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Preferred parameterisations on homogeneous curves

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Abstract. We show how to specify preferred parameterisations on a homogeneous curve in an arbitrary homogeneous space. We apply these results to limit the natural parameters on distinguished curves in parabolic geometries.

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1. Introduction

This article is motivated by the theory of *distinguished curves* in *parabolic geometries*, as developed in [2]. A parabolic geometry is, by definition, modelled on a homogeneous space of the form G/P where G is a real semisimple Lie group and P is a parabolic subgroup. (There is also a complex theory which corresponds to the choices of complex G's and P's with specific curvature restrictions for the holomorphic cases.) The notion of Cartan connection replaces the Maurer-Cartan form on G, viewed as a principal fibre bundle over G/P with structure group P, and much of the geometry of G/P automatically carries over to parabolic geometries in general (see also [4]). In particular, the curves on G/P obtained by exponentiating elements in the Lie algebra $\mathfrak{g}$ of G have counterparts in general obtained by *development* under the Cartan connection. These matters are thoroughly discussed in [2] and will not be repeated here. Suffice it to say that results concerning distinguished curves on G/P have immediate consequences for the corresponding general parabolic geometry. Here, we shall discuss only the homogeneous setting G/P. We shall also not touch problems related to higher dimensional analogues of curves since curvature obstructs the development in general.

2. Generalities on G/P

Firstly, let us discuss a general homogeneous space, namely a smooth manifold M equipped with the smooth transitive action of a real Lie group G. Each $X \in \mathfrak{g}$ gives a 1-parameter Lie subgroup $t \mapsto \exp(tX)$ of G and hence a parameterised

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curve $t \mapsto \exp(tX)m$ through $m \in M$, which we shall suppose to be non-constant. Conversely, without the parameterisation, such a curve is *homogeneous*, namely it is the orbit of a Lie subgroup of the symmetry group G.

To investigate homogeneous curves on M we may as well choose a basepoint $m_\circ \in M$ and consider only curves passing through $m_\circ$. All other homogeneous curves are obtained by translation under the action of G. Let P denote the stabiliser subgroup of $m_\circ$ so that $M = G/P$. We shall now suppose that G is semisimple and P is parabolic. In this case, there is a splitting

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$$

into subalgebras with $\mathfrak{n}$ nilpotent (as in [2]). This splitting is not canonical. It is, however, well-defined up to the Adjoint action of P and we obtain, therefore, a preferred subset

$$\bigl\{\mathrm{Ad}_p X \text{ s.t. } p \in P \text{ and } X \in \mathfrak{n}\bigr\} \subset \mathfrak{g}, \tag{1}$$

which we may use to generate homogeneous curves. Such curves are evidently non-constant but not all non-constant homogeneous curves arise in this way. These special curves are said to be *distinguished*. Equivalently, distinguished curves through the basepoint $m_\circ \in M$ are those of the form $t \mapsto p\exp(tX)m_\circ$ for some $p \in P$ and $X \in \mathfrak{n}$.

As an example, consider $G = \mathrm{SL}(3,\mathbb{R})$ with P the upper triangular matrices. We may take

$$\mathfrak{n} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}\right\}. \tag{2}$$

Then

$$t \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \exp\left(t\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \bmod P = \begin{pmatrix} 1+t & 1 & 0 \\ t & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bmod P$$

and

$$t \mapsto \exp\left(t\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\right) \bmod P = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t+\frac{1}{2}t^2 & t & 1 \end{pmatrix} \bmod P$$

are typical distinguished curves whereas

$$t \mapsto \exp\left(t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \bmod P = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \bmod P$$

is homogeneous but (with this parameterisation) not distinguished.

Suppose $t \mapsto \gamma(t) \in M$ is a distinguished curve with $\gamma(0) = m_\circ$ and let C denote its unparameterised image. In this article, we shall answer the question 'what are the possible reparameterisations of C as a distinguished curve?' A direct approach to this question is given in [2, §3]. Here, we shall reason indirectly by firstly establishing the following on general grounds.

Theorem 1. *Let C be an unparameterised distinguished curve passing through $m_\circ \in M = G/P$. The freedom in reparameterising C with origin at $m_\circ$ is of two possible types: –*

affine	$t \mapsto at$	*for* $a \neq 0$
projective	$t \mapsto at/(ct+1)$	*for* $a \neq 0$ *and* c *arbitrary.*

If we drop the requirement that the parameter be zero at $m_\circ$, then translation is also allowed so the freedom becomes

$$t \mapsto at + b \quad \text{or} \quad t \mapsto \frac{at+b}{ct+d},$$

respectively. The proof of Theorem 1 will be given in §4. Once this theorem is established, it is a matter of elementary computation to decide, for a given C, which type of freedom pertains. Examples will be given in §4. For the proof of Theorem 1 we shall need some general considerations as in the following section.

3. Lie algebras of vector fields in one dimension

The following is a classical topic and Theorem 3 is due to Lie [3] (see also [5]). We are grateful to Ian Anderson for pointing out to us the translation and commentary on Lie's article given by Ackerman and Hermann [1]. Nevertheless, we believe that it is useful to give an independent, elementary, and self-contained treatment.

Theorem 2. *Suppose $\mathfrak{g}$ is a finite-dimensional subalgebra of the Lie algebra of smooth vector fields on $\mathbb{R}$. Let x be the standard coördinate on $\mathbb{R}$ and suppose $\mathfrak{g} \ni \partial/\partial x$. Then $\mathfrak{g}$ is one of the following: –*

$$\mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}\right\} \quad \mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, e^{\lambda x}\frac{\partial}{\partial x}\right\} \quad \mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right\}$$

$$\mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, \sin(\lambda x)\frac{\partial}{\partial x}, \cos(\lambda x)\frac{\partial}{\partial x}\right\} \quad \mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}$$

$$\mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, \sinh(\lambda x)\frac{\partial}{\partial x}, \cosh(\lambda x)\frac{\partial}{\partial x}\right\}.$$

PROOF: If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \operatorname{span}\{\partial/\partial x\}$ and we are done. Next, if $\dim \mathfrak{g} = 2$, then $\mathfrak{g} = \operatorname{span}\{\partial/\partial x, g(x)\partial/\partial x\}$ for some smooth non-constant function $g(x)$. Now,

$$\left[\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] = g'(x)\frac{\partial}{\partial x}$$

so closure under Lie bracket implies $g'(x) = \mu + \lambda g(x)$. This is a differential equation we may solve: –

$$\begin{aligned} &g(x) = Ce^{\lambda x} + D \qquad &&\text{if } \lambda \neq 0 \\ \text{or}\quad &g(x) = \mu x + C &&\text{if } \lambda = 0. \end{aligned}$$

These are the two-dimensional subalgebras stated in the theorem.

Now suppose $\dim \mathfrak{g} = k + 1 \geq 3$ and choose a basis

$$\frac{\partial}{\partial x}, g_1(x)\frac{\partial}{\partial x}, \ldots, g_k(x)\frac{\partial}{\partial x}$$

of $\mathfrak{g}$. From closure under Lie bracket of $\partial/\partial x$ with the other basis vectors, we immediately encounter a system of ordinary differential equations with constant coefficients

$$g_i'(x) = \mu_i + \sum_{j=1}^{k} \lambda_{ij} g_j(x), \qquad \text{for } i = 1, \ldots, k.$$

We may conclude that the functions $g_i(x)$ and, therefore, all vector fields in $\mathfrak{g}$ are real-analytic.

Since $\dim \mathfrak{g} \geq 3$, there is a vector field $g(x)\partial/\partial x \in \mathfrak{g}$ with

$$g(x) = x^N + \cdots \quad \text{for some } N \geq 2.$$

Because $\mathfrak{g}$ is finite-dimensional, we may choose $g(x)$ with N maximal. But then

$$\begin{aligned} \mathfrak{g} \ni \left[\left[\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right], g(x)\frac{\partial}{\partial x}\right] &= \left[g'(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] \\ &= \left((g'(x))^2 - g(x)g''(x)\right)\frac{\partial}{\partial x} \\ &= \left(Nx^{2(N-1)} + \cdots\right)\frac{\partial}{\partial x}, \end{aligned}$$

contradicting maximality of N unless $N = 2$. Therefore, $\dim \mathfrak{g} = 3$ and

$$\mathfrak{g} = \operatorname{span}\left\{\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}, g'(x)\frac{\partial}{\partial x}\right\}, \tag{3}$$

where

$$g(x) = x^2 + ax^3 + \cdots. \tag{4}$$

But then $\mathfrak{g}$ contains the vector field

$$\left[g'(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] - 2g(x)\frac{\partial}{\partial x} = \left((g'(x))^2 - g(x)g''(x) - 2g(x)\right)\frac{\partial}{\partial x}$$
$$= \left(2ax^3 + \cdots\right)\frac{\partial}{\partial x},$$

again contradicting maximality of N unless $a = 0$. Now, in order for (3) to be closed under Lie bracket we must have

$$g''(x) = \left[\frac{\partial}{\partial x}, g'(x)\frac{\partial}{\partial x}\right] \in \operatorname{span}\left\{\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}, g'(x)\frac{\partial}{\partial x}\right\}$$

and to be, in addition, consistent with $a = 0$ in (4), we conclude that

$$g''(x) = 2 + \nu g(x), \quad \text{for some constant } \nu.$$

This differential equation, with initial conditions imposed by (4), has solutions

$$\begin{array}{lll} & g(x) = (2/\lambda^2)(\cos(\lambda x) - 1) & \text{if } \nu < 0 \\ \text{or} & g(x) = x^2 & \text{if } \nu = 0 \\ \text{or} & g(x) = (2/\lambda^2)(\cosh(\lambda x) - 1) & \text{if } \nu > 0. \end{array}$$

It remains to observe that (3) is, indeed, closed under Lie bracket in these cases. □

Notice that this proof is local: the same conclusion holds for vector fields on any open interval $(a, b) \subseteq \mathbb{R}$. As subalgebras of $\operatorname{Vect}(\mathbb{R})$, the various possibilities listed in the statement of Theorem 2 for different values of λ are distinct. Locally, however, the parameter λ may be eliminated and more besides. Precisely, there is a coördinate change near the origin: –

$$y = \frac{1 - e^{-\lambda x}}{\lambda} \quad \Rightarrow \quad e^{\lambda x}\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x} = (1 - \lambda y)\frac{\partial}{\partial y}$$

whence

$$\operatorname{span}\left\{\frac{\partial}{\partial x}, e^{\lambda x}\frac{\partial}{\partial x}\right\} \cong \operatorname{span}\left\{y\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\} \cong \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right\}$$

whilst the coördinate change $y = \tan((\lambda x)/2)$ gives

$$\frac{\partial}{\partial x} = \frac{\lambda}{2}(1 + y^2)\frac{\partial}{\partial y}, \ \sin(\lambda x)\frac{\partial}{\partial x} = \lambda y\frac{\partial}{\partial y}, \ \cos(\lambda x)\frac{\partial}{\partial x} = \frac{\lambda}{2}(1 - y^2)\frac{\partial}{\partial y}$$

whence

$$\operatorname{span}\left\{\frac{\partial}{\partial x}, \sin(\lambda x)\frac{\partial}{\partial x}, \cos(\lambda x)\frac{\partial}{\partial x}\right\} \cong \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}$$

and $y = \tanh((\lambda x)/2)$ gives

$$\frac{\partial}{\partial x} = \frac{\lambda}{2}(1 - y^2)\frac{\partial}{\partial y}, \ \sinh(\lambda x)\frac{\partial}{\partial x} = \lambda y\frac{\partial}{\partial y}, \ \cosh(\lambda x)\frac{\partial}{\partial x} = \frac{\lambda}{2}(1 + y^2)\frac{\partial}{\partial y}$$

whence

$$\operatorname{span}\left\{\frac{\partial}{\partial x}, \sinh(\lambda x)\frac{\partial}{\partial x}, \cosh(\lambda x)\frac{\partial}{\partial x}\right\} \cong \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}.$$

We have proved the following.

Theorem 3. *Suppose $\mathfrak{s}$ is a finite-dimensional subalgebra of the Lie algebra of vector fields in a neighbourhood of the origin in $\mathbb{R}$. Suppose $\mathfrak{s}$ contains a vector field that does not vanish at the origin. Then there is a neighbourhood U of the origin and a change of coördinates such that one of the following three possibilities holds.*

$$\begin{gathered}\mathfrak{s}|_U \cong \operatorname{span}\left\{\frac{\partial}{\partial x}\right\} \qquad \mathfrak{s}|_U \cong \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right\} \\ \mathfrak{s}|_U \cong \operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\}.\end{gathered} \tag{5}$$

4. Reparameterisations

Let C be an arbitrary smooth connected curve in a smooth manifold M homogeneous under the action $\rho : G \times M \to M$ of a connected Lie group G. There is a homomorphism of Lie algebras $\dot\rho : \mathfrak{g} \to \operatorname{Vect}(M)$ given by

$$\dot\rho(X)(m) = \frac{\partial}{\partial t}\big(\exp(-tX)m\big)|_{t=0}$$

and the *symmetry algebra* of C is defined by

$$\mathfrak{s} = \{X \in \mathfrak{g} \text{ s.t. } \dot\rho(X)(m) \text{ is tangent to } C \text{ for all } m \in C\}.$$

Clearly, $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ and C is homogeneous if and only if $\dot\rho(\mathfrak{s})|_C$ contains non-trivial vector fields at each point of C. In this case, we may invoke Theorem 3 to conclude that $\dot\rho(\mathfrak{s})|_C$ is at most three-dimensional and locally has one of the three forms listed in (5).

Now suppose that C is homogeneous and pick a basepoint $m_\circ \in C$. Suppose that $X \in \mathfrak{s} \subset \mathfrak{g}$ is nilpotent in $\mathfrak{g}$ and $\dot\rho(X)(m_\circ) \neq 0$. Then we shall say that

$$t \mapsto \exp(tX)m_\circ \in C$$

is a *preferred parameterisation* of C.

Theorem 4. *The freedom in reparameterising a homogeneous curve with a preferred parameter is one of two possible types: –*

affine	$t \mapsto at$	*for* $a \neq 0$
projective	$t \mapsto at/(ct+1)$	*for* $a \neq 0$ *and* c *arbitrary.*

Proof: Since X is nilpotent in $\mathfrak{g}$, certainly $\dot\rho(X)$ is nilpotent in $\dot\rho(\mathfrak{s})|_C$. By inspection, we may find the nilpotent elements in each of the local forms (3): –

$$\operatorname{span}\left\{\frac{\partial}{\partial x}\right\} \ni a\frac{\partial}{\partial x}$$

$$\operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right\} \ni a\frac{\partial}{\partial x}$$

$$\operatorname{span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\} \ni (p-qx)^2\frac{\partial}{\partial x}.$$

In the first two cases,

$$a\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \iff x = at,$$

which gives affine freedom, whilst in the third case

$$(p-qx)^2\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \iff x = \frac{p^2 t}{1+pqt},$$

which gives projective freedom. □

Proof of Theorem 1: The parameterisations on a distinguished curve have the form $t \mapsto \exp(tY)m_\circ$ where Y is P-conjugate to an element of $\mathfrak{n}$ in accordance with (1). Certainly, there is affine freedom in such a parameterisation because Y can be replaced by aY. But the allowed Y are, in particular, nilpotent. Therefore, the parameterisations on C as a distinguished curve are *ipso facto* preferred parameterisations on C as a homogeneous curve. Theorem 4 now implies that, if there is any additional freedom, it must be projective. But just one projective transformation, together with affine freedom, generates all projective freedom and the proof is complete. □

Theorem 1 is useful in practice. Consider the general distinguished curve $t \mapsto p\exp(tX)m_\circ$ for fixed $p \in P$ and $X \in \mathfrak{n}$. The dichotomy offered by Theorem 1 implies that if there are reparameterisations other than affine, then the specific projective freedom $t \mapsto t/(t+1)$ occurs. In this case, we may find $q \in P$ and $Y \in \mathfrak{n}$ such that

$$p\exp(tX) = q\exp\big(\frac{t}{t+1}Y\big) \bmod P, \quad \forall\, t$$

or, equivalently,

$$\exp\big(-\frac{t}{t+1}Y\big) r\exp(tX) \in P, \quad \forall\, t \tag{6}$$

where $r = q^{-1}p \in P$. The existence of suitable $r \in P$ and $Y \in \mathfrak{n}$ is a restriction on X. Furthermore, if we adopt the Levi decomposition $P = LU$ corresponding to our choice of $\mathfrak{n}$, then the L-component of r may be absorbed into Y. Hence, Theorem 1 has the following corollary.

Corollary 5. *Suppose $P = LU$ is a Levi decomposition of a parabolic subgroup P of a semisimple Lie group G. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$ be the associated decomposition of the Lie algebra of G. Then the distinguished curve $t \mapsto p\exp(tX) \bmod P$ admits a projective reparameterisation if and only if there are $r \in U$ and $Y \in \mathfrak{n}$ such that* (6) *holds.*

We close this article with a complete analysis of the distinguished curves in the real flag manifold $\mathrm{SL}(3,\mathbb{R})/P$ where P is the subgroup of upper triangular matrices. As already remarked in §2, we may take $\mathfrak{n}$ to be the strictly lower triangular matrices (2). We shall use Corollary 5 with U taken to be the upper triangular matrices with 1's along the diagonal. Consider, for example, the distinguished curve

$$t \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \exp\left(t \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \bmod P. \tag{7}$$

According to Corollary 5, it admits a projective reparameterisation if and only if we can find a, b, c, u, v, w such that

$$\exp\left(-\frac{t}{t+1} \begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ v & w & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Multiplying through by $(t+1)^2$ yields

$$\begin{pmatrix} (t+1)^2 & 0 & 0 \\ -t(t+1)u & (t+1)^2 & 0 \\ -t(t+1)v + \frac{1}{2}t^2uw & -t(t+1)w & (t+1)^2 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for the left hand side. Expanding and equating coefficients of t to zero in the sub-diagonal entries gives algebraic equations for a, b, c, u, v, w whose general solutions are

$$\begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ v & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

The existence of solutions shows that the distinguished curve (7) admits projective reparameterisations. On the other hand, this same exercise for the curve

$$t \mapsto \exp\left(t \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right) \bmod P$$

gives an inconsistent set of equations for a, b, c, u, v, w. According to Theorem 1 and Corollary 5, it admits only affine reparameterisations.

Notice that the criterion (6) of Corollary 5 depends only on the L-conjugacy class of $X \in \mathfrak{n}$. Therefore, to say which distinguished curves admit projective reparameterisations it suffices to say whether (6) is satisfied for $X \in \mathfrak{n}$ normalised under the Adjoint action of L. We obtain the following table of normal forms.

Normal form	Reparameterisation
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	projective
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ x & 1 & 0 \end{pmatrix}$	affine if $x \neq 0$

We have to be careful, however, with the decision which of the above normal forms give rise to different distinguished curves. In our case, the lines four through six in the table are in the same orbit of the Adjoint action of the entire P and so the distinguished curves indicated by these lines coincide. Indeed, a simple check reveals

$$(\exp Z)^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\exp Z = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

while the other case is symmetric. We should also like to remark, that the latter observation yields a sufficient condition for coincidences of classes of distinguished curves. There are also examples of such a coincidence where the corresponding L-orbits are not in the same orbit of P. In our case, however, the first three lines and the last two lines in the table obviously produce different curves.

This completes our analysis of distinguished curves in this real flag manifold. It is more efficient than the direct approach of [2] because Theorem 1 tells us, in advance, what sort of reparameterisation we may expect on a distinguished curve.

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