$. Then there are $x \in [a]_\theta$, $y, y' \in [b]_\theta$ and $z \in [c]_\theta$ such that $x \leq y$, $y' \leq z$. By (c) of Definition 3 there exists $q \in [c]_\theta$ such that $y \leq q$. Due to transitivity of "$\leq$", $x \leq q$, thus $[a]_\theta \leq/\theta\ [c]_\theta$, thus "$\leq/\theta$" is also transitive. □

Definition 4. Let $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, Q)$ be relational systems. Then a mapping $f\colon A \to B$ is called:

(a) *monotonous* if $\langle a, b\rangle \in R$ implies $\langle f(a), f(b)\rangle \in Q$;
(b) *strong homomorphism* if it is monotonous and for each $a, b \in A$ with $\langle f(a), f(b)\rangle \in Q$ there exist $c, d \in A$ such that $\langle c, d\rangle \in R$ and $f(c) = f(a)$, $f(d) = f(b)$;
(c) *U-morphism* if it is surjective and for each $x \in A$ we have

$$f(U_R(x)) = U_Q(f(x)).$$

Lemma 2. *Let $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, Q)$ be relational systems and a mapping $f\colon A \to B$ be a U-morphism. Then f is a strong homomorphism.*

Proof: Suppose that $a, b \in A$, $\langle a, b\rangle \in R$. Then $b \in U_R(a)$ and since f is a U-morphism, $f(b) \in f(U_R(a)) = U_Q(f(a))$, which gives $\langle f(a), f(b)\rangle \in Q$. Thus f is monotonous.

Suppose now that $\langle f(a), f(b)\rangle \in Q$. Then $f(b) \in U_Q(f(a)) = f(U_R(a))$, thus there exists $c \in U_R(a)$ with $f(c) = f(b)$. But $c \in U_R(a)$ implies $\langle a, c\rangle \in R$. Hence, f is a strong homomorphism. □

Theorem 2. *Let $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, Q)$ be relational systems and $f\colon A \to B$ a surjective mapping. The following are equivalent:*

(a) *f is a U-morphism;*
(b) *f is monotonous and for each $x, y \in A$ with $\langle f(x), f(y)\rangle \in Q$ there exists $z \in A$ such that $\langle x, z\rangle \in R$ and $f(y) = f(z)$.*

Proof: (a)⇒(b). By Lemma 2 we have that f is monotonous. Suppose that $\langle f(x), f(y)\rangle \in Q$. Then $f(y) \in U_Q(f(x)) = f(U_R(x))$, thus there is $z \in U_R(x)$ (i.e. $\langle x, z\rangle \in R$) such that $f(z) = f(y)$.
(b)⇒(a). Let $f\colon A \to B$ be a surjective and monotonous mapping. Then clearly

$$f(U_R(x)) \subseteq U_Q(f(x)).$$

Let $z \in U_Q(f(x))$. Then $z = f(w)$ for some $w \in A$, where $\langle f(x), f(w)\rangle \in Q$. By (b) there exists $c \in A$ such that $\langle x, c\rangle \in R$ and $f(c) = f(w) = z$. Thus $c \in U_R(x)$ proving the converse inclusion, i.e. f is a U-morphism. □

Theorem 3. *Let $\mathscr{A} = (A, R)$ be a relational system and θ be a compatible equivalence on $\mathscr{A}$. Then the canonical mapping $h_\theta\colon A \mapsto A/\theta$ given by $h_\theta(a) = [a]_\theta$ is a U-morphism.*

PROOF: Let θ be a compatible equivalence on $\mathscr{A} = (A, R)$. Suppose $\langle a, b\rangle \in R$, $a, b \in A$. Thus $[a]_\theta\ R/\theta\ [b]_\theta$ and hence h_θ is monotonous. Of course, h_θ is surjective. We only need to verify (b) of Theorem 2. Suppose $[x]_\theta\, R/\theta\, [y]_\theta$. Then there exist $a \in [x]_\theta$, $b \in [y]_\theta$ with $\langle a, b\rangle \in R$. By Definition 3, there exists $q \in A$ such that $\langle x, q\rangle \in R$ and $\langle q, b\rangle \in \theta$, i.e. $[q]_\theta = [b]_\theta$. Hence, (b) of Theorem 2 is satisfied. □

Theorem 4. *Let $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, Q)$ be relational systems and $f\colon A \to B$ a U-morphism. Then the induced equivalence*

$$\langle a, b\rangle \in \theta_f \quad \textit{iff} \quad f(a) = f(b)$$

is compatible with $\mathscr{A}$.

PROOF: Suppose $\langle x, y\rangle \in \theta_f$ and $\langle y, z\rangle \in R$. Then $f(x) = f(y)$ and by Lemma 2 we have $\langle f(y), f(z)\rangle \in Q$. Further, by Theorem 2 there exists $u \in A$ such that $\langle x, u\rangle \in R$ and $f(u) = f(z)$, i.e. $\langle u, z\rangle \in \theta_f$. Hence, condition (c) of Definition 3 is satisfied for $q = u$, i.e. θ_f is compatible with $\mathscr{A}$. □

We can finish our treatment concerning the problems in the introduction:

Corollary 1. *Let $(H, \le)$ be a quasiordered set and $\mathscr{H} = (H, \circ)$ the induced quasiorder hypergroup. Let θ be a congruence on $\mathscr{H}$. Then θ is a compatible equivalence on $(H, \le)$ and $\mathscr{H}/\theta$ is isomorphic to the quasiorder hypergroup induced by the quasiordered set $(H/\theta, \le/\theta)$.*

PROOF: By Lemma 1 the relation "$\le/\theta$" is reflexive; later on we will verify that it is also transitive.

First we will prove that the canonical mapping $h_\theta\colon (H, \le) \mapsto (H/\theta, \le/\theta)$ is a U-morphism.

For $x \le y$ we have $[x]_\theta \le/\theta\ [y]_\theta$, thus $[U_\le(x)]_\theta \subseteq U_{\le/\theta}([x]_\theta)$.

Let $[z]_\theta \in U_{\le/\theta}([x]_\theta)$. Then $[x]_\theta \le/\theta [z]_\theta$, which implies that there exist $x_1, z_1 \in H$ such that $\langle x, x_1\rangle \in \theta$, $\langle z, z_1\rangle \in \theta$, $x_1 \le z_1$. Therefore, as θ is a congruence on $\mathscr{H}$,

$$\langle a, b\rangle \in \theta \quad \Rightarrow \quad [a \circ a]_\theta = [b \circ b]_\theta \quad \Leftrightarrow \quad [U_\le(a)]_\theta = [U_\le(b)]_\theta.$$

Thus $[U_\le(x)]_\theta = [U_\le(x_1)]_\theta$, $[U_\le(z)]_\theta = [U_\le(z_1)]_\theta$. Further

$$U_\le(z_1) \subseteq U_\le(x_1) \quad \Rightarrow \quad [U_\le(z_1)]_\theta \subseteq [U_\le(x_1)]_\theta \quad \Rightarrow \quad [U_\le(z)]_\theta \subseteq [U_\le(x)]_\theta.$$

As $[z]_\theta \in [U_\leq(z)]_\theta$, we get $[z]_\theta \in [U_\leq(x)]_\theta$ and $U_{\leq/\theta}([x]_\theta) \subseteq [U_\leq(x)]_\theta$.

Together we have obtained that $[U_\leq(x)]_\theta = U_{\leq/\theta}([x]_\theta)$ and the canonical mapping $h_\theta\colon a \to [a]_\theta$ is a U-morphism.

In Theorem 4 let us put $f = h_\theta$, $A = H$, $B = H/\theta$, $R = \leq$ and $Q = \leq/\theta$. Then the induced equivalence θ_{h_θ} is compatible with $(H, \leq)$. But $\theta_{h_\theta} = \theta$. Now Theorem 1 implies that "$\leq/\theta$" is a quasiorder on H/θ.

Due to the fact that h_θ is U-morphism we get

$$\begin{aligned}[a \circ b]_\theta &= [U_\leq(a) \cup U_\leq(b)]_\theta = [U_\leq(a)]_\theta \cup [U_\leq(b)]_\theta \\ &= U_{\leq/\theta}([a]_\theta) \cup U_{\leq/\theta}([b]_\theta).\end{aligned} \tag{2}$$

The operations "$\circ_\theta$" (for the definition of a hyperoperation induced by the congruence θ on the quotient hypergroup H/θ see [9, p. 153]) and "$\star$" (compare (1)), where

$$\begin{aligned}&[a]_\theta \circ_\theta [b]_\theta = [a \circ b]_\theta, \\ &[a]_\theta \star [b]_\theta = U_{\leq/\theta}([a]_\theta) \cup U_{\leq/\theta}([b]_\theta),\end{aligned}$$

are the same due to (2). Thus Diagram (D1) (with $\theta = \psi$) commutes. □

Corollary 2. *Let ψ be a compatible equivalence on a quasiordered set $(H, \leq)$. Then ψ is a congruence on the quasiorder hypergroup $\mathscr{H}$ induced by $(H, \leq)$ and $\mathscr{H}/\psi$ is isomorphic to the quasiorder hypergroup induced by $(H/\psi, \leq/\psi)$.*

Proof: As ψ is compatible, by Theorem 1 we get that "$\leq/\psi$" is the quasiorder and by Theorem 3 we have $[U_\leq(x)]_\psi = U_{\leq/\psi}([x]_\psi)$. If $\langle a, c\rangle \in \psi$, $\langle b, d\rangle \in \psi$, then $[a]_\psi = [c]_\psi$, $[b]_\psi = [d]_\psi$, which implies

$$[U_\leq(a)]_\psi = U_{\leq/\psi}([a]_\psi) = U_{\leq/\psi}([c]_\psi) = [U_\leq(c)]_\psi.$$

Analogously $[U_\leq(b)]_\psi = [U_\leq(d)]_\psi$.

Then

$$\begin{aligned}[a \circ b]_\psi &= [U_\leq(a) \cup U_\leq(b)]_\psi = [U_\leq(a)]_\psi \cup [U_\leq(b)]_\psi \\ &= [U_\leq(c)]_\psi \cup [U_\leq(d)]_\psi = [U_\leq(c) \cup U_\leq(d)]_\psi = [c \circ d]_\psi,\end{aligned}$$

which means that ψ is the congruence on $(H, \circ)$. The commutativity of Diagram (D2) can be verified in the same way as in Corollary 1.

□

Example 2. Consider the quasiordered set $(H, \leq)$, where $H = \{a, b, c, d\}$ is depicted in Figure 2, and the equivalence θ determined by the partition $\{a, b\}$, $\{c, d\}$. It is easy to verify that θ does not satisfy condition (c) of Definition 3.

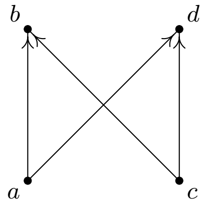


Figure 2

Although $(H/\theta, \leq/\theta)$ is still a quasiordered set (see Figure 3), θ is not a congruence on $\mathscr{H}$ induced by $(H, \leq)$ (as $[d \circ d]_\theta \neq [d \circ c]_\theta$, see Table 1):

$\circ$	a	b	c	d
a	$\{a, b, d\}$	$\{a, b, d\}$	H	$\{a, b, d\}$
b	$\{a, b, d\}$	$\{b\}$	$\{b, c, d\}$	$\{b, d\}$
c	H	$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$
d	$\{a, b, d\}$	$\{b, d\}$	$\{b, c, d\}$	$\{d\}$

Table 1

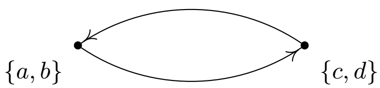
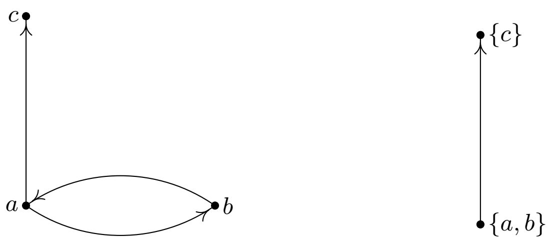


Figure 3

Example 3. Let $(H, \leq)$ be the quasiordered set in Figure 4(a):



(a) (b)

Figure 4

Then $(H, \leq)$ induces the quasiorder hypergroup $\mathscr{H} = (H, \leq)$ given by Table 2:

$\circ$	a	b	c
a	H	H	H
b	H	H	H
c	H	H	$\{c\}$

Table 2

The equivalence θ given by the partition $\{a, b\}$, $\{c\}$ is clearly a congruence on $\mathscr{H}$ and a compatible equivalence on $(H, \leq)$. The quotient quasiordered set $(H/\theta, \leq/\theta)$ is visualized in Figure 4(b) and $\mathscr{H}/\theta$ is determined by Table 3:

$\circ_\theta$	$\{a, b\}$	$\{c\}$
$\{a, b\}$	H/θ	H/θ
$\{c\}$	H/θ	$\{c\}$

Table 3

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