Nonreciprocal algebraic numbers of small measure

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Abstract. The main result of this paper implies that for every positive integer $d \geqslant 2$ there are at least $(d-3)^2/2$ nonconjugate algebraic numbers which have their Mahler measures lying in the interval $(1,2)$. These algebraic numbers are constructed as roots of certain nonreciprocal quadrinomials.

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1. Introduction

The *Mahler measure* of a polynomial with complex coefficients is defined as the modulus of the product of its leading coefficient and those of its roots that lie outside the unit circle (counted with multiplicities). The Mahler measure of an algebraic number is defined as the Mahler measure of its minimal polynomial in $\mathbb{Z}[x]$. Given a positive integer d and an interval of real numbers I, let $N(d,I)$ be the number of nonconjugate algebraic numbers of degree d having their Mahler measures lying in I. Clearly, the Mahler measures of two algebraic numbers conjugate over $\mathbb{Q}$ are equal, so the number of *all* such algebraic numbers will then be $dN(d,I)$.

There are several published upper bounds on the number $N(d,I)$ for $I=[1,T]$. M. Mignotte [8], [9] was the first who obtained such bounds. (He used a version of Siegel's lemma.) D.W. Boyd and H.L. Montgomery [2] found an asymptotic formulae for the number of not necessarily irreducible integer polynomials having all roots on the unit circle. (This corresponds to the case $T=1$.) On the other hand, S.J. Chern and J.D. Vaaler [3] found a nice asymptotic formulae for $N(d,[1,T])$ when T is large compared to d. However, the most difficult case occurs when T is fixed and d is large. Then the best result bounding $N(d,[1,T])$ from above is due to the author and S.V. Konyagin. It was shown in [6] that $N(d,[1,T]) < T^{(1+\varepsilon)d}$ for every fixed $\varepsilon > 0$ and $d > d(\varepsilon)$. This bound is apparently far from the true value of $N(d,[1,T])$.

Usually, all algebraic numbers having Mahler's measure greater than 1 and smaller, say, than 2 are of special interest, because of their connection to the

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question of Lehmer. So far no attempt has been made to obtain a lower bound for $N(d,(1,2))$ (see p. 1075 in [9]). The aim of this note is to derive at least some lower estimate for $N(d,(1,2))$.

In all what follows, let $\lambda_1 = 1.32497\ldots$ be the largest real root of $4x^8 - 5x^6 - 2x^4 - 5x^2 + 4 = 0$, and let $\lambda_2 = (\sqrt{47}+\sqrt{15})/4\sqrt{2} = 1.89657\ldots$ be the largest real root of $8x^4 - 31x^2 + 8 = 0$. Recall that an algebraic number α is called *reciprocal* if α^{-1} is conjugate to α over $\mathbb{Q}$ and is called *nonreciprocal* otherwise. We prove the following theorem.

Theorem. *For every $d \geqslant 2$, there are at least $(d-3)^2/2$ nonreciprocal nonconjugate units of degree d having their Mahler measures in the interval (λ_1, λ_2).*

Since $(\lambda_1, \lambda_2) = (1.32497\ldots, 1.89657\ldots) \subset (1,2)$, we clearly have $N(d,(1,2)) \geqslant (d-3)^2/2$.

The proof of the theorem will be given in Section 3. It is based on a few simple lemmas (see Section 2) which lead to the construction of many nonreciprocal quadrinomials of fixed degree. All this is based on the following result of W. Ljunggren [7]: if u, v, w are three distinct positive integers, then the polynomial $x^u \pm x^v \pm x^w \pm 1$ is reducible over $\mathbb{Q}$ if and only if it has a cyclotomic factor.

2. Auxiliary lemmas

Lemma 1. *Let z_1, z_2, z_3, z_4 be four complex numbers (clockwise) on the unit circle $|z| = 1$ which sum to zero. Then $z_1 + z_3 = z_2 + z_4 = 0$.*

Proof: It is sufficient to prove that $z_1 + z_3 = 0$. Let ℓ be a line passing through the origin and the midpoint of the line segment connecting z_1 and z_2. By projecting the sum $z_1 + z_2 + z_3 + z_4 = 0$ into ℓ, we deduce that ℓ passes through the midpoint of the line segment connecting z_3 and z_4. Furthermore, the distances between the origin and these midpoints must be equal. Therefore the points z_1, z_2, z_3, z_4 are the consecutive vertices of a rectangle. Of course, two degenerate situations, namely, $z_1 = z_2$, $z_3 = z_4$ and $z_1 = z_4$, $z_2 = z_3$ are also possible. However, in all three cases, we deduce that z_1 and z_3 are on a diameter of the unit circle, so $z_1 + z_3 = 0$, as claimed. □

Lemma 2. *Let μ be a root of unity and let u and v be two positive even and odd integers, respectively. Assume that $\mu^u = -1$. Then $\mu^v \neq \pm 1$.*

Proof: Without loss of generality we may assume that μ is a primitive mth root of unity $\mu = \exp(2\pi i/m)$. Then $\exp(2\pi i u/m) = -1$, so $u = m(2k+1)/2$ with $k \in \mathbb{Z}$. Since u is even, m is divisible by 4. Assuming that $\mu^v = \pm 1$ we deduce the equality $\mu^{2v} = \exp(4\pi i v/m) = 1$. Hence $v = ms/2$ with $s \in \mathbb{Z}$, so v must be even, a contradiction. □

Lemma 3. *Let u, v, w be three distinct positive integers. If they are all odd, then the polynomial $x^u + x^v + x^w \pm 1$ is irreducible over $\mathbb{Q}$. Furthermore, if u, v are even and w is odd then the polynomial $x^u + x^v \pm x^w + 1$ is irreducible over $\mathbb{Q}$.*

Proof: Let u, v, w be three distinct positive integers. By the result of W. Ljunggren [7], if the quadrinomial $q(x) = x^u + x^v \pm x^w \pm 1$ is reducible over $\mathbb{Q}$, then it has a cyclotomic factor. This can only happen if $q(\mu) = 0$ for some root of unity μ.

Suppose that u, v, w are all odd and $q(x) = x^u + x^v + x^w \pm 1$ is reducible. Rearranging u, v and w, if necessary, and combining the result of W. Ljunggren with Lemma 1 we deduce that $\mu^u + \mu^v = 0$ and $\mu^w \pm 1 = 0$. Assuming without loss of generality that $u > v$ we have that $\mu^{u-v} = -1$ (and $\mu^w = \pm 1$), contrary to Lemma 2. For the polynomial $x^u + x^v \pm x^w + 1$ with u, v even and w odd the argument is exactly the same. $\square$

Lemma 4. *Let $q(x) = x^u \pm x^v \pm x^w \pm 1$, where $u > v > w$ are three positive integers. Then the polynomial $q(x)q(1/x)x^u$ has at least 5 nonzero coefficients.*

Proof: Writing $q(x) = x^u + \tau_1 x^v + \tau_2 x^w + \tau_3$, where $\tau_1, \tau_2, \tau_3 \in \{-1, 1\}$, we have

$$\begin{aligned} q(x)q(1/x)x^u = x^u \big(&4 + \tau_3(x^u + x^{-u}) + \tau_2(x^{u-w} + x^{w-u}) + \tau_1(x^{u-v} + x^{v-u}) \\ &+ \tau_1\tau_3(x^v + x^{-v}) + \tau_1\tau_2(x^{v-w} + x^{w-v}) + \tau_2\tau_3(x^w + x^{-w})\big) . \end{aligned}$$

Clearly, we have 3 nonzero coefficients of powers x^j with $j = 0, u, 2u$. The polynomial also has 5 terms of the form $\pm x^j$, where $u < j < 2u$. Some of them may cancel, but, by a parity argument, at least one of these terms remains. This gives another nonzero coefficient. Since the polynomial $q(x)q(1/x)x^u$ is reciprocal, there is a corresponding nonzero term with j in the range $0 < j < u$. Summarizing, we have at least 5 nonzero coefficients. $\square$

The example of $q(x) = x^3 + x^2 + x - 1$ shows that

$$q(x)q(1/x)x^3 = -x^6 + x^4 + 4x^3 + x^2 - 1$$

has exactly 5 nonzero coefficients.

3. Quadrinomials have small Mahler measure

The claim of the theorem is evident for $d \leqslant 4$, so assume that $d \geqslant 5$. If d is even then, by Lemma 3, the polynomials $x^d + x^v \pm x^w + 1$ are irreducible whenever v and w are even and odd, respectively, in the range $1 \leqslant v, w \leqslant d - 1$. Clearly, there are precisely $2d(d-2)/4 = d(d-2)/2$ of such polynomials. They are all nonreciprocal.

If d is odd then, by Lemma 3 again, the polynomial $x^d+x^v+x^w+1$ is irreducible if v and w (in the range $1 \leqslant w < v \leqslant d-1$) are either both even or both odd. There are

$$2\binom{(d-1)/2}{2} = \frac{(d-1)(d-3)}{4}$$

of such polynomials. Similarly, the polynomial $x^d + x^v + x^w - 1$ is irreducible with v, w being both odd, whereas the polynomial $x^d - x^v - x^w - 1$ is irreducible with v, w being both even (with the same natural restrictions $1 \leqslant w < v \leqslant d-1$) which gives another $(d-1)(d-3)/4$ nonreciprocal polynomials. Summarizing, we obtain at least $(d-1)(d-3)/2$ distinct nonreciprocal irreducible polynomials of degree d. It follows that in both (even and odd) cases there are at least $(d-3)^2/2$ distinct nonreciprocal irreducible quadrinomials of degree d.

The Mahler measure M of a quadrinomial (with coefficients ± 1) can be bounded from above by combining Lemma 4 with Lemma 13 on p. 244 in [10]

$$M^2 + M^{-2} + \sqrt{(M^2 + M^{-2})^2 + 2} \leqslant 8.$$

(This inequality is stronger than that given by the inequality of J.V. Gonçalves: $M^2 + M^{-2} \leqslant 4$.) By an easy computation, we have that $M \leqslant \lambda_2 = (\sqrt{47} + \sqrt{15})/4\sqrt{2}$. Moreover, the inequality must be strict, because λ_2 is not an algebraic integer, so cannot be a Mahler measure. (See, for instance, [1], [4] for more necessary conditions on an algebraic number to be a measure.)

As for the lower bound, the inequality $M > \lambda_1$ holds for the Mahler measure of any nonreciprocal irreducible polynomial except for some very special trinomials. Indeed, C.J. Smyth [11] showed that every nonreciprocal Mahler measure must be at least $\lambda_0 = 1.32471\ldots$, where λ_0 is the positive solution of the equation $x^3 - x - 1 = 0$. Furthermore, in his thesis [12], he showed that any nonreciprocal number with Mahler measure equal to λ_0 must be conjugate to $\pm\lambda_0^{\pm 1/\ell}$, where ℓ is a positive integer. Hence, all other nonreciprocal algebraic numbers have their Mahler measures strictly greater than λ_0. Combining the results of [5] and [12] we showed in [4] that the interval $(\lambda_0, \lambda_1]$, where $\lambda_1 = 1.32497\ldots$ is the largest real root of $4x^8 - 5x^6 - 2x^4 - 5x^2 + 4 = 0$, contains no nonreciprocal measures at all. It follows that the Mahler measures of nonreciprocal algebraic numbers which are not conjugate to $\pm\lambda_0^{\pm 1/\ell}$ (including all roots of nonreciprocal irreducible quadrinomials) must be greater than λ_1, as claimed. This completes the proof of the theorem.

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