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On spaces with point-countable k-systems

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Abstract. This paper deals with the behavior of M-spaces, countably bi-quasi-k-spaces and singly bi-quasi-k-spaces with point-countable k-systems. For example, we show that every M-space with a point-countable k-system is locally compact paracompact, and every separable singly bi-quasi-k-space with a point-countable k-system has a countable k-system. Also, we consider equivalent relations among spaces entried in Table 1 in Michael's paper [15] when the spaces have point-countable k-systems.

Keywords: countably-bi-quasi-k-space, point-countable k-system, local compactness, metrizability

Classification: 54D55, 54E18, 54E35

1. Introduction and primary results

In this paper, we assume that *all spaces are Hausdorff and all maps are continuous onto*. By $\mathbb{R}$ and $\mathbb{N}$, we denote the real line and the set of all natural numbers, respectively.

Let $\mathcal{A}$ be a collection of subsets of a space X. By $\overline{\mathcal{A}}$, $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ we denote the collection $\{\overline{A} \mid A \in \mathcal{A}\}$, the union $\bigcup\{A \mid A \in \mathcal{A}\}$ and the intersection $\bigcap\{A \mid A \in \mathcal{A}\}$, respectively. It is necessary to recall the following definitions.

Definition 1.1. Let $\mathcal{P}$ be a cover of a space X.

(a) X is *determined* ([10]) by $\mathcal{P}$ if $H \subset X$ is closed if and only if $H \cap P$ is closed in P for every $P \in \mathcal{P}$.

(b) $\mathcal{P}$ is called a *k-network* ([16]) for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

(c) X is called an $\aleph_0$-space ([14]) if X is regular and has a countable k-network.

(d) $\mathcal{P}$ is called *point-countable* (*point-finite*) if every $x \in X$ is in at most countably many (finitely many) $P \in \mathcal{P}$.

(e) $\mathcal{P}$ is called a *k-system* ([1]) if every element of $\mathcal{P}$ is compact and X is determined by $\mathcal{P}$.

Definition 1.2 ([15])**.** Let X be a space.

(a) A decreasing sequence $\{A_n\}$ of non-empty subsets of X is a *k-sequence* (*q-sequence*) if $C = \bigcap_{n\geq 1} A_n$ is compact (countably compact) and every open subset U with $C \subset U$ contains A_m for some m.

(b) X is a *strict q-space* if every $x \in X$ has a q-sequence of open neighbourhoods.

(c) X is a *q-space* if every $x \in X$ has a sequence $\{U_n\}$ of open neighbourhoods such that $x_n \in U_n$ $(n \geq 1)$ implies that $\{x_n\}_n$ has a cluster point in X.
(d) X is a *bi-k*-space (*bi-quasi-k*-space) if, whenever a filter base $\mathcal{F}$ clusters at x, then $\mathcal{F}$ meshes with some k-sequence (q-sequence) $\{A_n\}$ in X (i.e., $F \cap A_n \neq \emptyset$ for each $F \in \mathcal{F}$ and each $n \in \mathbb{N}$).
(e) X is a *countably bi-sequential* (= *strongly Fréchet*) space if, whenever $\{F_n\}$ is a decreasing sequence with $x \in \bigcap_{n \geq 1} \overline{F_n}$, then there exist $x_n \in F_n$ such that $\{x_n\}_n \longrightarrow x$.
(f) X is a *countably bi-quasi-k* (*singly bi-quasi-k*)-space if, whenever $\{F_n\}$ is a decreasing sequence with $x \in \bigcap_{n \geq 1} \overline{F_n}$ $(x \in \overline{F})$, then there exists a q-sequence $\{A_n\}$ such that $x \in \bigcap_{n \geq 1} \overline{F_n \cap A_n}$ $(x \in \bigcap_{n \geq 1} \overline{F \cap A_n})$.
(g) X is a Fréchet space (k'-space) if, whenever $x \in \overline{F}$, then there exists a sequence $\{x_n\} \subset F$ (a compact subset $K \subset X$) such that $\{x_n\}_n \longrightarrow x$ $(x \in \overline{F \cap K})$.

Every first countable space satisfies conditions (b)–(g) of Definition 1.2. Every Fréchet space is a k'-space and every k'-space is a singly bi-quasi-k-space.

For undefined terms, the readers are referred to [7], [15].

Under the assumption that spaces have point-countable k-systems, we study the behavior of spaces in Table 1 in Michael's paper [15, p. 93] (below, we write simply Table 1 in this paper).

In Table 1, Michael [15, p. 94] showed that for paracompact spaces, corresponding entries in columns **E** and **F** are equivalent.

In §2, we prove that every countably bi-quasi-k, regular space X determined by a point-countable cover consisting of subspaces with point-countable bases has a point-countable base. Also, we show that for spaces determined by a point-countable cover consisting of locally compact, metrizable subsets, all entries in rows 1, 2, 3 and 4 in all columns except for column **C** in Table 1 are equivalent. On the other hand, there exists a space Y with a point-countable k-system consisting of metrizable subsets such that Y belongs to all entries in row 5, but does not belong to any entry in a row 4. We show also that every singly bi-quasi-k-space which has a point-finite k-system consisting of locally separable, metrizable closed subsets is metrizable.

In §3, we define the concept of point-countable weak k-systems and prove that for spaces with point-countable weak k-systems, corresponding entries in columns **B** and **F** in Table 1 are equivalent, and therefore columns **B**, **E** and **F** become identical. We show further that for spaces with point-countable k-systems, all entries in rows 2, 3 and 4 in columns **B**, **E** and **F** are equivalent. Moreover, in a class of M-spaces, we prove that the class of spaces with point-countable k-systems and the class of spaces with point-countable weak k-systems are equivalent, and that the finite product of M-spaces with point-countable k-systems has also a point-countable k-system. For metrizabilities of spaces, we show that every Moore (or Nagata) space with a point-countable weak k-system is metrizable.

For Tanaka's question [19, p. 203] whether a separable k', regular space with a point-countable k-system has a countable k-system, Li and Lin got the following affirmative answer for Hausdorff spaces.

Theorem 1.3 ([12]). *Every separable k'-space with a point-countable k-system has a countable k-system.*

In §4, referring to the proof of the above theorem, we generalize this theorem as follows:

Every separable singly-bi-quasi-k-space with a point-countable weak k-system has a countable k-system.

Finally, we recall some elementary facts which are used later on.

The following propositions can be proved in the same manner as Lemma 6 in [20].

Proposition 1.4. *Let a space X be determined by a point-countable cover $\mathcal{P}$. Then for each q-sequence $\{A_n\}$ in X, some A_m is contained in a finite union of elements of $\mathcal{P}$. Therefore, if $C \subset X$ is countably compact, then C is contained in a finite union of elements of $\mathcal{P}$.*

Proposition 1.5 ([20, Proposition 7]). *Let X be a countably bi-quasi-k-space. If X is determined by a point-countable closed cover $\mathcal{P}$ then for each $x \in X$, $\mathcal{P}$ contains a finite subcollection $\mathcal{F}$ such that $x \in \bigcap \mathcal{F}$ and $x \in \operatorname{int}(\bigcup \mathcal{F})$.*

Definition 1.6. A space X is *hemicompact* if there exists a sequence $\{K_n\}$ consisting of compact subsets such that every compact subset is contained in some K_m.

Every hemicompact regular space is paracompact.

The following well-known result follows from Proposition 1.4.

Proposition 1.7. *For a space X, the following conditions are equivalent:*

(1) *X has a countable k-system;*
(2) *X is a hemicompact k-space.*

2. Metrizability

In this section, we study the metrizations of spaces which are determined by point-countable covers consisting of metrizable subsets.

We begin with a well-known example.

Example 2.1 ([15, Example 10.1]). There exists an $\aleph_0$, Fréchet regular, non metrizable space Y with the following properties.

(1) Y is not countably bi-quasi-k.

(2) Y is not locally compact (not even q).
(3) Y has a countable k-system $\mathcal{P}$ consisting of metrizable subsets.
(4) The above cover $\mathcal{P}$ is also a k-network for Y.
(5) Y has no point-countable base.
(6) Y has no point-finite k-system.
(7) Y is not determined by any point-finite cover consisting of metrizable subsets.

Indeed, let X be the topological sum of a sequence $\{I_n\}$ of copies of the interval I, let $A = \{a_n = 0 \in I_n \mid n \in \mathbb{N}\}$, and let $Y = X/A$, the space obtained from X by identifying A to a point. Let $f : X \longrightarrow Y$ be the quotient map, and let $a = f(A)$. Then f is closed, so Y is an $\aleph_0$ ([14]), Fréchet regular space in which every point is a G_δ set. (1) follows from [15, Theorem 9.9] or Theorem 2.5 later. Now, let Y be a q-space. Then Y is strict q by [15, p. 103], so Y is countably bi-quasi-k. This contradiction implies (2). To prove (3) and (4), we show that Y is determined by a countable k-network consisting of compact metrizable subsets. For each $n \in \mathbb{N}$, let $\{B_{n,k} \mid k \in \mathbb{N}\}$ be a base for I_n such that each $\overline{B_{n,k}}$ is compact metrizable. Then $\mathcal{B} = \{B_{n,k} \mid n, k \in \mathbb{N}\}$ is a countable base for X. First, we show that $\mathcal{P} = \{f(\overline{B_{n,k}}) \mid n, k \in \mathbb{N}\}$ is a countable k-network consisting of compact metrizable subsets. Suppose $K \subset V$, where K is compact and V is open in Y. Then we have two cases.

Case 1. Let $a \notin K$. Since $f^{-1}(K) \subset f^{-1}(V)$ and $f^{-1}(K)$ is homeomorphic to K, $f^{-1}(K) \subset \overline{B_1} \cup \cdots \cup \overline{B_m} \subset f^{-1}(V)$ for some $\{B_1, \ldots, B_m\} \subset \mathcal{B}$. Hence, $K \subset f(\overline{B_1}) \cup \cdots \cup f(\overline{B_m}) \subset V$ for $\{f(\overline{B_i}) \mid 1 \le i \le m\} \subset \mathcal{P}$.

Case 2. Let $a \in K$. Then, we have that $f^{-1}(K \setminus \{a\}) \subset I_1 \cup \cdots \cup I_n$ for some n. Indeed, if $x_{n(i)} \in f^{-1}(K \setminus \{a\}) \cap I_{n(i)}$ for $\{n(1) < n(2) < \ldots\}$, then we put $U = \bigcup_{n \ge 1} A_n$, where $A_{n(i)} = [a_{n(i)}, x_{n(i)}) \subset I_{n(i)}$ $(i \ge 1)$ and $A_n = I_n (n \notin \{n(i) \mid i \ge 1\})$. Then $f^{-1}f(U) = U$ and $W = f(U)$ is an open neighbourhood of a such that $W \cap \{f(x_{n(i)})\}_i = \emptyset$. Therefore, a is not a cluster point of $\{f(x_{n(i)})\}_i$ and hence, $\{f(x_{n(i)})\}_i$ has no cluster point in Y. This contradicts to $\{f(x_{n(i)})\}_i \subset K$. Now, $L = f^{-1}(K) \cap (I_1 \cup \cdots \cup I_n)$ is compact and $L \subset f^{-1}(V)$. Hence, $L \subset \overline{B_1} \cup \cdots \cup \overline{B_k} \subset f^{-1}(V)$ for some $\{B_i \mid 1 \le i \le k\} \subset \mathcal{B}$. Since $f(L) = K \cap f(I_1 \cup \cdots \cup I_n)$, $K \setminus \{a\} \subset f(I_1 \cup \cdots \cup I_n)$ and $a \in f(I_1 \cup \cdots \cup I_n)$, we have that $f(L) = K$. Therefore $K \subset f(\overline{B_1}) \cup \cdots \cup f(\overline{B_k}) \subset V$ for $\{f(\overline{B_i}) \mid 1 \le i \le k\} \subset \mathcal{P}$, which implies that $\mathcal{P}$ is a k-network for Y. Next, X is determined by $\overline{\mathcal{B}}$ since X is determined by $\mathcal{B}$. Hence, since f is quotient, Y is determined by $\mathcal{P}$. To see (5), let Y have a point-countable base. Then the separable space Y is metrizable, which is a contradiction. To prove (6), suppose that Y has a point-finite k-system $\mathcal{P}$. Let $\{F_n\}$ be any decreasing sequence with $y \in \overline{F_n}$ $(n \ge 1)$. Since Y is Fréchet, for each $n \in \mathbb{N}$, some sequence $\{y_{n,k}\}_k \subset F_n$ converges to y. Therefore $K_n = \{y_{n,k}\}_k \cup \{y\} \subset \bigcup \mathcal{F}_n$ for some finite $\mathcal{F}_n \subset \mathcal{P}$ by Proposition 1.4.

Hence, some subsequence $S_n \subset \{y_{n,k}\}_k$ is contained in P_n for some $P_n \in \mathcal{F}_n$ $(n \geq 1)$, which implies that $y \in \overline{S_n} \subset P_n$. Then there exists $P_0 \in \mathcal{P}$ such that $P_{n(i)} = P_0$ for some increasing subsequence $\{n(i)\}$. Hence, for the compact subset P_0, $y \in \overline{S_{n(i)}} \subset \overline{F_{n(i)} \cap P_0}$ for each $i \in \mathbb{N}$. Therefore Y is strongly k', which contradicts to (1). Finally, to show (7), let Y be determined by a point-finite cover consisting of metrizable subsets. Since Y is Fréchet, Y is countably bi-sequential (this is proved without closedness of elements of $\mathcal{P}$) from Theorem 2.9 later, which contradicts to (1).

Example 2.1 asserts that there exists an $\aleph_0$, Fréchet space X with a countable k-system consisting of metrizable subsets, but X is not metrizable. On the other hand, Theorem 2.7 later asserts that every countably bi-quasi-k-space with a point-countable k-system consisting of metrizable subsets is metrizable. Prior to this result we give conditions for a countably bi-quasi-k-space to have a point-countable base.

The following lemma is known (see [21, Lemma 1]).

Lemma 2.2. *Consider the following conditions for a cover $\mathcal{P}$ of a space X.*

(1) *X is determined by $\mathcal{P}$.*

(2) *For every infinite sequence $\{x_n\}$ converging to x, some $P \in \mathcal{P}$ contains x and x_n frequently.*

Then (1)$\Longrightarrow$(2) *and,* (2)$\Longrightarrow$(1) *if X is a sequential space.*

A space X is of *countable tightness* if whenever $A \subset X$ and $x \in \overline{A}$, then $x \in \overline{C}$ for some countable subset $C \subset A$. As is well-known, every sequential space is of countable tightness.

The following lemma is proved in the same manner as Proposition 3.2 in [10], using Proposition 1.4.

Lemma 2.3. *Let X be a countably bi-quasi-k-space of countable tightness. If X is determined by a point-countable cover $\mathcal{P}$, then every $x \in X$ is in* $\mathrm{int}(\bigcup \mathcal{F})$ *for some finite $\mathcal{F} \subset \mathcal{P}$.*

Tanaka [21, Lemma 5] gave the following condition for a countably bi-sequential regular space to have a point-countable base.

Theorem 2.4. *Let X be a countably bi-sequential regular space which is determined by a point-countable cover $\mathcal{P}$. If every element of $\mathcal{P}$ is a locally separable, metrizable subset, then X is a locally separable space with a point-countable base. Thus X is metrizable.*

Franklin [8, Example 7.1] gave a countably bi-quasi-k, sequential regular space which is not countably bi-sequential. So, for countably bi-quasi-k-spaces, we prove a similar result to Theorem 2.4.

Theorem 2.5. *Let X be a countably bi-quasi-k, regular space which is determined by a point-countable cover $\mathcal{P}$. If every element of $\mathcal{P}$ has a point-countable base, then X has a point-countable base.*

Additionally, if every element of $\mathcal{P}$ is locally separable, then X is a locally separable, metrizable space.

Proof: First, notice that X is a sequential space by [10, Lemma 1.8]. Let $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ and let $\mathcal{B}_\alpha$ be a point-countable base for $P_\alpha(\alpha \in A)$. Then $\mathcal{B} = \{B \mid B \in \mathcal{B}_\alpha$ for some $\alpha \in A\}$ is a point countable cover of X. We show that for every open subset $W \subset X$, W is determined by the point-countable cover $\mathcal{B}(W) = \{B \in \mathcal{B} \mid B \subset W\}$. Indeed, suppose not. Let $H \subset W$ be a subset such that $H \cap B$ is closed in B for each $B \in \mathcal{B}(W)$ but H is not closed in W. Since W is sequential, H is not sequentially closed in W. Hence for some $z \in W$, H contains the sequence $\{z_n\}$ such that $\{z_n\}_n \longrightarrow z$ in W, $z \notin H$. By Lemma 2.2, some $P_\alpha \in \mathcal{P}$ contains $\{z_{n(i)}\}_i \cup \{z\}$, so $z \in P_\alpha \cap W$. Therefore $z \in B_0 \subset P_\alpha \cap W$ for some $B_0 \in \mathcal{B}_\alpha$, so that $B_0 \in \mathcal{B}(W)$. There exists some t such that $\{z_{n(i)} \mid i \geq t\} \cup \{z\} \subset B_0$ and $\{z_{n(i)} \mid i \geq t\} \subset B_0 \cap H$. Since $B_0 \cap H$ is closed in B_0 and $\{z_{n(i)} \mid i \geq t\} \longrightarrow z$ in B_0, $z \in B_0 \cap H$. This is a contradiction. Next, let $x \in U$, where U is open in X. By regularity of X, U is a countably bi-quasi-k, sequential space and U is determined by $\mathcal{B}(U)$. By Lemma 2.3, there exists a finite family $\mathcal{F} \subset \mathcal{B}(U)$ such that $x \in \operatorname{int}_U(\bigcup \mathcal{F}) = \operatorname{int}(\bigcup \mathcal{F}) \subset \bigcup \mathcal{F} \subset U$ (where for $C \subset U$, $\operatorname{int}_U(C)$ is the interior of C in U). Therefore, X has a point-countable base by [5, Theorem 6.2]. Moreover, let the element of $\mathcal{P}$ be locally separable. Then any element of $\mathcal{P}$ is locally separable, metrizable by [13, Theorem 6]. Hence X is locally separable, metrizable by Theorem 2.4. $\square$

Recall that a space X is *meta-Lindelöf* if every open cover of X has a point-countable open refinement.

Corollary 2.6 ([4, Theorem 4.28])**.** *Let X be a meta-Lindelöf regular space which is locally separable, locally metrizable. Then X is a metrizable space.*

Proof: X has a point-countable cover $\mathcal{P}$ consisting of separable metrizable open subsets and hence, X is determined by $\mathcal{P}$. Since X is first countable, X is metrizable by Theorem 2.5. $\square$

Theorem 2.7. *Let X be a countably bi-quasi-k-space which is determined by a point-countable cover $\mathcal{P}$. If each element of $\mathcal{P}$ is a locally compact, metrizable subset or a locally separable, metrizable closed subset, then X is a locally separable, metrizable space. In particular, if all elements of $\mathcal{P}$ are locally compact, metrizable, then X is locally compact.*

Proof: Let $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$, where each P_α is a locally compact, metrizable subset or a locally separable, metrizable closed subset. Let $\alpha \in A$. In the former case, there exists a point-countable base $\mathcal{B}_\alpha$ for P_α such that $\overline{\mathcal{B}_\alpha}$ is point-countable,

P_α is determined by $\overline{\mathcal{B}_\alpha}$ and $\overline{B} = \overline{B}^{P_\alpha}$ (closure of B in P_α) is compact metrizable for every $B \in \mathcal{B}_\alpha$. In the latter case, there exists a point-countable base $\mathcal{B}_\alpha$ for P_α such that $\overline{\mathcal{B}_\alpha}$ is point-countable, P_α is determined by $\overline{\mathcal{B}_\alpha}$ and $\overline{B}(\subset P_\alpha)$ is separable metrizable for every $B \in \mathcal{B}_\alpha$. Therefore by [10, Lemma 1.9], X is determined by a point-countable cover $\mathcal{C} = \{\overline{B} \mid B \in \mathcal{B}_\alpha$ for some $\alpha \in A\}$ consisting of separable metrizable closed subsets. To see that X is a locally separable, regular space, let $x \in X$. Then, by Proposition 1.5, $x \in \operatorname{int}(\bigcup \mathcal{F})$ for some finite $\mathcal{F} \subset \mathcal{C}$ and, $\bigcup \mathcal{F}$ is a separable metrizable closed subset (if all elements of $\mathcal{P}$ are locally compact, then $\bigcup \mathcal{F}$ is compact). Hence X is locally separable, regular (locally compact, respectively), which implies that X is metrizable by Theorem 2.5. □

Remark 2.8. Example 2.1 asserts that in Theorem 2.7, the condition that X is countably bi-quasi-k cannot be changed to be "Fréchet $\aleph_0$".

For the metrizability of singly bi-quasi-k-spaces, the following theorem is related to the question [23, Question 1.2] whether a regular space X with a point-finite (or point-countable) k-system consisting of metrizable subsets is a σ-space.

Theorem 2.9. *Let X be a singly bi-quasi-k-space which is determined by a point-finite cover $\mathcal{P}$. Then the following hold.*

(1) *If every element of $\mathcal{P}$ is a locally compact, metrizable subset, then X is a locally compact, metrizable space.*
(2) *If every element of $\mathcal{P}$ is a locally separable, metrizable closed subset, then X is a locally separable, metrizable space.*

Proof: Let the condition of (1) or (2) hold. From the proof of Theorem 2.7, X is determined by a point-countable cover $\mathcal{C}$ consisting of separable metrizable closed subsets. Therefore by Theorem 2.7, it is sufficient to show that X is countably bi-sequential. We first show that X is Fréchet. Suppose that $x \in \overline{F}$. Then $x \in \overline{F \cap A_n}$ $(n \geq 1)$ for some q-sequence $\{A_n\}$. By Proposition 1.4, some A_k is contained in the union of a finite subcollection $\{C_1, \dots, C_m\} \subset \mathcal{C}$. Let $B = C_1 \cup \dots \cup C_m$. Since B is Fréchet and $x \in \overline{F \cap A_k} \subset B$, there exists a sequence $\{x_n\} \subset F \cap A_k$ such that $\{x_n\}_n \longrightarrow x$. Hence X is Fréchet. To see that X is countably bi-sequential, let $\{F_n\}$ be a decreasing sequence with $x \in \overline{F_n}$ $(n \geq 1)$. For each $n \in \mathbb{N}$, we can choose a sequence $\{x_{n,k}\}_k \subset F_n$ such that $\{x_{n,k}\}_k \longrightarrow x$. For each $n \in \mathbb{N}$, $\{x\} \cup S_n \subset P_n$ for some $P_n \in \mathcal{P}$ and some subsequence $S_n \subset \{x_{n,k}\}_k$ by Lemma 2.2. Since $\mathcal{P}$ is a point-finite cover, there exists $P_0 \in \mathcal{P}$ such that $P_{n(i)} = P_0$ for some sequence $n(1) < n(2) < \dots$. Then $\{x\} \cup S_{n(i)} \subset P_0$ $(i \geq 1)$. Since P_0 is a first countable space, we can choose $z_i \in S_{n(i)} \subset F_{n(i)}$ such that $\{z_i\}_i \longrightarrow x$. Consequently, X is countably bi-sequential. □

Remark 2.10. For the necessity of local separability of each element of a cover $\mathcal{P}$ of the space X in Theorem 2.7 or 2.9, Tanaka [18, Example 3.2] showed that there exists a first countable Tychonoff space which is determined by a point-finite cover consisting of metrizable open and closed subsets, but not normal. On the other hand, Stone [17, Theorem 5] showed that a regular space X is metrizable if X has a point-countable cover consisting of locally separable, metrizable open subsets.

The following example shows that in Theorem 2.9, we cannot change "singly bi-quasi-k" for "sequential".

Example 2.11 ([10, Example 9.3]). A two-to-one quotient map $f : M \longrightarrow Y$, with M the topological sum of compact metric spaces, and Y separable, Tychonoff, not meta-Lindelöf. Also, Y is a sequential space with a point-finite k-system consisting of metrizable subsets. But, Y is not singly bi-quasi-k by Theorem 2.9.

3. Local compactness

Example 2.1 asserts that there exists an $\aleph_0$, Fréchet space with a countable k-system, which is not locally compact. But, we have the following theorem among countably bi-quasi-k-spaces.

Theorem 3.1. (1) *If X is a countably bi-quasi-k-space with a point-countable k-system, then X is a locally compact space.*

(2) *If X is a countably bi-quasi-k, hemicompact regular space, then X is a locally compact space with a countable k-system.*

PROOF: Since (1) is evident from Proposition 1.5, we prove (2). By paracompactness of X, the closure of every countably compact subset is compact, so X is countably bi-k from [15, p. 94] and hence a k-space. Thus, from Proposition 1.7, X has a countable k-system and consequently is locally compact. □

Remark 3.2. (1) The separable completely metrizable, non locally compact space $X = [0,1] \setminus \{1/n \,|\, n \geq 2\}$ with the relative topology of $\mathbb{R}$ has no point-countable k-system. This implies that we cannot weaken "hemicompact" to "σ-compact" in Theorem 3.1(2).

(2) Let X be an uncountable discrete space. Then X is a locally compact metrizable space with a point-finite k-system, but X is not hemicompact.

Theorem 3.3. *If X is a singly bi-quasi-k-space with a point-finite k-system $\mathcal{P}$, then X is a locally compact space.*

PROOF: Suppose that X has no compact neighbourhood at some point $x \in X$. Let $\{P \in \mathcal{P} \,|\, x \in P\} = \{P_1, \ldots, P_k\}$ and $E = \bigcup_{i=1}^{k} P_i$. Then $x \in \overline{X \setminus E}$ and hence, $x \in \overline{A_n \cap (X \setminus E)}$ for some q-sequence $\{A_n\}$. By Proposition 1.4, some A_m is contained in $\bigcup_{i=1}^{l} Q_i$ for some finite $\mathcal{Q} = \{Q_1, \ldots, Q_l\} \subset \mathcal{P}$. Then, $G = X \setminus \bigcup\{Q \in \mathcal{Q} \,|\, x \notin Q\}$ is an open neighbourhood of x such that $G \cap A_m \cap (X \setminus E) = \emptyset$, which is a contradiction. □

We note that the condition "singly bi-quasi-k" of a space X in Theorem 3.3 cannot be weakened to be "sequential" by Example 2.11.

Let us define the concept of "weak k-systems".

Definition 3.4. (a) A subset A of a space X is called *relatively compact* if the closure of A is compact.

(b) A cover $\mathcal{P}$ of a space X is called a *weak k-system* if X is determined by $\mathcal{P}$ and every element of $\mathcal{P}$ is relatively compact.

Clearly, every space with a point-countable weak k-system is a k-space.

A space X is called *σ-para-Lindelöf* if every open cover of X has a σ-locally countable open refinement.

Every σ-para-Lindelöf space is meta-Lindelöf.

Proposition 3.5. (1) *Every locally compact, meta-Lindelöf space X has a point-countable weak k-system.*

(2) *Every locally compact, σ-para-Lindelöf space X has a point-countable k-system.*

(3) *Every locally compact locally separable, meta-Lindelöf space X has a point-countable k-system.*

PROOF: Since (1) is evident, we prove (2). For any $x \in X$, let $V(x)$ be a compact neighbourhood of x. Then, X has a σ-locally countable open refinement $\mathcal{P} = \bigcup_{n\geq 1} \mathcal{P}_n$ of $\{V(x) \,|\, x \in X\}$ with $\mathcal{P}_n$ locally countable. Therefore, $\overline{\mathcal{P}} = \bigcup_{n\geq 1} \overline{\mathcal{P}}_n$ is a point-countable cover of X consisting of compact subsets. Since X is determined by $\mathcal{P}$, X is also determined by $\overline{\mathcal{P}}$. For (3), for any $x \in X$, let $V(x)$ be an open neighbourhood of x, where $\overline{V(x)}$ is compact and $V(x)$ is separable. Then, X has a star-countable open refinement $\mathcal{P}$ of $\{V(x) \,|\, x \in X\}$ by [4, Theorem 4.28]. Hence, X is determined by a point-countable cover $\overline{\mathcal{P}}$ consisting of compact subsets. □

I do not know whether a space with a point-countable weak k-system has a point-countable k-system.

The following example shows that the class of paracompact spaces and the class of spaces with point-countable k-systems are exclusive.

Example 3.6. (1) Let $p \in \beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. Then the subspace $X = \mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ is hemicompact paracompact. Since X is not a k-space, X has no point-countable weak k-system.

(2) The space Y in Example 2.11 has a point-finite k-system, but Y is not paracompact.

Definition 3.7. (a) A space X is an *M-space* if there exists a quasi-perfect map $f : X \longrightarrow Y$ onto a metrizable space Y.

(b) A space X is a *p-space* ([2]) if X is Tychonoff and there exists a sequence $\{\mathcal{G}_n\}$ of open collections in βX such that $X \subset \bigcup \mathcal{G}_n$ $(n \geq 1)$ and for each $x \in X$, $\bigcap_{n\geq 1} St(x, \mathcal{G}_n) \subset X$.

It is well-known that every locally compact space or Moore Tychonoff space is a p-space, every p-space is a strict q-space [3, Theorem 1.3]. Also, every locally compact paracompact space is an M-space, and every M-space is a strict q-space.

We now consider the relations between entries in Table 1 ([15, p. 93]) for spaces with point-countable k-systems.

Michael showed that for paracompact spaces, corresponding entries in columns **E** and **F** in Table 1 coincide for each row ([15, p. 94]). On the other hand, he gave a paracompact M-space (hence, singly bi-quasi-k-space) which is not k' ([15, Example 10.5]). Hence, in the realm of paracompact spaces, for $n = 1, 2, 3, 4$ and 5, an entry in row n in **F** is not necessarily an entry of the same row in **B**.

Theorem 3.8. *In Michael's Table* 1, *the following facts hold.*

(1) *In the realm of spaces with a point-countable weak k-system, corresponding entries in columns* **B** *and* **F** *coincide for each row.*
(2) *In the realm of spaces with a point-countable k-system, all entries in rows* 2, 3 *and* 4 *in columns* **B** *and* **F** *are equivalent.*
(3) *If X is a countably bi-quasi-k-space with a point-countable k-system consisting of metrizable subsets, then X is a locally compact, metrizable space.*

Proof: (1): In rows 2, 4, 5 and 6, corresponding entries in columns **B** and **F** are coincident by Proposition 1.4. We show the coincidence in row 1. Let X be an M-space and let $\mathcal{P}$ be a point-countable weak k-system of X and let $x \in X$. Since X is a strict q-space, there exists a q-sequence $\{U_n\}$ of open neighbourhoods of x. Then, some U_m is contained in $\bigcup \mathcal{F}$ for some finite $\mathcal{F} \subset \mathcal{P}$ by Proposition 1.4. Therefore, $\overline{U_m}$ is a compact neighbourhood of x, which implies that X is locally compact. Next, let $f : X \longrightarrow Y$ be a quasi-perfect map onto a metrizable space Y. Since $f^{-1}(y)$ is a countably compact closed subset for any $y \in Y$, $f^{-1}(y)$ is contained in $\bigcup \mathcal{F}$ for some finite $\mathcal{F} \subset \mathcal{P}$, so that $f^{-1}(y)$ is compact. Since f is a perfect map, X is paracompact. Next by [19], we show the coincidence in row 3. Let X be a bi-quasi-k-space and let $\mathcal{P}$ be a point-countable weak k-system of X. For any filter base $\mathcal{F}$ with $x \in \bigcap \overline{\mathcal{F}}$, some q-sequence $\{A_n\}$ meshes with $\mathcal{F}$. Therefore, some $\overline{A_m}$ is compact, and the k-sequence $\{B_n\}$, where $B_n = \overline{A_m}$ for each $n \in \mathbb{N}$, meshes with $\mathcal{F}$. This implies that X is a bi-k-space. Also, let $\mathcal{K}$ be the set of all compact subsets of X, Z be the topological sum of $\mathcal{K}$ and $f : Z \longrightarrow X$ be a natural map. To see that f is a bi-quotient map, let $x \in X$ and let $\mathcal{F}$ be a filter base on X with $x \in \bigcap \overline{\mathcal{F}}$. Then, there exists a k-sequence $\{A_n\}$ such that $x \in \overline{F \cap \overline{A_n}}$ for each $F \in \mathcal{F}$ and each $n \in \mathbb{N}$. Since some A_m is contained in the union of some finite family of $\mathcal{P}$, $\overline{A_m} \in \mathcal{K}$ and $g = f \,|\, \overline{A_m}$ is a homeomorphism. Thus, f is bi-quotient by [19, Lemma 2.1(3)]. Then it follows that X is locally compact. (2) follows from Theorem 3.1(1) and, (3) follows from Theorem 2.7. □

Remark 3.9. (1): With respect to (2) or (3) of Theorem 3.8, we note that the space Y in Example 2.1 has a countable k-system consisting of metrizable subsets, and has all conditions in row 5 in Table 1, but Y does not satisfy any of the conditions in row 4.

(2): Theorem 3.3 asserts that for spaces with a point-finite k-system, all entries in rows 2, 3, 4 and 5 in columns **B**, **E** and **F** in Table 1 are equivalent. On the other hand, the space Y in Example 2.11 has a point-finite k-system and satisfies all conditions in row 6 except for column **C** in Table 1, but Y does not satisfy any of the conditions in row 5.

(3): The next example shows that there exists a space X which has a point-finite k-system and satisfies the condition in row 2 of a column **B**, but X does not satisfy any of the conditions in row 1 (compare with Theorem 3.3 or Theorem 3.8(2)).

The following example is given by modifying Example 4.3 in [4].

Example 3.10. There exists a locally compact, metacompact subparacompact space X with a point-finite k-system consisting of the one-point compactifications of discrete spaces such that X^2 is a locally compact space with a point-finite k-system, but X is not paracompact nor an M-space.

Indeed, let $X = \omega_1 \times \omega_0 \setminus \{(0,0)\}$ as a set. Let $H_n = \omega_1 \times \{n\}$ $(n \geq 1)$ and $V_\alpha = \{\alpha\} \times \omega_0 (0 < \alpha < \omega_1)$. Define a topology on X as follows: For $n \geq 1$, neighbourhoods of $(0,n)$ must contain $(0,n)$ and all but finitely many points of H_n. For $0 < \alpha < \omega_1$, neighbourhoods of $(\alpha, 0)$ must contain $(\alpha, 0)$ and all but finitely many points of V_α. All other points of X are isolated. Since each H_n or V_α is compact, X is a locally compact T_2-space. Hence, X is determined by a point-finite cover $\mathcal{P} = \{H_n \,|\, n \geq 1\} \cup \{V_\alpha \,|\, 0 < \alpha < \omega_1\}$ consisting of compact open subsets (X^2 is also determined by $\{P \times P' \,|\, P, P' \in \mathcal{P}\}$). Next, metacompactness of X is evident and subparacompactness of X follows from $X = (\bigcup_{n \geq 1} H_n) \cup ((0,\omega_1) \times \{0\})$ and [4, Theorem 7.3]. Also, X is not normal since two disjoint closed subsets $A = \{(0,n) \,|\, n \geq 1\}$ and $\{(\alpha, 0) \,|\, 0 < \alpha < \omega_1\}$ cannot be separated by open subsets in X. Finally, if X is an M-space, then X is paracompact from Theorem 3.8(1), which is a contradiction.

Question 3.11. Is every normal locally compact space with a point-countable k-system paracompact ?

A space X is called a *Nagata* space ([6, Definition 5.1]) if, for any $x \in X$, there exists a sequence $\{g_n(x)\}$ of open neighbourhoods of x such that (i) $g_{n+1}(x) \subset g_n(x)$ and (ii) if $g_n(x) \cap g_n(x_n) \neq \emptyset$ $(n \geq 1)$, then x is a cluster point of $\{x_n\}_n$.

Every Nagata space is paracompact perfectly normal, and the above equivalent condition was given by [11, Theorem 5].

Theorem 3.12. (1) *Every Nagata space X with a point-countable weak k-system is a locally compact, metrizable space.*

(2) *Every developable space X with a point-countable weak k-system is a locally separable, metrizable space.*

PROOF: (1): Since X is a strict q-space, X is locally compact by Proposition 1.4. Hence X is metrizable from [24, Theorem 18].

(2): Let $\mathcal{P}$ be a point-countable weak k-system of X. Then for any $P \in \mathcal{P}$, $\overline{P}$ is a compact Moore space and hence, P is separable metrizable. Since X is first countable, some open neighbourhood U of x is contained in $\bigcup \mathcal{F}$ for some finite $\mathcal{F} \subset \mathcal{P}$ by Proposition 1.4. Then $\bigcup \overline{\mathcal{F}}$ is regular, so that X is a regular space. Therefore, X is locally separable, metrizable by Theorem 2.5. □

Theorem 3.13. *Let X be a hemicompact regular space. Then in Table* 1*, all entries in rows* 1, 2, 3 *and* 4 *in columns* **B** *and* **F** *are equivalent, and corresponding entries in columns* **B** *and* **F** *coincide in rows* 5 *and* 6.

PROOF: Let X be a countably bi-quasi-k-space. Since X is Lindelöf regular, X is paracompact. Also X is locally compact by Theorem 3.1(2). Next, let X be a quasi-k-space, then X is a k-space by paracompactness and [15, p. 94]. Finally, let X be a singly bi-quasi-k-space, then X is a singly bi-k-space. Therefore, X has a countable k-system by Proposition 1.7. Hence X is a k'-space by Theorem 3.8. □

We note that the space Y in Example 2.1 is a hemicompact Fréchet $\aleph_0$, regular space with a countable k-system. Also, Y satisfies all the conditions in rows 5 and 6 in Table 1, but none of the conditions in rows 1, 2, 3 and 4.

Theorem 3.14. *Consider the following conditions for a space X.*

(1) *X is an M-space with a point-countable k-system.*
(2) *X is an M-space with a point-countable weak k-system.*
(3) *X is a locally compact, paracompact space.*
(4) *X is a locally compact space with a point-countable k-system.*
(5) *X is a locally compact space with a point-countable weak k-system.*
(6) *X is a p-space with a point-countable k-system.*
(7) *X is a p-space with a point-countable weak k-system.*
(8) *X is a countably bi-quasi-k-space with a point-countable k-system.*
(9) *X is a countably bi-quasi-k-space with a point-countable weak k-system.*

Then the following implications hold.

(1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3)$\Longrightarrow$(4)$\Longleftrightarrow$(6)$\Longleftrightarrow$(8) *and* (4)$\Longrightarrow$(5)$\Longleftrightarrow$(7)$\Longrightarrow$(9).

PROOF: The implications (1)$\Longrightarrow$(2), (4)$\Longrightarrow$(6)$\Longrightarrow$(8) and (4)$\Longrightarrow$(5)$\Longrightarrow$(7)$\Longrightarrow$(9) are evident. (3)$\Longrightarrow$(1) and (3)$\Longrightarrow$(4) follows from Proposition 3.5. (8)$\Longrightarrow$(4) follows from Theorem 3.1. Next, let X be a p-space with a point-countable weak k-system. Then X is a strict q-space and hence, X is locally compact by Theorem 3.8(1). This implies (7)$\Longrightarrow$(5). Finally, (2)$\Longrightarrow$(3) also follows from Theorem 3.8(1). □

Remark 3.15. It is well-known that in the realm of paracompact spaces, M-spaces and p-spaces are equivalent. On the other hand, in Theorem 3.14, (6) does not always imply (1) and, (4) does not always imply (3) by Example 3.10.

In §4, we will see that for a class of separable spaces, all statements of Theorem 3.14 are equivalent.

Question 3.16. Under what conditions, does a space with a point-countable weak k-system have a point-countable k-system ?

Tanaka [20, Example 3] showed that there exists a paracompact space X with a point-finite k-system consisting of metrizable subsets, but X^2 has no k-system. On the other hand, in [22, Theorem 6] he proved that the product space $X \times Y$ of singly bi-quasi-k-spaces determined by point-countable covers consisting of locally compact closed subsets is also determined by a point-countable cover consisting of locally compact closed subsets.

For the product of spaces with point-countable weak k-systems, the next corollary follows from Theorem 3.8.

Corollary 3.17. (1) *Every countable product of M-spaces with point-countable weak k-systems is a paracompact Čech-complete M-space.*

(2) *Every finite product of M-spaces with point-countable weak k-systems is an M-space with a point-countable k-system.*

Proof: (1) Let X_n be an M-space with a point-countable weak k-system for each $n \in \mathbb{N}$. Then, by Theorem 3.14, each X_n is locally compact, paracompact M and hence, there exists a perfect map $f_n : X_n \longrightarrow Y_n$ onto a locally compact metrizable space Y_n. Therefore, the product map of $\{f_n\}_n$ from $X = \prod_{n\geq 1} X_n$ to a completely metrizable space $\prod_{n\geq 1} Y_n$ is perfect. Hence, X is paracompact Čech-complete M.

(2) Let X and Y be M-spaces with point-countable weak k-systems. Then $X \times Y$ is locally compact, paracompact M by the above. Hence $X \times Y$ has a point-countable k-system by Theorem 3.14. □

We note that the countable infinite power $\mathbb{R}^\infty$ of $\mathbb{R}$ has no point-countable weak k-system, because $\mathbb{R}^\infty$ is not locally compact.

Question 3.18. [1] Does the square X^2 of a locally compact space X with a point-countable weak k-system $\mathcal{P}$ have a point-countable weak k-system ?

Corollary 3.19. *For a space X, the following conditions are equivalent.*

(1) *X is an M-space with a countable k-system.*
(2) *X is a p-space with a countable k-system.*
(3) *X is a regular hemicompact M-space.*

[1] Quite recently, Y. Tanaka gave a partial answer as follows: If X is a sequential space, or every element of $\mathcal{P}$ is a k-space, then the answer is affirmative.

(4) *X is a locally compact, hemicompact space.*
(5) *X is a countably bi-quasi-k-space with a countable k-system.*
(6) *X is a countably bi-quasi-k, hemicompact regular space.*
(7) *There exists a perfect map f from X onto a hemicompact metrizable (hence, separable locally compact, metrizable) space Y.*

Proof: (1)$\Longrightarrow$(2) follows from Theorem 3.8. (2)$\Longrightarrow$(3) is evident. (3)$\Longrightarrow$(4): Since X is countably bi-quasi-k, X is locally compact by Theorem 3.1. (4)$\Longrightarrow$(5) holds by Proposition 1.7. (5)$\Longrightarrow$(6): Since X is locally compact, X is regular. (6)$\Longrightarrow$(7): Since X is locally compact, paracompact, X is paracompact M. Hence, there exists a perfect map f from X onto a metrizable space Y, so Y is hemicompact. (7)$\Longrightarrow$(1): Since Y is locally compact, X is locally compact, hemicompact. Hence, X is an M-space with a countable k-system. □

4. Separable spaces

Theorem 1.3 can be weakened as follows.

Theorem 4.1. *Let X be a separable singly bi-quasi-k-space. Then the following conditions are equivalent.*
(1) *X has a point-countable k-system.*
(2) *X has a point-countable weak k-system.*
(3) *X has a countable k-system.*

Proof: (1)$\Longrightarrow$(2) and (3)$\Longrightarrow$(1) are evident.

(2)$\Longrightarrow$(3): By Theorem 3.8(1), X is a k'-space. Let $\mathcal{K}$ be a point-countable weak k-system. For a countable dense subset D of X, let $\mathcal{P} = \{\overline{P} \mid P \in \mathcal{K}$ and $P \cap D \neq \emptyset\}$. Then X is determined by $\mathcal{P}$ in view of the proof of Theorem in [12]. Hence, X has a countable k-system. □

Remark 4.2. In Theorem 4.1, the condition "singly bi-quasi-k" of a space X is necessary.

Indeed, the space Y in Example 2.11 is a separable sequential regular space with a point-finite k-system consisting of metrizable subsets, but Y has no countable k-system because Y is not Lindelöf.

Theorem 4.3. *If X is a separable space, then all conditions in Theorem 3.14 are equivalent.*

Proof: It is sufficient to show (9)$\Longrightarrow$(3). Since X has a countable k-system by Theorem 4.1, X is locally compact by Theorem 3.1. Since X is regular, X is paracompact. □

Theorem 4.4. *Let X be a Lindelöf space. Then all conditions from (1) to (8) in Theorem 3.14 are equivalent. Moreover, these are equivalent to the following condition.*

(10) *X is a strict q-space with a point-countable weak k-system.*

PROOF: It is sufficient to show (7)$\Longrightarrow$(2) and (3)$\Longleftrightarrow$(10). (7)$\Longrightarrow$(2): Since X is a Tychonoff space, X is a paracompact p-space. Hence X is an M-space. (10)$\Longrightarrow$(3): Since X is locally compact by Theorem 3.8(1), X is paracompact. (3)$\Longrightarrow$(10) is evident. □

Theorem 4.5. *Let X be a separable countably bi-quasi-k-space. Then the following conditions are equivalent.*

(1) *X has a point-countable weak k-system.*
(2) *X is a locally compact, hemicompact space.*
(3) *X is a locally compact, paracompact space.*
(4) *X is a locally compact, metacompact space.*
(5) *X is a locally compact, meta-Lindelöf space.*
(6) *X is a hemicompact regular space.*

PROOF: (1)$\Longrightarrow$(2) follows from Theorems 4.1 and 3.1. (2)$\Longrightarrow$(3)$\Longrightarrow$(4)$\Longrightarrow$(5) is evident. We show (5)$\Longrightarrow$(6). By Proposition 3.5(1) and Theorem 4.1, X has a countable k-system. Since X is locally compact, X is a hemicompact regular space by Proposition 1.7. Finally, (6)$\Longrightarrow$(1) follows from Theorem 3.1(2). □

Remark 4.6. Theorem 4.5 asserts that in the realm of separable spaces with point-countable weak k-systems, all entries in rows 1, 2, 3 and 4 in columns **B**, **E** and **F** in Table 1 are equivalent.

On the other hand, the space Y in Example 2.1 satisfies all the conditions in row 5 of all columns, but Y is not countably bi-quasi-k.

We note that Burke [4, Corollary 6.12] gave a first countable separable Lindelöf regular space Z such that Z^2 is not paracompact.

Corollary 4.7. *Let X and Y be countably bi-quasi-k, separable* (*Lindelöf*) *spaces with point-countable weak k-systems* (*point-countable k-systems*). *Then the product $X \times Y$ is a locally compact space with a countable k-system* (*hence, paracompact*).

PROOF: X, Y are locally compact, hemicompact by Theorem 4.5. Hence, so is $X \times Y$. Therefore $X \times Y$ has a countable k-system. If X, Y are Lindelöf countably bi-quasi-k-spaces with point-countable k-systems, then X, Y are locally compact by Theorem 3.1(1). Since X, Y are Lindelöf, they are hemicompact. Hence $X \times Y$ is locally compact, hemicompact and hence, it has a countable k-system. □

For separable spaces, the class of locally compact spaces and the class of spaces with point-countable k-systems are exclusive.

Example 4.8. (1) The space Y in Example 2.1 is a separable space with a countable k-system which is not locally compact.

(2) There exists a separable locally compact, countably compact space X which has no point-countable weak k-system.

Indeed, let $X = \beta\mathbb{N} \setminus \{p\}$ ($p \in \beta\mathbb{N} \setminus \mathbb{N}$) be the subspace of $\beta\mathbb{N}$. Then X is a separable locally compact, countably compact space by [7, Theorem 3.6.14]. Suppose that X has a point-countable weak k-system. Then X is compact by Proposition 1.4, which is a contradiction.

(3) Let Ψ be the separable locally compact Moore space in [9, 5I]. Then Ψ has no point-countable weak k-system. In fact, if Ψ has a point-countable weak k-system, then Ψ is metrizable by Theorem 3.12. This contradiction implies that Ψ has no point-countable weak k-system.

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