A critical point result for non-differentiable indefinite functionals

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Abstract. In this paper, two deformation lemmas concerning a family of indefinite, non necessarily continuously differentiable functionals are proved. A critical point theorem, which extends the classical result of Benci-Rabinowitz [14, Theorem 5.29] to the above-mentioned setting, is then deduced.

Keywords: locally Lipschitz continuous and indefinite functionals, deformation lemmas, critical point theorems

Classification: 35A15, 49J35, 58E05

1. Introduction

The critical point theory for smooth functionals in a Banach space is at present well established and excellent monographs devoted to various aspects of it are already available; we mention for instance [14], [11], [15], [5].

In 1981, through techniques of non-smooth analysis previously introduced by F.H. Clarke (see [7]), K.-C. Chang treated the case of functionals that are only locally Lipschitz continuous, generalizing both the famous Mountain Pass Theorem (briefly, MPT) of Ambrosetti-Rabinowitz [14, Theorem 2.2] and the Saddle Point Theorem [14, Theorem 4.6] to this more general framework; vide Theorems 3.4 and 3.3 of [6], respectively. Later on, in 1997, D. Motreanu and C. Varga made the same for the result of Du [10, Theorem 2.1], where the strict inequality occurring in the MPT has been weakened to allow also equality; see [13, Theorem 2.1]. Finally, the very recent paper [2] provides extensions of a 'dual version' [10, Theorem 2.2] of the MPT besides the Generalized MPT [14, Theorem 5.3] to the above-mentioned setting.

The main purpose of the present work is to establish a version of the classical Benci-Rabinowitz's result [14, Theorem 5.29] for functionals which, roughly speaking, can be non-smooth but only locally Lipschitz continuous at all possible critical points having critical value given by the usual minimax procedure, thus strengthening the analogy between the two theories. The approach of Ding [9]

Work performed under the auspices of G.N.A.M.P.A. of I.N.D.A.M. and partially supported by M.I.U.R. of Italy, 2003.

is adopted here. Consequently, we work in the case of linking sets and with the strict inequality weakened to permit also equality. However, several new non-trivial difficulties, mainly arising from the fact that the derivative of the involved functional now exhibits a multifunction (the so-called generalized gradient), have to be overcome. We first prove two deformation results (Lemmas 3.1 and 3.2 below) which extend Proposition A.18 in [14] and Lemma 3.2 of [9], respectively, to our framework. From a technical point of view, it represents the most difficult part of the paper and is presented in Section 3. These results are then exploited in Section 4 to establish the existence of critical points for non necessarily continuously differentiable indefinite functionals, even when 'less than or equal to' takes the place of 'less than' in the standard 'mountain pass' inequality; vide Theorem 4.1. Some applications of this result are examined in [1].

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\mathrm{int}(V)$ for the interior of V, $\overline{V}$ for the closure of V, $\mathrm{co}(V)$ for the convex hull of V. When V is nonempty, $x \in X$, and $\delta > 0$, we define $\mathrm{diam}(V) := \sup\{\|y - z\| : y, z \in V\}$ besides

$$B(x,\delta) := \{z \in X : \|z - x\| < \delta\}, \quad B_\delta := B(0,\delta), \quad \overline{B}_\delta := \overline{B(0,\delta)},$$
$$d(x,V) := \inf_{z \in V} \|x - z\|, \quad N_\delta(V) := \{z \in X : d(z,V) \le \delta\}.$$

Given $x, z \in X$, the symbol $[x,z]$ indicates the line segment joining x to z, namely

$$[x,z] := \{(1-t)x + tz : t \in [0,1]\}.$$

We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $h : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighbourhood V_x of x as well as a constant $L_x \ge 0$ such that

$$|h(u) - h(z)| \le L_x \|u - z\| \quad \forall\, u, z \in V_x.$$

If $x, z \in X$, we write $h^0(x;z)$ for the generalized directional derivative of h at the point x along the direction z, which means

$$h^0(x;z) := \limsup_{u \to x,\, t \to 0^+} \frac{h(u+tz) - h(u)}{t}\,.$$

It is known [7, Proposition 2.1.1] that h^0 turns out upper semicontinuous on $X \times X$. The symbol $\partial h(x)$ indicates the generalized gradient of the function h at x, i.e.

$$\partial h(x) := \left\{ x^* \in X^* : \langle x^*, z \rangle \le h^0(x;z)\ \forall\, z \in X \right\},$$

while, for any nonempty set $V \subseteq X$,

$$\partial h(V) := \bigcup_{x\in V} \partial h(x).$$

Since Proposition 2.1.2 of [7] ensures that $\partial h(x)$ is nonempty, convex, and weak*-compact, it makes sense to put

$$m_h(x) := \min\{\|x^*\|_{X^*} : x^* \in \partial h(x)\}.$$

The following compactness condition of Palais-Smale type at a given level $d \in \mathbb{R}$ will be adopted in this paper.

$(\mathrm{PS})_{h,d}$ *Every sequence $\{x_n\} \subseteq X$ satisfying $h(x_n) \to d$ and $m_h(x_n) \to 0$ possesses a convergent subsequence.*

Moreover, $(\mathrm{PS})_h$ will simply denote $(\mathrm{PS})_{h,d}$ for any $d \in \mathbb{R}$; see [6, Definition 2].

We say that $x \in X$ is a critical point of h when $0 \in \partial h(x)$, which clearly signifies $h^0(x;z) \geq 0$ for all $z \in X$. If $d \in \mathbb{R}$, we write

$$K_d(h) := \{x \in X : h(x) = d,\ x \ \text{ is a critical point of } \ h\}$$

in addition to

$$h_d := \{x \in X : h(x) \leq d\}, \quad h^d := \{x \in X : h(x) \geq d\}.$$

The real number d is called a critical value of h provided $K_d(h) \neq \emptyset$.

Lemma 2.1. *Let X be reflexive and let $h : X \to \mathbb{R}$ be locally Lipschitz continuous. If $\overline{\partial h(V)}$ is compact in X^* for any bounded subset V of X then*

- (h_1) *h maps bounded sets into bounded sets,*
- (h_2) *$\forall\, x \in X$, $\varepsilon > 0$ there exists a $\delta > 0$ such that $\partial h(z) \subseteq N_\varepsilon(\partial h(x))$ $\forall\, z \in B(x,\delta)$.*

Proof: Assertion (h_1) is achieved once we show that the image through h of each closed ball centered at the origin of X is bounded. Arguing by contradiction, assume there exists a $\delta_0 > 0$ such that $h(\overline{B}_{\delta_0})$ turns out unbounded from above (a similar reasoning applies when $h(\overline{B}_{\delta_0})$ is unbounded from below). Then $h(x_n) \to +\infty$ along some sequence $\{x_n\} \subseteq \overline{B}_{\delta_0}$. Since X is reflexive, passing to a subsequence if necessary, we may suppose $x_n \rightharpoonup x$ in X, with $x \in \overline{B}_{\delta_0}$. Now, pick $n \in \mathbb{N}$. The Mean Value Theorem [5, Theorem 7.1.1] furnishes two points $z_n \in [x, x_n]$, $y_n^* \in \partial h(z_n)$ such that

$$h(x_n) - h(x) = \langle y_n^*, x_n - x\rangle. \tag{1}$$

By eventually taking a subsequence we have $y_n^* \to y^*$ in X^*, as $\{y_n^*\} \subseteq \partial h(\overline{B}_{\delta_0})$ and the latter set is relatively compact. So, owing to (1), $h(x_n) \to h(x)$, which is absurd.

Let us next prove (h_2). If the assertion were false for some $x \in X$ and $\varepsilon > 0$ then one could construct two sequences $\{x_n\} \subseteq X$, $\{y_n^*\} \subseteq X^*$ with the properties $x_n \to x$, $y_n^* \in \partial h(x_n)$ for every $n \in \mathbb{N}$,

$$d(y_n^*, \partial h(x)) > \varepsilon \quad \forall n \in \mathbb{N}. \tag{2}$$

Taking account of the hypotheses, and passing to a subsequence when necessary, we would get $y_n^* \to y^*$ in X^*. By Proposition 2.1.5 in [7] this would imply $y^* \in \partial h(x)$, whereas from (2) it follows $d(y^*, \partial h(x)) \geq \varepsilon$. □

3. Two deformation results

Henceforth, X will denote a real Hilbert space with inner product $(\cdot, \cdot)$ and induced norm $\|x\| = (x, x)^{1/2}$, $x \in X$, while the function $b : X \to \mathbb{R}$ will always be assumed to satisfy the following conditions:

(b_1) *b is locally Lipschitz continuous on X;*

(b_2) *$\overline{\partial b(V)}$ is compact for any bounded subset V of X.*

Now, let $h : X \to \mathbb{R}$ be defined by

$$h(x) = \frac{1}{2}(Lx, x) + b(x) \quad \forall x \in X, \tag{3}$$

where $L : X \to X$. We shall suppose also that:

(L) *L is linear, bounded, and self-adjoint.*

(b_3) *There exists an $\varepsilon_0 > 0$ such that*

$$\operatorname{diam}(\partial b(x)) = 0, \quad x \in (h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h),$$

for some $d \in \mathbb{R}$.

Remark 3.1. Because of (b_3) the set $\partial b(x)$ reduces to a singleton whenever $x \in (h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h)$. Consequently, via Proposition 2.2.4 in [7] and the second conclusion of Lemma 2.1 we easily realize that b turns out continuously differentiable at each interior point of $(h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h)$.

Thanks to (b_1) besides (L), the function h is locally Lipschitz continuous. Hence, arguing exactly as in the proof of [6, Lemma 3.2] provides the following

Proposition 3.1. *If $d \in \mathbb{R}$, assumptions* (b_1) *and* (L) *hold true, while the function h given by* (3) *fulfils* $(\mathrm{PS})_{h,d}$ *then for every $\delta > 0$ there exists an $\overline{\varepsilon} > 0$ such that*

$$\inf\left\{m_h(x) : x \in (h_{d+\overline{\varepsilon}} \cap h^{d-\overline{\varepsilon}}) \setminus N_\delta(K_d(h))\right\} > 0. \tag{4}$$

Remark 3.2. When $K_d(h) = \emptyset$ we have $N_\delta(K_d(h)) = \emptyset$ and the number $\overline{\varepsilon}$ does not depend on δ. Thus, in this case the above inequality takes the form

$$\inf\left\{m_h(x) :\, x \in h_{d+\overline{\varepsilon}} \cap h^{d-\overline{\varepsilon}}\right\} > 0. \tag{5}$$

To simplify notations, define, for any $d \in \mathbb{R}$, $\delta > 0$, $\varepsilon > 0$,

$$S_{\delta,\varepsilon}(h,d) := (h_{d+\varepsilon} \cap h^{d-\varepsilon}) \setminus N_\delta(K_d(h)), \quad S_\varepsilon(h,d) := h_{d+\varepsilon} \cap h^{d-\varepsilon}.$$

Moreover, write

$$\mu_{\delta,\varepsilon} := \inf\left\{m_h(x) :\, x \in S_{\delta,\varepsilon}(h,d)\right\}, \tag{6}$$

$$\mu_\varepsilon := \inf\left\{m_h(x) :\, x \in S_\varepsilon(h,d)\right\} \tag{7}$$

as long as no confusion can arise. By (4) and (5) one evidently has $\mu_{\delta,\overline{\varepsilon}} > 0$ for all $\delta > 0$, $\mu_{\overline{\varepsilon}} > 0$.

Recall that a function $w : X \to X$ is said to be compact when it maps bounded sets into relatively compact sets.

Proposition 3.2. *Let hypotheses* (L) *and* (b_1)–(b_3) *be satisfied. Then to each $\delta > 0$, $\varepsilon \in]0, \varepsilon_0/3[$, $r > 0$ there corresponds a locally Lipschitz continuous compact function $w : X \to X$ such that*

$$\|w(x) - \partial b(x)\| < r \quad \forall\, x \in S_{\delta,\varepsilon}(h,d). \tag{8}$$

Proof: Fix δ, ε, and r as above. For every $x \in X$ put

$$T(x) := \alpha(x)\partial b(x) \text{ if } x \in S_{\delta/3,3\varepsilon}(h,d), \quad T(x) := 0 \text{ otherwise},$$

where $\alpha : X \to [0,1]$ denotes any continuous function fulfilling

$$\alpha(x) = 1 \quad \forall\, x \in S_{\delta,\varepsilon}(h,d), \qquad \alpha(x) = 0 \quad \forall\, x \in X \setminus S_{\delta/2,2\varepsilon}(h,d).$$

Observe that $T : X \to X$ is well defined, because

$$S_{\delta/3,3\varepsilon}(h,d) \subseteq (h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h)$$

and (b_3) holds. Moreover, if $x_0 \in S_{\delta/3,3\varepsilon}(h,d)$ then there exists an open neighbourhood V_0 of x_0 which is contained in $(h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h)$. In fact, letting $V_0 := h^{-1}(]d - \varepsilon_0, d + \varepsilon_0[) \setminus N_{\delta/3}(K_d(h))$ we clearly have

$$S_{\delta/3,3\varepsilon}(h,d) \subseteq V_0 \subseteq (h_{d+\varepsilon_0} \cap h^{d-\varepsilon_0}) \setminus K_d(h).$$

From diam $(\partial b(x)) = 0$ for all $x \in V_0$ besides the second conclusion in Lemma 2.1 it follows that ∂b is (single-valued and) continuous at x_0. As $x_0 \in S_{\delta/3,3\varepsilon}(h,d)$ was arbitrary, the function ∂b turns out continuous on $S_{\delta/3,3\varepsilon}(h,d)$. Since $\alpha(x) = 0$ in $X \setminus S_{\delta/2,2\varepsilon}(h,d)$ we see that also T is continuous on $S_{\delta/3,3\varepsilon}(h,d)$. Therefore, $T \in C^0(X,X)$. Let us next show that the function T turns out compact. Clearly, we only need to verify that $T(V)$ is relatively compact for every bounded set $V \subseteq S_{\delta/3,3\varepsilon}(h,d)$. Pick $\{y_n\} \subseteq T(V)$. Then

$$y_n = T(x_n) = \alpha(x_n)\partial b(x_n)$$

for some $x_n \in V$, $n \in \mathbb{N}$. Since $0 \le \alpha(x_n) \le 1$, passing to a subsequence if necessary, we may suppose $\alpha(x_n) \to \alpha^*$ in $\mathbb{R}$. By eventually taking a subsequence one has $\partial b(x_n) \to y^*$, because $\{x_n\}$ is bounded and (b$_2$) holds. Therefore, $T(x_n) \to \alpha^* y^*$, namely $y_n \to y$ with $y := \alpha^* y^*$. This means that $\overline{T(V)}$ is compact. Now, Proposition A.23 of [14] can be applied, and we obtain a locally Lipschitz continuous compact function $w : X \to X$ such that $\|w(x) - T(x)\| < r$ in X, from which the conclusion follows. $\square$

Define, for any $s \in \mathbb{R}_0^+$,

$$\sigma(s) := \begin{cases} 1 & \text{if } s \in [0,1], \\ \dfrac{1}{s} & \text{otherwise.} \end{cases} \tag{9}$$

Proposition 3.3. *Let the assumptions of Proposition* 3.2 *and* $(\mathrm{PS})_{h,d}$ *be satisfied. Then to each* $\delta > 0$ *there corresponds an* $\overline{\varepsilon} \in]0, \varepsilon_0/3[$ *such that if* $r \in]0, \mu_{\delta,\overline{\varepsilon}}/2[$ *then the function*

$$v(x) := \sigma(\|Lx + w(x)\|)\,[Lx + w(x)]\,, \quad x \in X, \tag{10}$$

where w *is given by Proposition* 3.2 *for* $\varepsilon := \overline{\varepsilon}$*, is locally Lipschitz continuous. Moreover, one has*

$$\|v(x)\| \le 1 \quad \forall\, x \in X, \tag{11}$$

$$(Lx + \partial b(x), v(x)) > \min\{\mu_{\delta,\overline{\varepsilon}}(\mu_{\delta,\overline{\varepsilon}} - r), \mu_{\delta,\overline{\varepsilon}} - 2r\} \quad \forall\, x \in S_{\delta,\overline{\varepsilon}}(h,d). \tag{12}$$

Proof: Fix $\delta > 0$. Using Proposition 3.1 yields an $\overline{\varepsilon} > 0$ such that $\mu_{\delta,\overline{\varepsilon}} > 0$, with $\mu_{\delta,\overline{\varepsilon}}$ as in (6). Without loss of generality we can suppose $3\overline{\varepsilon} < \varepsilon_0$. Now, pick $r \in]0, \mu_{\delta,\overline{\varepsilon}}/2[$. Proposition 3.2 provides a locally Lipschitz continuous and compact function $w : X \to X$ fulfilling (8). The locally Lipschitz continuity of v

immediately follows from the properties of L, w, besides σ. Since (11) is obvious, it remains to check that v satisfies (12). If $x \in S_{\delta,\overline{\varepsilon}}(h,d)$ then by (8) we have

$$\mu_{\delta,\overline{\varepsilon}} \leq \|Lx + \partial b(x)\| \leq \|Lx + w(x)\| + \|w(x) - \partial b(x)\| < \|Lx + w(x)\| + r,$$

thus

$$\|Lx + w(x)\| > \mu_{\delta,\overline{\varepsilon}} - r. \tag{13}$$

Two situations may now occur. When $\|Lx + w(x)\| \leq 1$, exploiting (8) as well as (9) leads to

$$\begin{aligned}(Lx + \partial b(x), v(x)) &= (Lx + \partial b(x), Lx + w(x)) \\ &\geq \|Lx + \partial b(x)\|(\|Lx + \partial b(x)\| - \|w(x) - \partial b(x)\|) \geq \mu_{\delta,\overline{\varepsilon}}(\mu_{\delta,\overline{\varepsilon}} - r).\end{aligned}$$

If $\|Lx + w(x)\| > 1$ then through (9), (8), and (13) we obtain

$$\begin{aligned}(Lx + \partial b(x), v(x)) &= \frac{1}{\|Lx + w(x)\|}(Lx + \partial b(x), Lx + w(x)) \\ &\geq \|Lx + w(x)\| - \|w(x) - \partial b(x)\| > \mu_{\delta,\overline{\varepsilon}} - 2r.\end{aligned}$$

Therefore, in either case (12) holds true. □

Remark 3.3. For $K_d(h) = \emptyset$, inequality (12) becomes

$$(Lx + \partial b(x), v(x)) > \min\{\mu_{\overline{\varepsilon}}(\mu_{\overline{\varepsilon}} - r), \mu_{\overline{\varepsilon}} - 2r\} \quad \forall\, x \in S_{\overline{\varepsilon}}(h,d), \tag{14}$$

with $\mu_{\overline{\varepsilon}}$ given by (5).

Remark 3.4. Reading the preceding proof we immediately realize that the more general condition $r \in]0, \mu_{\delta,\overline{\varepsilon}}[$ can take the place of $r \in]0, \mu_{\delta,\overline{\varepsilon}}/2[$. However, this latter guarantees also that the right-hand side of (12) is positive, which will be useful later in the paper.

The next deformation lemma extends Proposition A.18 of [14] to the framework of the present paper; see also [6, Theorem 3.1]. Write

$$\hat{\mu}_r := \min\{\mu_{\overline{\varepsilon}}(\mu_{\overline{\varepsilon}} - r), \mu_{\overline{\varepsilon}} - 2r\}, \quad r \in]0, \mu_{\overline{\varepsilon}}/2[\,. \tag{15}$$

Lemma 3.1. *Let conditions* (b$_1$)–(b$_3$), (3), (L), *and* (PS)$_{h,d}$ *be satisfied. If* $K_d(h) = \emptyset$ *while* $r \in]0, \mu_{\overline{\varepsilon}}/2[$ *then for every* $\varepsilon \in]0, \overline{\varepsilon}[$ *there exists* $\eta \in C^0([0,1] \times X, X)$ *such that*

- (i$_1$) $\|\eta(t,x) - x\| \leq t$ *in* $[0,1] \times X$,
- (i$_2$) $\eta(t,x) = x \ \forall\, t \in [0,1]$ *provided* $x \notin S_{\overline{\varepsilon}}(h,d)$,
- (i$_3$) $\forall x \in X$ *the function* $t \mapsto h(\eta(t,x))$ *is monotonically decreasing on* $[0,1]$,
- (i$_4$) $h(x) - h(\eta(1,x)) > \hat{\mu}_r$ *whenever* $\eta(t,x) \in S_\varepsilon(h,d)$ *for almost all* $t \in [0,1]$,
- (i$_5$) $\eta(t,x) = \exp(\theta(t,x)L)x + K(t,x)$, $(t,x) \in [0,1] \times X$, *where* $\theta \in C^0([0,1] \times X)$ *is bounded and* $K : [0,1] \times X \to X$ *is compact.*

PROOF: Fix $r \in]0, \mu_{\overline{\varepsilon}}/2[$. Through Proposition 3.3 we obtain a locally Lipschitz continuous function $v : X \to X$ fulfilling (14) besides (11). Next, pick $\varepsilon \in]0, \overline{\varepsilon}[$ and define $V(x) := \alpha(x)v(x)$, $x \in X$, where $\alpha : X \to [0,1]$ denotes a locally Lipschitz continuous function such that

$$\alpha(x) = 1 \quad \forall\, x \in S_{\varepsilon}(h,d), \qquad \alpha(x) = 0 \quad \forall\, x \in X \setminus S_{\overline{\varepsilon}}(h,d). \tag{16}$$

The basic existence-uniqueness theorem for ordinary differential equations in Banach spaces ensures that the Cauchy problem

$$\frac{d\eta}{dt} = -V(\eta), \quad \eta(0,x) = x, \tag{17}$$

with $x \in X$, has a unique solution $\eta(\cdot, x) : \mathbb{R} \to X$, while the continuous dependence of solutions to (17) on the initial datum x forces $\eta \in C^0([0,1] \times X, X)$. Now, integrating (17) and using (11) yields immediately (i_1).

If $x \in X \setminus S_{\overline{\varepsilon}}(h,d)$ then, by (16), $V \equiv 0$ on some neighbourhood of x. Consequently, $\eta(t,x) = x$ for all $t \in [0,1]$, which shows (i_2).

Let us next verify (i_3). To this end, fix $x \in X$ and write $\varphi(t) := h(\eta(t,x))$, $t \in [0,1]$. One evidently has

$$h(\eta(t'',x)) - h(\eta(t',x)) = \int_{t'}^{t''} \varphi'(s)\, ds \quad \forall\, t', t'' \in [0,1] \tag{18}$$

because $\varphi : [0,1] \to \mathbb{R}$ is locally Lipschitz continuous. The conclusion is thus achieved once we prove that $\varphi'(t) \le 0$ almost everywhere in $[0,1]$. Thanks to [6, Proposition 9] as well as (17) it results

$$\varphi'(t) \le -\min\{(x^*, V(\eta(t,x))) : x^* \in \partial h(\eta(t,x))\} \text{ at almost all } t \in [0,1]. \tag{19}$$

When $\eta(t,x) \in S_{\varepsilon}(h,d)$, taking account of (16) and (14) we get

$$(x^*, V(\eta(t,x))) = (L\eta(t,x) + \partial b(\eta(t,x)), v(\eta(t,x))) > \hat{\mu}_r.$$

By (19) it implies

$$\varphi'(t) < -\hat{\mu}_r < 0. \tag{20}$$

If $\eta(t,x) \in X \setminus S_{\varepsilon}(h,d)$ then, owing to (14) besides (16),

$$(x^*, V(\eta(t,x))) = \alpha(\eta(t,x))(x^*, v(\eta(t,x))) \ge 0, \quad x^* \in \partial h(\eta(t,x)).$$

In view of (19) this forces $\varphi'(t) \le 0$. Hence $\varphi'(t) \le 0$ for almost every $t \in [0,1]$, and (i_3) follows.

Finally, since (i_4) is an easy consequence of (18) and (20), it remains to verify (i_5). Through (10) we can write $V(x) = \omega(x)\,[Lx + w(x)]$, with

$$\omega(x) := \alpha(x)\sigma(\|Lx + w(x)\|) \quad \forall\, x \in X.$$

Problem (17) thus becomes

$$\frac{d\eta}{dt} + \omega(\eta)L\eta = -\omega(\eta)w(\eta), \quad \eta(0,x) = x.$$

Arguing exactly as in the proof of [14, Proposition A.18] yields

$$\eta(t,x) = \exp(\theta(t,x)L)x + K(t,x), \quad (t,x) \in [0,1] \times X,$$

where $\theta(t,x) := -\int_0^t \omega(\eta(s,x))\,ds$ and

$$K(t,x) := -\int_0^t [\exp\left(\theta(s,x) - \theta(t,x)\right) L]\,\omega(\eta(s,x))w(\eta(s,x))\,ds.$$

Obviously, the function $\theta : [0,1] \times X \to \mathbb{R}$ turns out continuous besides bounded. Since (i_1) holds, w is compact, while $0 \le \omega(x) \le 1$ in X, the same technique exploited at p. 86 of [14] guarantees here that the function $K : [0,1] \times X \to X$ is compact. □

We now come to the deformation result below, which extends Lemma 2.1 of [9] to our framework. Set

$$\hat{\mu}_{\delta,r} := \min\{\mu_{\delta,\bar{\varepsilon}}(\mu_{\delta,\bar{\varepsilon}} - r), \mu_{\delta,\bar{\varepsilon}} - 2r\}, \quad r \in]0, \mu_{\delta,\bar{\varepsilon}}/2[\,. \tag{21}$$

Moreover, write id for the identity map of X.

Lemma 3.2. *Let* (b_1)–(b_3), (3), (L), *and* $(\mathrm{PS})_{h,d}$ *be satisfied. If* A, B *are two nonempty closed subsets of* X *such that*

$$A \cap B = \emptyset, \quad A \subseteq h^d, \quad B \subseteq h_d, \quad K_d(h) \cap B = \emptyset \tag{22}$$

then there exist $\eta \in C^0(\mathbb{R} \times X, X)$ *in addition to* $\varepsilon > 0$ *with the following properties:*

(j_1) $\eta(t,\cdot)$, $t \in \mathbb{R}$, *is a homeomorphism and* $\eta(0,\cdot) = \mathrm{id}$;
(j_2) $\eta(t,\cdot)|_A = \mathrm{id}|_A \;\forall\, t \in \mathbb{R}$;
(j_3) $\forall x \in B$ *the function* $t \mapsto h(\eta(t,x))$ *turns out monotonically decreasing on* $[0,1]$;
(j_4) $\eta(1,B) \subseteq h_{d-\varepsilon}$;
(j_5) *assertion* (i_5) *of Lemma* 3.1 *holds.*

Proof: Thanks to condition $(\mathrm{PS})_{h,d}$ the set $K_d(h)$ is compact. Thus, using the assumption $K_d(h) \cap B = \emptyset$ we can find a $\delta > 0$ such that $N_\delta(K_d(h)) \cap N_\delta(B) = \emptyset$. Now, define

$$C := X \setminus (S_{\overline{\varepsilon}/2}(h,d) \cap N_{\delta/2}(B)), \quad D := S_{\overline{\varepsilon}/3}(h,d) \cap N_{\delta/3}(B),$$

where $\overline{\varepsilon} > 0$ is given by Proposition 3.3, and pick any locally Lipschitz continuous function $\alpha : X \to [0,1]$ fulfilling

$$\alpha(x) = 0 \quad \forall\, x \in C, \qquad \alpha(x) = 1 \quad \forall\, x \in D. \tag{23}$$

If $r \in]0, \mu_{\delta,\overline{\varepsilon}}/2[$ then Proposition 3.3 provides a locally Lipschitz continuous function $v : X \to X$ that complies with (11)–(12). Finally, write, whenever $x \in X$,

$$V(x) := \begin{cases} \dfrac{\delta}{3}\alpha(x)v(x) & \text{if } x \in S_{\overline{\varepsilon}}(h,d) \cap N_\delta(B), \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Clearly, V is locally Lipschitz continuous also. Moreover, $\|V(x)\| \le \delta/3$ in X. Through the basic existence-uniqueness theorem for ordinary differential equations in Banach spaces we therefore obtain a function $\zeta \in C^0(\mathbb{R} \times X, X)$ satisfying

$$\frac{d\zeta(t,x)}{dt} = -V(\zeta(t,x)), \quad \zeta(0,x) = x \quad \forall\,(t,x) \in \mathbb{R} \times X. \tag{25}$$

Let $B_1 := \zeta([0,1] \times B)$. If $x \in B$ then

$$\|\zeta(t,x) - x\| = \Big\| \int_0^t \frac{d\zeta(s,x)}{ds}\, ds \Big\| = \Big\| \int_0^t V(\zeta(s,x))\, ds \Big\| \le \frac{\delta}{3}, \quad t \in [0,1],$$

namely

$$B_1 \subseteq N_{\delta/3}(B). \tag{26}$$

We claim that the set B_1 is closed. To verify this, pick a sequence $\{y_n\} \subseteq B_1$ converging to some $y \in X$. Since $y_n = \zeta(t_n, x_n)$ with $(t_n, x_n) \in [0,1] \times B$, by eventually taking a subsequence we can suppose $t_n \to t$ in $[0,1]$. Define $z_n = \zeta(t, x_n)$, $n \in \mathbb{N}$, and observe that, on account of (25),

$$\|y_n - z_n\| = \Big\| \int_t^{t_n} \frac{d\zeta(s,x_n)}{ds}\, ds \Big\| \le \frac{\delta}{3}|t_n - t| \quad \forall\, n \in \mathbb{N}.$$

Hence, $z_n \to y$. Through the properties of ζ we thus achieve $x_n = \zeta(-t, z_n)$, $n \in \mathbb{N}$, $\zeta(-t, z_n) \to \zeta(-t, y)$. Setting $x = \zeta(-t, y)$ one has $x_n \to x$, the point x belongs to B because B is closed, while

$$y = \zeta(t,x) \in \zeta([0,1] \times B) = B_1,$$

which represents the desired conclusion. Our next goal is to show that

$$\forall\, x \in B \text{ the function } \ t \mapsto h(\zeta(t,x)) \ \text{ turns out decreasing on } \ [0,1]. \tag{27}$$

Fix $x \in B$ and write $\varphi(t) := h(\zeta(t,x))$, $t \in [0,1]$. Arguing as in the proof of Lemma 3.1 we only need the inequality $\varphi'(t) \le 0$ for almost every $t \in [0,1]$. Thanks to [6, Proposition 9] besides (25) it results

$$\varphi'(t) \le -\min\{(x^*, V(\zeta(t,x))) :\, x^* \in \partial h(\zeta(t,x))\} \ \text{ for almost all } \ t \in [0,1]. \tag{28}$$

If $\zeta(t,x) \in S_{\overline{\varepsilon}}(h,d) \cap N_\delta(B)$ then, bearing in mind (24), the choice of δ, and (12), we get

$$(L\zeta(t,x) + \partial b(\zeta(t,x)), V(\zeta(t,x))) \ge \frac{\delta}{3}\alpha(\zeta(t,x))\hat{\mu}_{\delta,r} \ge 0, \tag{29}$$

with $\hat{\mu}_{\delta,r}$ given by (21). Due to (28) this implies $\varphi'(t) \le 0$. A simple reasoning yields $\varphi'(t) = 0$ whenever $\eta(t,x) \in X \setminus (S_{\overline{\varepsilon}}(h,d) \cap N_\delta(B))$. Hence, in either case, $\varphi'(t) \le 0$, and the assertion follows. We now proceed to verify that

$$A \cap B_1 = \emptyset. \tag{30}$$

If (30) were false one could find $(t_0, x_0) \in]0,1] \times B$ fulfilling $\zeta(t_0, x_0) \in A$. Because of assumption (22) and (27) it forces

$$h(\zeta(t,x_0)) = d \quad \forall\, t \in [0, t_0]. \tag{31}$$

Using (26) we thus obtain $\zeta(t,x_0) \in D$, $t \in [0,t_0]$, which leads to

$$h(\zeta(t_0,x_0)) = h(\zeta(0,x_0)) + \int_0^{t_0} \varphi'(s)\,ds \le d - \frac{\delta}{3}\hat{\mu}_{\delta,r} t_0 < d \tag{32}$$

by means of (23), (29), and (28). However, (32) contradicts (31) written for $t = t_0$.

Note that from (30) it follows $d(x,A) + d(x,B_1) > 0$ at each point $x \in X$. Let $A_1 = \{x \in X :\, \alpha_1(x) \le 1/2\}$, where

$$\alpha_1(x) := \frac{d(x,A)}{d(x,A) + d(x,B_1)}\,, \quad x \in X.$$

Since the function α_1 is evidently continuous, A_1 turns out closed. Moreover, one has $A \subseteq \operatorname{int}(A_1)$ as well as $A_1 \cap B_1 = \emptyset$. Pick a locally Lipschitz continuous function $\hat{\alpha} : X \to [0,1]$ satisfying

$$\hat{\alpha}(x) = 0 \quad \forall\, x \in A_1, \qquad \hat{\alpha}(x) = 1 \quad \forall\, x \in B_1,$$

and define

$$\hat{V}(x) := \hat{\alpha}(x)V(x), \quad x \in X, \tag{33}$$

with V given by (24). It is clear that the function $\hat{V} : X \to X$ is locally Lipschitz continuous, $\|\hat{V}(x)\| \leq \delta/3$ in X, $\hat{V}|_{A_1} \equiv 0$, while $\hat{V}|_{B_1} = V|_{B_1}$. If $\eta : \mathbb{R} \times X \to X$ denotes the solution to the Cauchy problem

$$\frac{d\eta(t,x)}{dt} = -\hat{V}(\eta(t,x)), \quad \eta(0,x) = x, \tag{34}$$

where $x \in X$, then classical results concerning ordinary differential equations in Banach spaces ensure that $\eta(t,\cdot) : X \to X$ is a homeomorphism, thus showing assertion (j_1).

To verify (j_2) we first point out that for every $x \in A$ it results $x \in \mathrm{int}(A_1)$ and so $\hat{V} \equiv 0$ on some neighbourhood of x. Because of (34) this yields $\eta(t,x) = x$, $t \in \mathbb{R}$, which represents the desired conclusion.

As $\hat{V}|_{B_1} = V|_{B_1}$ we then have

$$-\hat{V}(\zeta(t,x)) = -V(\zeta(t,x)) = \frac{d\zeta(t,x)}{dt} \quad \forall\,(t,x) \in [0,1] \times B.$$

By the uniqueness of the solutions to (34) it follows

$$\zeta(t,x) = \eta(t,x) \quad \text{in } [0,1] \times B. \tag{35}$$

Hence, assertion (j_3) is an immediate consequence of (27).

Let us next prove (j_4). Put

$$\varepsilon := \min\left\{\frac{\overline{\varepsilon}}{3}, \frac{\delta}{3}\hat{\mu}_{\delta,r}\right\} \tag{36}$$

and suppose, contrary to our claim, that there exists an $x_0 \in B$ fulfilling

$$h(\eta(1,x_0)) > d - \varepsilon. \tag{37}$$

Since $x_0 \in B$, (35) with $x = x_0$, (27), (22), besides (37) lead to

$$d - \varepsilon < h(\eta(t,x_0)) \leq d \quad \forall\, t \in [0,1].$$

Gathering (34) and (33) together provides

$$\|\eta(t,x_0) - x_0\| = \Big\| \int_0^t \frac{d\eta(s,x_0)}{ds}\, ds \Big\| = \Big\| \int_0^t \hat{V}(\eta(s,x_0))\, ds \Big\| \leq \frac{\delta}{3}, \quad t \in [0,1].$$

Therefore, $\eta(t,x_0)\in D$ because $\varepsilon\le\overline{\varepsilon}/3$. Thanks to (35), (28), (24), (23), the choice of δ, and (12) we get

$$\frac{dh(\eta(t,x_0))}{dt}\le-\frac{\delta}{3}\hat{\mu}_{\delta,r}\ \text{ almost everywhere in }\ [0,1].$$

In view of (36) this forces

$$h(\eta(1,x_0))-h(x_0)\le-\frac{\delta}{3}\hat{\mu}_{\delta,r}\le-\varepsilon.$$

As $x_0\in B$ we actually have

$$h(\eta(1,x_0))\le h(x_0)-\varepsilon\le d-\varepsilon,$$

which contradicts (37). The proof of assertion (j_4) is thus complete.

Finally, if one writes for any $x\in X$,

$$\omega(x):=\begin{cases}\dfrac{\delta}{3}\hat{\alpha}(x)\alpha(x)\sigma(\|Lx+w(x)\|) & \text{when } \ x\in S_{\overline{\varepsilon}}(h,d)\cap N_\delta(B),\\ 0 & \text{otherwise,}\end{cases}$$

then Problem (34) takes the form

$$\frac{d\eta}{dt}+\omega(\eta)L\eta=-\omega(\eta)w(\eta),\quad \eta(0,x)=x.$$

Hence, the same arguments exploited to achieve (i_5) of Lemma 3.1 ensure here that (j_5) is true. □

Remark 3.5. Examining this proof we realize that the conclusion of Lemma 3.2 could be more precisely stated as follows:

There exists a $\delta>0$ such that to each $r\in]0,\mu_{\delta,\overline{\varepsilon}}/2[$ there corresponds a function $\eta\in C^0(\mathbb{R}\times X,X)$ enjoying properties (j_1)–(j_5), *with ε as in* (36).

The section concludes with two elementary examples, showing that very natural non-smooth functionals h can be treated through the above results. In both cases one has $X:=\mathbb{R}$, $L\equiv 0$, besides $d:=0$.

Example 3.1. Define $b(x):=|x|$ for all $x\in X$. Then (b_1) is obviously satisfied, while (b_2), (b_3), and $(\mathrm{PS})_{h,d}$ immediately come out from the expression of ∂b, namely (vide [7, p. 28])

$$\partial b(x)=\{-1\}\quad\forall\, x<0,\qquad \partial b(0)=[-1,1],\qquad \partial b(x)=\{1\}\quad\forall\, x>0.$$

Example 3.2. Let $b:X\to\mathbb{R}$ given by, for every $x\in X$,

$$b(x):=x^2\sin\frac{1}{x}\quad\text{if } x\neq 0,\qquad b(0):=0.$$

The function b is differentiable, but not continuously, on X. However, it fulfils (b_1), because b' turns out bounded. Since

$$\partial b(x)=\left\{2x\sin\frac{1}{x}-\cos\frac{1}{x}\right\}\quad\text{if } x\neq 0,\qquad \partial b(0)=[-1,1],$$

an easy argument guarantees that (b_2), (b_3), and $(\mathrm{PS})_{h,d}$ hold true too.

4. Existence of critical points

In this section we establish a version of Benci-Rabinowitz's result (see [4, Theorem 1.4] as well as [14, Theorem 5.29]) where the involved functional is not necessarily continuously differentiable on the whole space. We further weaken the usual strict inequality to allow also equality, which is already known concerning the smooth case [9, Theorem 2.1]. Besides those of [4], [14], [9], significant results on the subject were obtained in [3], [12], [8].

The next definition of linking will be adopted here; vide for instance [14, p. 31]. Let $X = X_1 \oplus X_2$ with $X_2 = X_1^{\perp}$, the orthogonal complement of the subspace X_1, and let $P_i : X \to X_i$ be the projector of X onto X_i, $i = 1, 2$. Put

$$\begin{aligned}\mathcal{S} := \{\Phi \in C^0([0,1] \times X, X) :\ & \Phi(0,\cdot) = \mathrm{id},\\ & P_2\Phi(t,x) = P_2x - K(t,x), \quad \text{where} \ \ K : [0,1] \times X \to X_2 \ \ \text{is compact}\}.\end{aligned}$$

If $S, Q \subseteq X$ are nonempty, S is closed, $Q \subseteq \tilde{X}$, a given subspace of X, while ∂Q denotes the boundary of Q in $\tilde{X}$, then we say that S and ∂Q link when for every $\Phi \in \mathcal{S}$ fulfilling $\Phi(t, \partial Q) \cap S = \emptyset$, $t \in [0, 1]$, one has $\Phi(t, Q) \cap S \neq \emptyset$, $t \in [0, 1]$. As regards meaningful examples of linking sets we cite [14, Examples 5.22 and 5.26].

The symbol Γ indicates the family of $\gamma \in C^0([0,1] \times X, X)$ with the properties

(Γ_1) $\gamma(0, x) = x \ \forall\, x \in X$,

(Γ_2) $\gamma(t, x) = x$, $(t, x) \in [0, 1] \times \partial Q$, *and*

(Γ_3) $\gamma(t,x) = \exp(\theta(t,x)L)\, x + K(t,x) \ \forall\, (t,x) \in [0,1] \times X$, *where* $\theta \in C^0([0,1] \times X)$ *while* $K : [0,1] \times X \to X$ *is compact.*

The following assumption will be taken up in addition to those listed before.

(L′) $Lx = L_1P_1x + L_2P_2x$, $x \in X$, *where* $L_i : X_i \to X_i$ *is linear, bounded, and self-adjoint,* $i = 1, 2$.

Finally, write

$$f(x) := \frac{1}{2}(Lx, x) + b(x) \quad \forall\, x \in X.$$

Theorem 4.1. *Let* (b$_1$)–(b$_2$), (L′), *and* $(\mathrm{PS})_f$ *be satisfied. Suppose* S *links with* ∂Q, $S \subseteq X_1$, Q *is bounded, while* $\partial Q \cap S = \emptyset$. *If, moreover,*

(f$_1$) $f|_{\partial Q} \leq a \leq f|_S$ *for some* $a \in \mathbb{R}$,

(f$_2$) *setting*

$$c := \inf_{\gamma \in \Gamma} \sup_{x \in Q} f(\gamma(1, x))$$

there exists $\varepsilon_0 > 0$ *such that* $\operatorname{diam}(\partial b(x)) = 0$ *for all* $x \in (f_{c+\varepsilon_0} \cap f^{c-\varepsilon_0}) \setminus K_c(f)$,

then one has

(k$_1$) $c \geq a$,

(k$_2$) $K_c(f) \setminus \partial Q \neq \emptyset$, *and*

(k$_3$) $K_c(f) \cap S \neq \emptyset$ *provided* $c = a$.

Proof: We first note that $c < +\infty$. Indeed, the function $\gamma(t,x) := x$ on $[0,1]\times X$ belongs to Γ while, taking into account (h_1) of Lemma 2.1, the set $f(Q)$ turns out bounded because so is Q. Let us show assertion (k_1). Thanks to the assumptions, Proposition 5.32 in [14] can be applied. Consequently, for every $\gamma \in \Gamma$ there is a point $x_\gamma \in Q$ fulfilling $\gamma(1,x_\gamma) \in S$. By (f_1) it implies $\sup_{x\in Q} f(\gamma(1,x)) \geq a$. As γ was arbitrary, we actually have $c = \inf_{\gamma\in\Gamma}\sup_{x\in Q} f(\gamma(1,x)) \geq a$, which represents the desired conclusion. Two situations may now occur, namely $c > a$ or $c = a$.

If $c > a$ then (f_1) forces $K_c(f)\setminus\partial Q = K_c(f)$. Suppose, contrary to ($\mathrm{k}_2$), that $K_c(f) = \emptyset$. Proposition 3.1 provides an $\overline{\varepsilon} > 0$ satisfying $\mu_{\overline{\varepsilon}} > 0$, where $\mu_{\overline{\varepsilon}}$ is given by (5) for $h := f$ and $d := c$. Without loss of generality, we may assume

$$\overline{\varepsilon} < \min\left\{\frac{c-a}{2}, \frac{\varepsilon_0}{3}\right\}. \tag{38}$$

Therefore, both h and d satisfy all the hypotheses of Lemma 3.1. Pick $r \in]0,\mu_{\overline{\varepsilon}}/2[$ as well as $\varepsilon \in]0,\min\{\overline{\varepsilon},\hat{\mu}_r/2\}[$, with $\hat{\mu}_r$ as in (15). Through the above-mentioned result we obtain a function $\eta \in C^0([0,1]\times X, X)$ enjoying properties (i_1)–(i_5). One has

$$\eta(1, f_{c+\varepsilon}) \subseteq f_{c-\varepsilon}. \tag{39}$$

In fact, if (39) were false then $\eta(1,x_0) \notin f_{c-\varepsilon}$ for some $x_0 \in f_{c+\varepsilon}$. On account of ($\mathrm{i}_3$) and ($\mathrm{i}_1$) this would imply

$$c-\varepsilon < f(\eta(1,x_0)) \leq f(\eta(t,x_0)) \leq f(\eta(0,x_0)) = f(x_0) \leq c+\varepsilon \quad \forall\, t\in[0,1],$$

i.e. $\eta(t,x_0) \in S_\varepsilon(f,c)$, $t\in[0,1]$. Owing to the choice of ε besides (i_4) we would arrive at

$$c-\varepsilon < f(\eta(1,x_0)) < f(x_0) - \hat{\mu}_r \leq c+\varepsilon-\hat{\mu}_r < c-\varepsilon,$$

which is clearly absurd. Hence, (39) holds. Let us next verify that $\eta \in \Gamma$. Conditions (Γ_1) and (Γ_3) immediately follow from (i_1) and (i_5), respectively. So, it remains to check (Γ_2). If $x \in \partial Q$ then $x \notin S_{\overline{\varepsilon}}(f,c)$ because, by (38) in addition to (f_1),

$$f(x) \leq a = c-(c-a) < c - \frac{c-a}{2} < c-\overline{\varepsilon}.$$

Using (i_2) yields $\eta(t,x) = x$ for all $t\in[0,1]$. As x was arbitrary, (Γ_2) is true. Choose any $\gamma_\varepsilon \in \Gamma$ such that

$$\sup_{x\in Q} f(\gamma_\varepsilon(1,x)) < c+\varepsilon. \tag{40}$$

The same arguments adopted in [14, p. 33] ensure here that $(t,x)\mapsto\eta(t,\gamma_\varepsilon(t,x))$, $(t,x)\in[0,1]\times X$, belongs to Γ. Therefore, $c \leq \sup_{x\in Q} f(\eta(1,\gamma_\varepsilon(1,x)))$. On the other hand, gathering (40) and (39) together provides $\eta(1,\gamma_\varepsilon(1,Q)) \subseteq f_{c-\varepsilon}$, which is impossible.

We now come to the case $c = a$. Since $\partial Q \cap S = \emptyset$, the conclusion will be achieved once one verifies (k_3). Suppose on the contrary $K_c(f) \cap S = \emptyset$ and define $h := -f$, $d := -c$, $A := \partial Q$, $B := S$. As the hypotheses of Lemma 3.2 are evidently fulfilled, we get a function $\eta \in C^0(\mathbb{R} \times X, X)$ besides a number $\varepsilon > 0$ with properties (j_1)–(j_5). Let $\gamma_\varepsilon \in \Gamma$ satisfy (40) and let

$$\overline{\gamma}_\varepsilon(t,x) := \eta(t,\cdot)^{-1}(\gamma_\varepsilon(t,x)) \quad \forall\,(t,x) \in [0,1] \times X.$$

One clearly has $\overline{\gamma}_\varepsilon \in \Gamma$. Indeed, condition ($\Gamma_1$) immediately follows from (j_1) while an elementary argument based on (j_2) and the fact that $\eta(t,\cdot)^{-1}(x) = \eta(-t,x)$, $(t,x) \in \mathbb{R} \times X$, yields ($\Gamma_2$). Finally, ($\Gamma_3$) is a simple consequence of (j_5), because $\gamma_\varepsilon \in \Gamma$ while $\overline{\gamma}_\varepsilon(t,x) = \eta(-t,\gamma_\varepsilon(t,x))$ in $[0,1] \times X$; see [14, p. 33] for more details. At this point, Proposition 5.32 of [14] provides an $x_\varepsilon \in Q$ such that $\overline{\gamma}_\varepsilon(1,x_\varepsilon) \in S$. So, owing to (40) besides (j_4) of Lemma 3.2, we get

$$c + \varepsilon \le f(\eta(1,\overline{\gamma}_\varepsilon(1,x_\varepsilon))) = f(\gamma_\varepsilon(1,x_\varepsilon)) < c + \varepsilon,$$

which is absurd. This completes the proof. □

When $b \in C^1(X)$, Theorem 4.1 takes the following form, which coincides with Theorem 2.1 of [9] and hence includes Benci-Rabinowitz's result as a special case.

Theorem 4.2. *Let $b \in C^1(X)$ and let* (b_2), (L′), $(\mathrm{PS})_f$ *be satisfied. Suppose S links with ∂Q, $S \subseteq X_1$, Q is bounded, while $\partial Q \cap S = \emptyset$. If, moreover,* ($f_1$) *of* Theorem 4.1 *holds then f possesses a critical value $c \ge a$.*

Proof: Set, as before, $c := \inf_{\gamma\in\Gamma} \sup_{x\in Q} f(\gamma(1,x))$. The same reasonings made to establish Theorem 4.1 guarantee here that $c \ge a$. Now, the conclusion is a direct consequence of that result because (f_2) holds true. □

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(*Received* February 10, 2004)