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On the existence of true uniform ultrafilters

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Abstract. We shall show that there is an ultrafilter on singular κ with countable cofinality, which cannot be reached from the set of all subuniform ultrafilters by iterating the closure of sets of size $< \kappa$.

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The notation used in the present note is a standard one and follows e.g. [CN].

Definition. Let κ be an infinite cardinal; consider κ equipped with a discrete topology. In the space $\beta(\kappa)$, define subspaces Z_α by transfinite recursion as follows. $Z_0 = \kappa$; for a limit ordinal α, let $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$; and $Z_{\alpha+1} = \bigcup\{\mathrm{cl}_{\beta\kappa}(T) : T \in [Z_\alpha]^{<\kappa}\}$. Call an ultrafilter q on κ to be *true uniform*, if $q \in \beta\kappa \setminus \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$.

It is self-evident that for regular κ, an ultrafilter on κ is true uniform if and only if it is uniform. Also, one can easily produce a uniform ultrafilter on a singular κ, which is not true uniform. During the 2003 Summer Conference on Topology and its Applications, A. Dow asked a question, attributed to Wistar W. Comfort, if there are true uniform ultrafilters also on singular cardinals. The aim of this short note is to answer Comfort's question in the affirmative for the case of a singular cardinal with countable cofinality.

Fix a cardinal $\kappa > \mathrm{cf}(\kappa) = \omega$ and choose a strictly increasing sequence of regular cardinals $\langle \kappa_n : \alpha \in \omega \rangle$ converging to κ. Decompose κ as $\kappa = \bigcup_{n<\omega} R_n$, where $R_0 = \kappa_0$, $R_{n+1} = \kappa_{n+1} \setminus \kappa_n$.

Our method of the proof resembles the standard technique for constructing remote points, see e.g., [vD].

Definition. Let M be an infinite set, $0 < n < \omega$. A family $\mathcal{A} \subseteq \mathcal{P}(M)$ is called *precisely uniformly n-linked family on M*, if for every $\mathcal{A}' \in [\mathcal{A}]^n$, $|\bigcap \mathcal{A}'| = |M|$, and for every $\mathcal{A}'' \in [\mathcal{A}]^{n+1}$, $\bigcap \mathcal{A}'' = \emptyset$.

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Observation. *Let μ be an infinite cardinal, $0 < n < \omega$. Then there is a precisely uniformly n-linked family on μ of size μ.*

Indeed, identify μ with $X = [\mu]^n \times \mu$. For each $\alpha \in \mu$, let $A_\alpha = \{K \times \mu : K \in [\mu]^n, \alpha \in K\}$. Put $\mathcal{A} = \{A_\alpha : \alpha \in \mu\}$. The family $\mathcal{A}$ on X is as required.

On every set R_n, choose an n-linked family $\mathcal{A}_n = \{A_{n,\eta} : \eta \in \kappa_n\}$ with the properties from Observation. For a mapping $g \in \prod_{n\in\omega} \kappa_n$, denote by $L(g)$ the set $\{\xi \in \kappa : \text{If } \xi \in R_n, \text{ then } \xi \leq g(n)\}$.

Given two mappings $f, g \in \prod_{n\in\omega} \kappa_n$, define $A(f,g)$ to be the set $\bigcup_{n\in\omega} A_{n,f(n)} \setminus L(g)$. Observe that the family $\{A(f,g) : f, g \in \prod_{n\in\omega} \kappa_n\}$ generates a uniform filter on κ. Given $k \in \omega$ and $f_i, g_i \in \prod_{n\in\omega} \kappa_n$ for $i < k$, choose $n \in \omega$, $n > k$. Since the family $\mathcal{A}_n$ is n-linked and $k < n$, the intersection $\bigcap_{i<k} A_{n,f_i(n)}$ has cardinality κ_n. Since each set $\{\xi \in R_n : \xi < g_i(n)\}$ is of size $< \kappa_n$, we have that $\bigcap_{i<k} A_{n,f_i(n)} \setminus \bigcup_{i<k}\{\xi \in R_n : \xi < g_i(n)\}$ is of size κ_n and so the intersection $\bigcap_{i<k} A(f_i, g_i)$ has full cardinality κ. — Let $\mathcal{F}$ denote the filter generated by $\{A(f,g) : f, g \in \prod_{n\in\omega} \kappa_n\}$.

We claim that every uniform ultrafilter on κ extending $\mathcal{F}$ is true uniform.

For $h, g \in \prod_{n\in\omega} \kappa_n$, denote by $\mathcal{A}(h,g)$ the collection $\{A(f,g') : h \leq f, g \leq g'\}$.

We shall reach this goal by assigning to each ultrafilter $q \in \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$ two functions $h_q, g_q \in \prod_{n\in\omega} \kappa_n$ such that for each $A \in \mathcal{A}(h_q, g_q)$, $A \notin q$.

Let $q \in Z_1$. Then q is a free subuniform ultrafilter on κ and two cases are possible:

Case 1: There is some $m \in \omega$ such that $R_m \in q$. The family $\mathcal{A}_m$ is precisely m-linked, so the set $J = \{\eta < \kappa_m : A_{m,\eta} \in q\}$ contains at most m elements. Put $h_q(m) = \max J + 1$, $h_q(n) = 0$ for all $n \in \omega \setminus \{m\}$ and let g_q be the constant function with value 0.

Apparently, if $A \in \mathcal{A}(h_q, g_q)$, then $A \notin q$.

Case 2: Not Case 1. Then there is some $m \in \omega$ and a set $U \subseteq \kappa$ such that $|U| < \kappa_m$ and for every $n \in \omega$, $U \setminus R_n \in q$. Once $n \geq m$, then $U \cap \kappa_n$ cannot be cofinal in κ_n, so there is a function $g_q \in \prod_{n\in\omega} \kappa_n$ such that $U \setminus \bigcup_{i<m} R_i \subseteq L(g_q)$. Put $h_q(n) = 0$ for all $n \in \omega$. Again, if $A \in \mathcal{A}(h_q, g_q)$, then $A \notin q$, because $U \setminus \bigcup_{i<m} R_i$ is disjoint from A.

Let $\alpha > 1$ be an ordinal and assume that for every $\beta < \alpha$ and every ultrafilter $q \in Z_\beta$ there is a pair of functions $h_q, g_q \in \prod_{n\in\omega} \kappa_n$ such that for every $A(f,g) \in \mathcal{A}(h_q, g_q)$, $A(f,g) \notin q$.

There is nothing to prove if α is a limit ordinal. So assume $\alpha = \beta + 1$ and let an ultrafilter q belong to $Z_\alpha \setminus Z_\beta$. Then there is a cardinal $\kappa_m < \kappa$ and a set $P \subseteq Z_\beta$, $|P| < \kappa_m$, such that $q \in \mathrm{cl}_{\beta\kappa}(P)$.

Observe that for every $n \in \omega$, $R_n \notin q$. This follows immediately from the fact that $\beta \geq 1$ and $q \notin Z_\beta$. Define $h_q(n) = 0$ for $n < m$, $h_q(n) = \sup\{h_p(n) : p \in P\}$

for $n \geq m$, $n < \omega$. Similarly, let $g_q(n) = 0$ for $n < m$, $g_q(n) = \sup\{g_p(n) : p \in P\}$ for $n \geq m$.

Pick an arbitrary $A \in \mathcal{A}(h_q, g_q)$. Then $A = A(f, g)$ for some $f \geq h_q$ and $g \geq g_q$.

Let $P_0 = \{p \in P : \bigcup_{n<m} R_m \in p\}$, $P_1 = P \setminus P_0$. Since $\bigcup_{n<m} R_m \notin q$, we obtain $q \notin \mathrm{cl}_{\beta\kappa}(P_0)$, and so $q \in \mathrm{cl}_{\beta\kappa}(P_1)$. Let $p \in P_1$ be arbitrary. If we define functions f' and g' by the rule $f'(n) = h_p(n)$ for $n < m$ and $f'(n) = f(n)$ for $n \geq m$, $g'(n) = g_p(n)$ for $n < m$ and $g'(n) = g(n)$ for $n \geq m$, then $A(f', g') \in \mathcal{A}(h_p, g_p)$ and consequently $A(f', g') \notin p$. Since $A(f, g) \subseteq A(f', g') \cup \bigcup_{n<m} R_n$, $A(f, g) \notin p$. Since this holds for all $p \in P_1$, $A(f, g) \notin q$.

Hence every ultrafilter which extends the filter $\mathcal{F}$ is true uniform and we have just proved

Theorem. *There is a true uniform ultrafilter on every singular cardinal with countable cofinality.*

The reader undoubtedly observed that countable cofinality of a cardinal in question was crucial for the proof just given. So, a part of Comfort's question remains open: Are there true uniform ultrafilters also on singular cardinals with uncountable cofinality?

References

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