Aull-paracompactness and strong star-normality of subspaces in topological spaces

Kaori Yamazaki

Abstract. We prove for a subspace Y of a T_1-space X, Y is (strictly) Aull-paracompact in X and Y is Hausdorff in X if and only if Y is strongly star-normal in X. This result provides affirmative answers to questions of A.V. Arhangel'skii–I.Ju. Gordienko [3] and of A.V. Arhangel'skii [2].

Keywords: Aull-paracompactness of Y in X, strong star-normality of Y in X

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1. Introduction and results

All spaces are assumed to be T_1-spaces. In [3], A.V. Arhangel'skii and I.Ju. Gordienko say that a subspace Y of a space X is *Aull-paracompact in* (resp. *paracompact in*) X if for every open collection $\mathcal{U}$ of X with $Y \subset \bigcup \mathcal{U}$ (resp. open cover $\mathcal{U}$ of X), there exists an open collection $\mathcal{V}$ of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y in X (see also [2]). Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. A.V. Arhangel'skii and I.Ju. Gordienko [3] also say that Y is *strictly Aull-paracompact in* X if in the above definition of Aull-paracompactness of Y in X the family $\mathcal{V}$ can be chosen to satisfy one condition more: $\mathcal{V}$ is σ-discrete in Y (see also [2]). From the definitions, it is easy to see that Aull-paracompactness of Y in X implies paracompactness of Y in X ([3], [2]), and the converse does not necessarily hold.

For a point $y \in X$ and collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, the set $\bigcup\{U : y \in U \in \mathcal{U}\}$ is denoted by $\mathrm{St}(y, \mathcal{U})$. We also say that $\mathcal{V}$ is a Δ-*refinement of* $\mathcal{U}$ *at* y if there exists $U \in \mathcal{U}$ such that $\mathrm{St}(y, \mathcal{V}) \subset U$. In [3], a subspace Y of a space X is also said to be *strongly star-normal in* (resp. *star-normal in*) X if for every open collection $\mathcal{U}$ of X with $Y \subset \bigcup \mathcal{U}$ (resp. open cover $\mathcal{U}$ of X), there exists an open collection $\mathcal{V}$ of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a Δ-refinement of $\mathcal{U}$ at each point of $\bigcup \mathcal{V}$ (resp. each point of Y) (see also [2]).

Related to these properties, A.V. Arhangel'skii–I.Ju. Gordienko [3] asked a question as follows:

Question 1 (A.V. Arhangel'skii–I.Ju.Gordienko [3, Question 9])**.** Is it true that if Y is strongly star-normal in X, then Y is paracompact in X?

The same question was asked again by A.V. Arhangel'skii in [2, Problem 12] for Tychonoff spaces X.

In this paper, we give an affirmative answer to Question 1 above (see Corollary 2.1). Lemmas 1.1 and 1.2 below provide the essential part of the proof.

Now, let us recall the terminology which will be used in lemmas. Let X_Y denote the space obtained from the topological space X, with the topology generated by $\{U : U \text{ is open in } X \text{ or } U \subset X - Y\}$. The spaces X_Y were introduced in [1] and [2] for the study of relative normality, and also used in [6] as a key tool for the study of products of relative topological properties. A space X is said to be *fully normal* if every open cover $\mathcal{U}$ of X has an open Δ-refinement $\mathcal{V}$, i.e. $\mathcal{V}$ is an open cover of X such that for every $x \in X$ there is $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}) \subset U$. A space X is *paracompact* if every open cover of X has a locally finite open refinement; note that X is not necessarily assumed to be Hausdorff.

Lemma 1.1. *For a space X and a subspace Y of X, Y is Aull-paracompact in X if and only if X_Y is paracompact. If in addition X_Y is Hausdorff, Aull-paracompactness of Y in X and strict Aull-paracompactness of Y in X coincide with each other.*

Lemma 1.2. *For a space X and a subspace Y of X, Y is strongly star-normal in X if and only if X_Y is fully normal.*

Recall from [1] and [2] that Y is *Hausdorff in X* if every two distinct points y_0, y_1 of Y are separated by disjoint open subsets of X. Clearly, Y is Hausdorff in X if and only if X_Y is Hausdorff.

As is well known, a space X is paracompact Hausdorff if and only if X is fully normal ([5]). Hence, the following theorem is immediately obtained by Lemmas 1.1 and 1.2.

Theorem 1.3. *Let X be a space and Y a subspace of X. Then, the following statements are equivalent:*

(1) *Y is strongly star-normal in X;*
(2) *Y is strictly Aull-paracompact in X and Y is Hausdorff in X;*
(3) *Y is Aull-paracompact in X and Y is Hausdorff in X;*
(4) *X_Y is paracompact Hausdorff.*

Furthermore, Theorem 1.3 gives an affirmative answer to Question 2 below posed by A.V.Arhangel'skii–I.Ju.Gordienko [3, Question 4], the same question was asked again in [2, Problem 8] for Tychonoff spaces X (see Corollary 2.2):

Question 2 (A.V. Arhangel'skii–I.Ju. Gordienko [3, Question 4])**.** Suppose that Y is properly metrizable in a space X. Is then Y strongly star-normal in X? Is then star-normal in X?

2. Proofs and applications

For a space X and a subspace A of X, $\mathrm{int}_X A$ stands for the interior of A in X.

PROOF OF LEMMA 1.1: Let X be a space and Y a subspace of X.

To prove the "if" part, assume X_Y is paracompact. Let $\mathcal{U}$ be an open collection of X with $Y \subset \bigcup \mathcal{U}$. Set $\mathcal{U}' = \mathcal{U} \cup \{\{x\} : x \in X - \bigcup \mathcal{U}\}$. Since $\mathcal{U}'$ is an open cover of X_Y, there exists a locally finite open cover $\mathcal{W}$ of X_Y such that $\mathcal{W}$ refines $\mathcal{U}'$. Set $\mathcal{V} = \{\mathrm{int}_X W : W \in \mathcal{W}, \mathrm{int}_X W \cap Y \neq \emptyset\}$. We shall show that $\mathcal{V}$ is the required partial refinement. It is easy to see that $\mathcal{V}$ is an open collection of X covers Y and that $\mathcal{V}$ is locally finite at each point of Y. To prove $\mathcal{V}$ is a partial refinement of $\mathcal{U}$, take $W \in \mathcal{W}$ satisfying that $\mathrm{int}_X W \cap Y \neq \emptyset$. Since $\mathcal{W}$ refines $\mathcal{U}'$, there is $U \in \mathcal{U}'$ such that $W \subset U$. We shall show that $U \in \mathcal{U}$; otherwise $U = \{y\}$ for some point $y \in X - \bigcup \mathcal{U}$. Then, $\emptyset \neq \mathrm{int}_X W \cap Y \subset U \cap Y = \{y\} \cap Y \subset (X - \bigcup \mathcal{U}) \cap Y = \emptyset$, a contradiction. Hence, Y is Aull-paracompact in X.

To prove the "only if" part, assume Y is Aull-paracompact in X. Let $\mathcal{U}$ be an open cover of X_Y. Since $\{\mathrm{int}_X U : U \in \mathcal{U}\}$ is an open collection of X which covers Y, there exists an open collection $\mathcal{V}$ of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\{\mathrm{int}_X U : U \in \mathcal{U}\}$ and $\mathcal{V}$ is locally finite at each point of Y in X. For every $y \in Y$, take an open subset O_y of X such that $y \in O_y$, $O_y \subset \bigcup \mathcal{V}$ and O_y intersects at most finitely many elements of $\mathcal{V}$. Put $W = \bigcup_{y \in Y} O_y$, and set $\mathcal{W} = \{V \cap W : V \in \mathcal{V}\} \cup \{\{x\} : x \in X - W\}$. Then, $\mathcal{W}$ is the required locally finite open cover of X_Y which refines $\mathcal{U}$.

To prove the additional part, recall the fact that every open cover of a paracompact Hausdorff space has a σ-discrete locally finite open refinement (see [4, Theorem 2.8]). Hence, in the case X_Y is Hausdorff, in the proof of the "if" part above, we can take $\mathcal{W}$ as a σ-discrete locally finite open refinement. This completes the proof. □

PROOF OF LEMMA 1.2: Let X be a space and Y a subspace of X.

To prove the "if" part, assume X_Y is fully normal. Let $\mathcal{U}$ be an open collection of X with $Y \subset \bigcup \mathcal{U}$. Set $\mathcal{U}' = \mathcal{U} \cup \{\{x\} : x \in X - \bigcup \mathcal{U}\}$. Since $\mathcal{U}'$ is an open cover of X_Y, there exists an open cover $\mathcal{W}$ of X_Y which is a Δ-refinement of $\mathcal{U}'$. Set $\mathcal{V} = \{\mathrm{int}_X W : W \in \mathcal{W}, \mathrm{int}_X W \cap Y \neq \emptyset\}$. Then, $\mathcal{V}$ is the required open collection of X. The proof that $\mathcal{V}$ is an open Δ-refinement of $\mathcal{U}$ at each point of $\bigcup \mathcal{V}$ is similar to that of the "if" part of Lemma 1.1. So, we left it to the reader.

To prove the "only if" part, assume Y is strongly star-normal in X. Let $\mathcal{U}$ be an open cover of X_Y. Let $\mathcal{U}' = \{\mathrm{int}_X U : U \in \mathcal{U}\}$. Since $\mathcal{U}'$ is an open collection of X which covers Y, there exists an open collection $\mathcal{V}$ of X such that $\mathcal{V}$ is a Δ-refinement of $\mathcal{U}'$ at each point of $\bigcup \mathcal{V}$. Set $\mathcal{W} = \mathcal{V} \cup \{\{x\} : x \in X - \bigcup \mathcal{V}\}$. Then, $\mathcal{W}$ is an open cover of X_Y which is a Δ-refinement of $\mathcal{U}$. This completes the proof. □

By Lemmas 1.1 and 1.2, we immediately have Theorem 1.3.

By Theorem 1.3, we have the following result, which answers Question 1 affirmatively.

Corollary 2.1. *For a space X and a subspace Y of X, if Y is strongly star-normal in X, then Y is Aull-paracompact in* (*hence, paracompact in*) *X.*

It is easy to see that the converse of Corollary 2.1 does not necessarily hold unless Y is Hausdorff in X. In fact, an easy example shows that strict Aull-paracompactness of Y in X does not imply the normality of Y in X.

Now, note that if Y is properly metrizable in X, then Y is Hausdorff in X (for example, use [3, Theorem 5]). Hence, by combining Theorem 1.3 with [3, Theorem 7] (see also [2, Theorem 11.10]) we also give an affirmative answer to Question 2 as follows.

Corollary 2.2. *Suppose that Y is properly metrizable in a space X. Then, Y is strongly star-normal in* (*hence star-normal in*) *X.*

Finally, let us apply Theorem 1.3 for the study of absolute embedding. A subspace Y of a space X is said to be *weakly P-embedded* in X if every continuous pseudo-metric on Y can be extended to a pseudo-metric on X which is continuous at each point of Y ([6]). Now, recall two facts listed below, which have been obtained in this paper and in the previous papers [6] and [7]: Fact 1 follows from Theorem 1.3 and [6, Lemma 4.6], and Fact 2 follows from [7, Theorems 3.1, 3.2 and 3.3].

Fact 1. For a Hausdorff space X and a subspace Y of X, the following statements are equivalent:
(1) Y is (strictly) Aull-paracompact in X;
(2) Y is strongly star-normal in X;
(3) Y is paracompact and Y is weakly P-embedded in X.

Fact 2. A Hausdorff space Y is weakly P-embedded in every larger Hausdorff space X if and only if either Y is compact or every real-valued continuous function on Y is constant. A regular (resp. Tychonoff) space Y is weakly P-embedded in every larger regular (resp. Tychonoff) space X if and only if Y is almost compact or Lindelöf.

Here, a space Y is called *almost compact* if for every two disjoint zero-sets of Y at least one of them is compact.

Facts 1 and 2 immediately induce the following new characterizations of absolute embeddings:

A Hausdorff space Y is Aull-paracompact in (*or equivalently, strictly Aull-paracompact in, strongly star-normal in*) *every larger Hausdorff space X if and only if Y is compact.*

A regular (resp. Tychonoff) space Y is Aull-paracompact in (or equivalently, strictly Aull-paracompact in, strongly star-normal in) every larger regular (resp. Tychonoff) space X if and only if Y is Lindelöf.

In Theorem 1.3 and Corollary 2.2 of the original version of the paper, spaces X were assumed to be Hausdorff. In the present version, without changing the original proofs, we improve these results by using the notion of the Hausdorffness of Y in X. When the original version was already submitted, we were informed that E. Grabner, G. Grabner, K. Miyazaki and J. Tartir independently proved similar results by the direct proofs proceeding as in [5], assuming that all spaces are Hausdorff.

References

[1] Arhangel'skii A.V., *Relative topological properties and relative topological spaces*, Topology Appl. **70** (1996), 87–99.

[2] Arhangel'skii A.V., *From classic topological invariants to relative topological properties*, Sci. Math. Japon. **55** (2002), 153–201.

[3] Arhangel'skii A.V., Gordienko I.Ju., *Relative symmetrizability and metrizability*, Comment. Math. Univ. Carolinae **37** (1996), 757–774.

[4] Burke D.K., *Covering properties*, in: K. Kunen, J.E. Vaughan (eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 347–422.

[5] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] Hoshina T., Yamazaki K., *Weak C-embedding and P-embedding, and product spaces*, Topology Appl. **125** (2002), 233–247.

[7] Yamazaki K., *Absolute weak C-embedding in Hausdorff spaces*, Topology Appl. **131** (2003), 273–279.

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

E-mail: kaori@math.tsukuba.ac.jp

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