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Cardinal inequalities implying maximal resolvability

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Abstract. We compare several conditions sufficient for maximal resolvability of topological spaces. We prove that a space X is maximally resolvable provided that for a dense set $X_0 \subset X$ and for each $x \in X_0$ the π-character of X at x is not greater than the dispersion character of X. On the other hand, we show that this implication is not reversible even in the class of card-homogeneous spaces.

Keywords: maximally resolvable space, base at a point, π-base, π-character

Classification: 54A10, 54A25

1. Preliminaries

The paper is a continuation of studies in [BT]. We will use the following notation (see e.g. [Ho], [J]). As usual, $|X|$ denotes the cardinality of X and let $|\mathbb{R}| = \mathfrak{c}$. Suppose $(X, \mathcal{T})$ is a topological space. Then

- $w(X)$ denotes the weight of X:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\},$$

- $\Delta(X)$ – the dispersion character of X:

$$\Delta(X) = \min\{|U| : U \in \mathcal{T} \setminus \{\emptyset\}\},$$

- $\chi(X, x)$ – the character of a space X at a point x:

$$\chi(X, x) = \min\{|\mathcal{B}(x)| : \mathcal{B}(x) \text{ is a base of } X \text{ at } x\},$$

- $\chi(X)$ – the character of X:

$$\chi(X) = \sup\{\chi(X, x) : x \in X\},$$

- $\pi w(X)$ – the π-weight of X:

$$\pi w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base of } X\},$$

- $\pi\chi(X, x)$ – the π-character of a space X at a point x:

$$\pi\chi(X, x) = \min\{|\mathcal{B}| : \mathcal{B} \subset \mathcal{T} \setminus \{\emptyset\} \ \wedge \ \forall U \in \mathcal{T}, x \in U \Rightarrow \exists B \in \mathcal{B}\ B \subset U\},$$

- $\pi\chi(X)$ – the π-character of X:

$$\pi\chi(X) = \sup\{\pi\chi(X,x) \colon x \in X\}.$$

Let κ be a cardinal greater than 1. We say that X is *κ-resolvable* if it can be decomposed into κ pairwise disjoint dense subsets; X is called *maximally resolvable* (in short $\mathrm{MR}(X)$) if it is $\Delta(X)$-resolvable (see [CGF], [B]); X is called *cardinality-homogeneous* (*card-homogeneous*, shortly) if $\Delta(X) = |X|$.

All considered spaces are dense-in-itself. We study the following properties of a space X:

$\mathrm{P}(X)$: $w(X) \leq \Delta(X)$;

$\mathrm{P}'(X)$: $\chi(X) \leq \Delta(X)$;

$\mathrm{P}''(X)$: $\exists X_0 \subset X \big(\mathrm{cl}(X_0) = X \ \wedge\ \forall x \in X_0\ (\chi(X,x) \leq \Delta(X))\big)$;

$\mathrm{P}_\pi(X)$: $\pi w(X) \leq \Delta(X)$;

$\mathrm{P}'_\pi(X)$: $\pi\chi(X) \leq \Delta(X)$;

$\mathrm{P}''_\pi(X)$: $\exists X_0 \subset X\ \big(\mathrm{cl}(X_0) = X \ \wedge\ \forall x \in X_0\ (\pi\chi(X,x) \leq \Delta(X))\big)$.

Some of those conditions were considered in connection with resolvability of X. For example, the following facts were proved:

Fact 1 ([CGF])**.** *If a topological space X is card-homogeneous then* $\mathrm{P}(X)$ *implies* $\mathrm{MR}(X)$.

Fact 2 ([CGF], [B])**.** *If X is card-homogeneous then* $\mathrm{P}_\pi(X)$ *implies* $\mathrm{MR}(X)$.

Fact 3 ([BT])**.** *If X is card-homogeneous then* $\mathrm{P}''(X)$ *implies* $\mathrm{MR}(X)$.

It is clear that the statement $\mathrm{P}''_\pi(X)$ is the most general among considered conditions. The aim of this note is to show that $\mathrm{P}''_\pi(X)$ implies $\mathrm{MR}(X)$, and that $\mathrm{MR}(X)$ does not imply $\mathrm{P}_\pi(X)$ even for card-homogeneous spaces. These theorems will be proved in the final sections of the paper. We start with some construction and next we compare the introduced properties.

2. Small ideals with big cofinality

Let κ be an infinite cardinal. For $E \subset \kappa$ define $1E = E$ and $(-1)E = \kappa \setminus E$. A family $\mathcal{A} \subset \mathcal{P}(\kappa)$ is called *strongly independent* if $|\bigcap_{i=0}^m \varepsilon_i E_i| = \kappa$ for any sequence $E_0, \ldots, E_m$ of distinct elements of $\mathcal{A}$ and any sequence $\varepsilon_0, \ldots, \varepsilon_m$ of numbers from $\{-1, 1\}$. A theorem by Fichtenholz, Kantorovitch and Hausdorff (see [M]) states that there exists a strongly independent family $\mathcal{A} \subset \mathcal{P}(\kappa)$ of cardinality 2^κ. A family $\mathcal{F} \subset \mathcal{P}(\kappa)$ is called a *base* of an ideal $\mathcal{I} \subset \mathcal{P}(\kappa)$ if $\mathcal{F} \subset \mathcal{I}$ and each set $A \in \mathcal{I}$ is contained in a set $B \in \mathcal{F}$. The cardinal $\mathrm{cf}(\mathcal{I})$ stands for the minimal cardinality of a base of $\mathcal{I}$.

Theorem 4. *For each infinite cardinal κ there is an ideal $\mathcal{I} \subset \mathcal{P}(\kappa)$ such that $\bigcup \mathcal{I} = \kappa$ and $\mathrm{cf}(\mathcal{I}) = 2^\kappa$.*

PROOF: Consider a strongly independent family $\mathcal{A} \subset \mathcal{P}(\kappa)$ of cardinality 2^κ and let $\mathcal{I} \subset \mathcal{P}(\kappa)$ stand for the ideal generated by $\mathcal{A}$. (Thus $\mathcal{I} = \{F \subset \bigcup \mathcal{B} \colon \mathcal{B} \in [\mathcal{A}]^{<\omega}\}$, where $[\mathcal{A}]^{<\omega}$ denotes the family of all finite subsets of $\mathcal{A}$.) We may assume that $\bigcup \mathcal{A} = \kappa$ (adding $\kappa \setminus \bigcup \mathcal{A}$ to one of the sets from $\mathcal{A}$). Thus $\bigcup \mathcal{I} = \kappa$. Suppose that $\mathcal{F}$ is a base of $\mathcal{I}$ such that $|\mathcal{F}| = \lambda$ and $\omega \leq \lambda < 2^\kappa$. For each $F \in \mathcal{F}$ pick a family $\mathcal{A}_F \in [\mathcal{A}]^{<\omega}$ with $F \subset \bigcup \mathcal{A}_F$. Thus $|\bigcup_{F \in \mathcal{F}} \mathcal{A}_F| \leq \lambda$ and since $|\mathcal{A}| = 2^\kappa > \lambda$, we can find an $A_* \in \mathcal{A} \setminus \bigcup_{F \in \mathcal{F}} \mathcal{A}_F$. Pick an $F_* \in \mathcal{F}$ such that $A_* \subset F_*$. Hence $A_* \subset F_* \subset \bigcup \mathcal{A}_{F_*}$. On the other hand, by the strong independence of $\mathcal{A}$, we have

$$|A_* \setminus \bigcup \mathcal{A}_{F_*}| = |A_* \cap \bigcap_{A \in \mathcal{A}_{F_*}} (-1)A| = \kappa,$$

a contradiction. □

For an ideal $\mathcal{I} \subset \mathcal{P}(X)$ and $Y \subset X$ denote $\mathcal{I} \mid Y = \{A \cap Y \colon A \in \mathcal{I}\}$.

Corollary 5. *There is an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{I} = \mathbb{R}$, $\mathcal{I}$ consists of nowhere dense subsets of $\mathbb{R}$ and $\mathrm{cf}(\mathcal{I} \mid C) = 2^{\mathfrak{c}}$ for each perfect set $C \subset \mathbb{R}$.*

PROOF: Let C_α, $\alpha < \mathfrak{c}$, be an enumeration of all nowhere dense perfect subsets of $\mathbb{R}$. By a Bernstein-type construction we find a family $\{B_\alpha \colon \alpha < \mathfrak{c}\}$ of pairwise disjoint sets such that $\bigcup_{\alpha < \mathfrak{c}} B_\alpha = \mathbb{R}$ and $B_\alpha \subset C_\alpha$, $|B_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. By Theorem 4, for each $\alpha < \mathfrak{c}$ pick an ideal $\mathcal{I}_\alpha \subset \mathcal{P}(B_\alpha)$ with $\mathrm{cf}(\mathcal{I}_\alpha) = 2^{\mathfrak{c}}$. Let $\mathcal{I}$ consist of all sets $A \subset \mathbb{R}$ such that $A \cap B_\alpha \in \mathcal{I}_\alpha$ for each $\alpha < \mathfrak{c}$. So $\mathcal{I} \mid B_\alpha = \mathcal{I}_\alpha$ and thus $\mathrm{cf}(\mathcal{I} \mid B_\alpha) = 2^{\mathfrak{c}}$ (hence $\mathrm{cf}(\mathcal{I} \mid C_\alpha) = 2^{\mathfrak{c}}$) for all $\alpha < \mathfrak{c}$. □

3. Relationships between considered properties

Theorem 6. *For any dense-in-itself topological space X the following implications hold*

$$\begin{array}{ccccc}
\mathrm{P}(X) & \longrightarrow & \mathrm{P}'(X) & \longrightarrow & \mathrm{P}''(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{P}_\pi(X) & \longrightarrow & \mathrm{P}'_\pi(X) & \longrightarrow & \mathrm{P}''_\pi(X)
\end{array}$$

Moreover, all considered implications are not reversible.

PROOF: All implications considered in Theorem 6 are obvious. The following examples show that those implications do not reverse. □

Example 7 (see [BT]). Let $D(\mathfrak{c})$ be the discrete space of size $\mathfrak{c}$ and let $\mathbb{Q}$ be the space of all rationals with the Euclidean topology. Put $X_1 = D(\mathfrak{c}) \times \mathbb{Q}$ with the product topology. Then $w(X_1) = \pi w(X_1) = \mathfrak{c}$, $\Delta(X_1) = \omega$, $\chi(X_1) = \pi\chi(X_1) = \omega$. Hence $\mathrm{P}'(X) \not\to \mathrm{P}_\pi(X)$ (and consequently, $\mathrm{P}''(X) \not\to \mathrm{P}_\pi(X)$, $\mathrm{P}'_\pi(X) \not\to \mathrm{P}_\pi(X)$ and $\mathrm{P}'(X) \not\to \mathrm{P}(X)$).

Example 8. Let $\approx$ be the equivalence relation on $\mathbb{R} \times \mathbb{Q}$ defined by the formula $\langle x, y\rangle \approx \langle x', y'\rangle$ iff $\langle x, y\rangle = \langle x', y'\rangle$ or $y = y' = 0$. Let X_2 be the space $(\mathbb{R}\times\mathbb{Q})/_\approx$ with the topology introduced by a complete system of neighbourhoods (a hedgehog-type space). If $y \neq 0$ then define neighbourhoods of $\langle x, y\rangle_\approx$ as $U_n(\langle x, y\rangle_\approx) = \{x\} \times \left(y - \frac{|y|}{n}, y + \frac{|y|}{n}\right)$, $n \in \mathbb{N}$. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ be the ideal of countable sets. Neighbourhoods of the point $\langle 0, 0\rangle_\approx$ are the sets of the form $U_I(\langle 0, 0\rangle_\approx) = (\mathbb{R} \setminus I) \times \mathbb{Q}/_\approx \cup \{\langle 0, 0\rangle_\approx\}$ where $I \in \mathcal{I}$. Then $X_2 \setminus \{\langle 0, 0\rangle_\approx\}$ is dense in X_2 and $\Delta(X_2) = \omega$. For all $\langle x, y\rangle \not\approx \langle 0, 0\rangle$ we have $\chi(X_2, \langle x, y\rangle_\approx) = \pi\chi(X_2, \langle x, y\rangle_\approx) = \omega$, $\chi(X_2, \langle 0, 0\rangle_\approx) = \mathfrak{c}$, $\pi\chi(X_2, \langle 0, 0\rangle_\approx) = \omega_1 > \omega$. Hence $\mathrm{P}''(X) \not\to \mathrm{P}'_\pi(X)$ (so $\mathrm{P}''_\pi(X) \not\to \mathrm{P}'_\pi(X)$.

Example 9. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ be an ideal of nowhere dense sets with $\mathrm{cf}(\mathcal{I}) = 2^\mathfrak{c}$ (as in Corollary 5), $\mathcal{T}^*$ be the Hashimoto topology on $\mathbb{R}$ with respect to $\mathcal{I}$ (see [Ha]), i.e. the family of all sets of the form $U\setminus I$ where U is open in the Euclidean topology and $I \in \mathcal{I}$. Let $X_3 = (\mathbb{R}, \mathcal{T}^*)$. Then X_3 is card-homogeneous, $\Delta(X_3) = \mathfrak{c}$, $w(X_3) = 2^\mathfrak{c}$, $\pi w(X_3) = \pi\chi(X_3) = \omega$ and $\chi(X_3, x) = 2^\mathfrak{c}$ for all $x \in \mathbb{R}$. Hence $\mathrm{P}_\pi(X) \not\to \mathrm{P}''(X)$ (so $\mathrm{P}'_\pi(X) \not\to \mathrm{P}''(X)$ and $\mathrm{P}''_\pi(X) \not\to \mathrm{P}''(X)$).

Example 10. Let C be the Cantor ternary set, and $\mathcal{I}$ be an ideal of subsets of C with $\mathrm{cf}(\mathcal{I}) = 2^\mathfrak{c}$ (see Theorem 4). Define a topology $\mathcal{T}$ on $\mathbb{R}$ by a complete system of the neighbourhoods. If $x \in C$ then neighbourhoods of x are of the form $(x-\delta, x+\delta)\setminus I$ where $\delta > 0$, and $I \in \mathcal{I}$, $x \notin I$. If $x \notin C$ then the neighbourhoods of x are of the form $(x - \delta, x + \delta)$ where $\delta > 0$. Let $X_4 = (\mathbb{R}, \mathcal{T})$. Then X_4 is card-homogeneous, $\Delta(X_4) = \mathfrak{c}$, and the set $A = \mathbb{R} \setminus C$ is dense in X_4. We have $\chi(X_4, x) = \omega$ for all $x \in A$, and $\chi(X_4, x) = 2^\mathfrak{c}$ for all $x \in C$. Moreover $\pi w(X_4) = \pi\chi(X_4) = \omega$. Hence $\mathrm{P}''(X) \not\to P'(X)$.

Theorem 11. *In the class of card-homogeneous spaces the following relations hold*

$$\begin{array}{ccccc}
\mathrm{P}(X) & \longleftrightarrow & \mathrm{P}'(X) & \longrightarrow & \mathrm{P}''(X) \\
 & & & & \downarrow \\
\mathrm{P}_\pi(X) & \longleftrightarrow & \mathrm{P}'_\pi(X) & \longleftrightarrow & \mathrm{P}''_\pi(X)
\end{array}$$

Moreover, the implications $\mathrm{P}'(X) \to \mathrm{P}''(X)$ *and* $\mathrm{P}''(X) \to \mathrm{P}''_\pi(X)$ *do not reverse.*

Proof: Example 10 shows that $\mathrm{P}''(X) \nrightarrow \mathrm{P}'(X)$, and Example 9 yields $\mathrm{P}''_\pi(X) \nrightarrow \mathrm{P}''(X)$.

The proof of $\mathrm{P}'(X) \rightarrow \mathrm{P}(X)$: Suppose that for each $x \in X$, $\mathcal{B}(x)$ is a base of X at a point x such that $|\mathcal{B}(x)| \leq |X|$. Then $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}(X)$ is a base of X with $|\mathcal{B}| \leq |X|$. In a similar way we prove the implication $\mathrm{P}''_\pi(X) \rightarrow \mathrm{P}_\pi(X)$. $\square$

Remark 12. Theorem 11 solves a problem which follows Remark 4 in [BT].

Theorem 13. *If X is a dense-in-itself metrizable space then $\mathrm{P}'(X)$ is true and the following relations hold*

$$\begin{tikzcd} \mathrm{P}(X) \arrow[r] \arrow[d, leftrightarrow] & \mathrm{P}'(X) \arrow[r, leftrightarrow] & \mathrm{P}''(X) \arrow[d, leftrightarrow] \\ \mathrm{P}_\pi(X) \arrow[r] & \mathrm{P}'_\pi(X) \arrow[r, leftrightarrow] & \mathrm{P}''_\pi(X) \end{tikzcd}$$

Moreover, the implications $\mathrm{P}(X) \rightarrow \mathrm{P}'(X)$ and $\mathrm{P}_\pi(X) \rightarrow \mathrm{P}'_\pi(X)$ do not reverse.

Proof: Observe that if X is metrizable and dense in itself then $\Delta(X) \geq \omega$ and $\chi(X) = \omega$. Thus $\mathrm{P}'(X)$ holds, and consequently $\mathrm{P}''(X)$, $\mathrm{P}'_\pi(X)$ and $\mathrm{P}''_\pi(X)$ hold too. Example 7 shows that $\mathrm{P}'(X) \nrightarrow \mathrm{P}(X)$ and $\mathrm{P}'(X) \nrightarrow \mathrm{P}_\pi(X)$ (so $\mathrm{P}'_\pi(X) \nrightarrow \mathrm{P}_\pi(X)$).

To prove the implication $\mathrm{P}_\pi(X) \rightarrow \mathrm{P}(X)$ fix a π-base $\mathcal{B}$ of X with $|\mathcal{B}| \leq \Delta(X)$. For each $B \in \mathcal{B}$ choose an $x_B \in B$. Then the set $D = \{x_B : B \in \mathcal{B}\}$ is dense in X and $|D| \leq \Delta(X)$, thus the family of all open balls with the center at $x \in D$ and radii $1/n$, $n \in \mathbb{N}$, forms a base of X of size $\leq \Delta(X)$. $\square$

Corollary 14. *In the class of metrizable card-homogeneous spaces all six considered conditions hold.*

4. $\mathrm{P}''_\pi(X)$ implies $\mathrm{MR}(X)$

Lemma 15 ([BT, Lemma 5]). *For every dense-in-itself topological space X with $|X| = \kappa$ there exist pairwise disjoint open and card-homogeneous sets G_α, $\alpha < \kappa$, such that $X = \mathrm{cl}(\bigcup_{\alpha<\kappa} G_\alpha)$.*

Theorem 16. *For each dense-in-itself topological space X, the condition $\mathrm{P}''_\pi(X)$ implies $\mathrm{MR}(X)$.*

Proof: The proof of this theorem is analogous to the proof of Theorem 6 in [BT]. Let X_0 be a dense subset of X with $\pi\chi(X,x) \leq \Delta(X)$ for each $x \in X_0$. By Lemma 15 there exists a family of pairwise disjoint open and card-homogeneous sets G_α, $\alpha < |X|$, such that $X = \mathrm{cl}(\bigcup_\alpha G_\alpha)$. Then $\mathrm{P}''_\pi(G_\alpha)$ for each α and, by Theorem 11, $\mathrm{P}_\pi(G_\alpha)$ holds for $\alpha < |X|$. By Fact 2, all G_α are maximally resolvable. Note that $\Delta(G_\alpha) \geq \Delta(X)$, so G_α can be decomposed into dense subsets $D_{\alpha,\beta}$, $\beta < \Delta(X)$. Put $D_\beta = \bigcup_{\alpha<|X|} D_{\alpha,\beta}$ for $\beta < \Delta(X)$. Then the sets D_β are pairwise disjoint and dense in X. □

5. $\mathrm{MR}(X)$ for card-homogoneous spaces does not imply $\mathrm{P}_\pi(X)$

We shall prove that the implication given in Fact 2 cannot be reversed.

Theorem 17. *There exists a card-homogeneous topological space X which is maximally resolvable but does not satisfy condition* $\mathrm{P}_\pi(X)$.

Proof: We will construct X as a countable dense subspace of the Cantor cube $\{0,1\}^{\mathfrak{c}}$. (The existence of such subspaces follows from Hewitt-Marczewski-Pondiczery Theorem [E].) Let $\mathcal{B}$ be a countable base of the space $\{0,1\}^\omega$, let $\mathfrak{B}$ be the family of all finite subsets of pairwise disjoint sets from $\mathcal{B}$, and let $\mathcal{G}$ be the family of all functions $g\colon A \to \{0,1\}$, such that:

1. $(\exists\, \mathcal{B}_A \in \mathfrak{B})\ A = \bigcup \mathcal{B}_A$;
2. $(\forall\, B \in \mathcal{B}_A)\ g|B$ is constant.

The family $\mathcal{G}$ is countable, so put $\mathcal{G} = \{g_n : n < \omega\}$. Let $\{g_{n,m} : n, m < \omega\}$ be a sequence such that $g_{n,m} = g_n$ for $n, m < \omega$. Fix a bijection $\varphi\colon \omega \to \omega \times \omega$, $\varphi = (\varphi_1, \varphi_2)$, and choose inductively a one-to-one sequence $f_n\colon \{0,1\}^\omega \to \{0,1\}$ with

$$g_{\varphi(n)} \subset f_n \text{ for each } n.$$

Let $X = \{f_n : n < \omega\}$ and, for $m < \omega$, $X_m = \{f_k \in X : \varphi_2(k) = m\}$. Then all X_m's are dense in $\{0,1\}^{\mathfrak{c}}$. Indeed, fix an $m < \omega$ and a basic open set $U \subset \{0,1\}^{\mathfrak{c}}$. There exists a function $\psi_U\colon T \to \{0,1\}$ where T is a finite subset of $\{0,1\}^\omega$, with $f \in U$ iff $\psi_U \subset f$. Since $\{0,1\}^\omega$ is a Hausdorff space, there is n with $\psi_U \subset g_n$. Let $k = \varphi^{-1}(n,m)$. Then $f_k \in X_m \cap U$.

Thus X is a countable dense subspace of $\{0,1\}^{\mathfrak{c}}$. Moreover X is card-homogeneous, $\Delta(X) = \omega$, and, since X_m are pairwise disjoint, X is maximally resolvable. Finally, observe that X has no countable π-base, thus $\mathrm{P}_\pi(X)$ does not hold. Indeed, suppose that $\{V_n : n < \omega\}$ is a π-base of X. We may assume that all V_n are of the form $U_n \cap X$ where U_n is a basic open set in $\{0,1\}^{\mathfrak{c}}$ determined by a function $\psi_n\colon T_n \to \{0,1\}$ with T_n being a finite subset of $\{0,1\}^\omega$ (i.e., $f \in U_n$ iff $\psi_n \subset f$). Fix $t_0 \in \{0,1\}^\omega \setminus \bigcup_n T_n$. Then $H = \{f \in X : f(t_0) = 0\}$ is non-empty open in X, and no V_n is contained in H. □

References

[B] Bella A., *The density topology is extraresolvable*, Atti Sem. Mat. Fis. Univ. Modena **48** (2000), 495–498.

[BT] Bienias J., Terepeta M., *A sufficient condition for maximal resolvability of topological spaces*, Comment. Math. Univ. Carolinae **45** (2004), 139–144.

[CGF] Comfort W.W., Garcia-Ferreira S., *Resolvability: a selective survey and some new results*, Topology Appl. **74** (1996), 149–167.

[E] Engelking R., *General Topology*, PWN, Warsaw, 1977.
[Ha] Hashimoto H., *On the $*$-topology and its application*, Fund. Math. **91** (1976), 5–10.
[Ho] Hodel R., *Cardinals functions I*, in: Handbook of Set-Theoretic Topology, Elsevier, Amsterdam, 1984, pp. 1–61.
[J] Juhasz I., *Cardinals functions II*, in: Handbook of Set-Theoretic Topology, Elsevier, Amsterdam, 1984, pp. 63–109.
[M] Monk J.D., *Appendix on set theory*, in: Handbook of Boolean Algebras, vol. 3, Elsevier, Amsterdam, 1989, pp. 1215–1233.

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