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Combinatorial trees in Priestley spaces

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In honour of Věra Trnková on the occasion of her 70th birthday.

Abstract. We show that prohibiting a combinatorial tree in the Priestley duals determines an axiomatizable class of distributive lattices. On the other hand, prohibiting n-crowns with $n \geq 3$ does not. Given what is known about the diamond, this is another strong indication that this fact characterizes combinatorial trees. We also discuss varieties of 2-Heyting algebras in this context.

Keywords: distributive lattice, Priestley duality, poset, first-order definable

Classification: Primary 06D55, 06A11, 54F05; Secondary 06D20, 03C05

Introduction

Priestley duality provides a link between distributive lattices and certain ordered topological spaces. Properties of a lattice L can sometimes be expressed by configurations, or their absence, in the space X associated with L. For instance, it is a well known classical fact that L is a Boolean algebra if and only if every prime filter is maximal, and this condition can be reformulated by saying that the space X contains no two-element chain. In 1991 Adams and Beazer generalized this result by presenting first order formulas in L equivalent to the absence of an n-element chain in X ([1]). Much earlier in 1974 ([10]), Monteiro proved that L is relatively normal if and only if X contains no V, i.e., no three-element poset $\{x, y, z\}$ with $x < y, z$, and y incomparable to z.

This article is a continuation of our investigation into results of this type. We began by investigating configurations with a top element. We showed that prohibiting such configurations P can be expressed by first order formulas, indeed by a finite number of such, if and only if P is a tree, producing the formulas in [3] and proving the negative in [5].

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In this paper we prove some results concerning the general case. Here the role of trees is played by acyclic configurations. (To distinguish them we refer to the former as CS-trees and the latter as combinatorial trees; see Section 2). The fact that the diversity of the combinatorial trees greatly exceeds that of the CS-trees, together with the fact that, in the negative case, the diamond may be absent from a combinatorial tree while cycles of another sort may be present instead, creates considerable difficulties, and not only of a technical nature. The new cycles are the n-crowns, and to make matters even more complicated, a 2-crown can sometimes indicate a genuine cycle and sometimes not — see Figure 1.

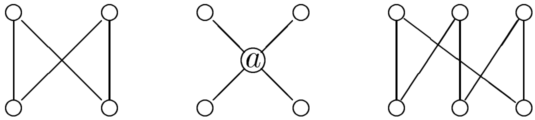

Figure 1. A 2-crown, a 2-crown induced in a combinatorial tree (remove a), and a 3-crown

We prove that prohibiting a combinatorial tree determines an axiomatizable class (Section 3), i.e., is first-order definable in the lattice, and that n-crowns with $n \geq 3$ do not (Section 5). With the exception of the cases confused by the simultaneous appearance of the "genuine" 2-crowns and the 2-crowns induced by an element separating its two antichains, we present a characterization of the prohibitions that produce varieties of 2-Heyting algebras (Section 4).

The results are not as complete as those on the configurations with tops. However, we have chosen to confine ourselves to what we present here because our results which go further differ in the techniques used and in the nature of the links with other problems (see 5.6). And we prefer to keep the length of this article within reasonable bounds.

1. Preliminaries

1.1. For subsets M, resp. elements x of a poset $(P, \leq)$, we write $\downarrow M = \{y \mid y \leq m \in M\}$, $\uparrow M = \{y \mid y \geq m \in M\}$, $\downarrow x = \downarrow\{x\}$ and $\uparrow x = \uparrow\{x\}$. An $M \subseteq (P, \leq)$ is said to be *decreasing*, resp. *increasing*, if $\downarrow M = M$, resp. $\uparrow M = M$. An *interval* is a subset of the form $[x, y] \equiv \uparrow x \cap \downarrow y$ for $x \leq y$ in P.

1.2. Recall that a *Priestley space* is an ordered compact space $(X, \tau, \leq)$ such that for any two $x, y \in X$ with $x \nleq y$ there is a closed open increasing $U \subseteq X$ such that $x \in U$ and $y \notin U$. The category of Priestley spaces and monotone continuous maps will be denoted by

$$\mathbf{PSp}.$$

Further recall the famous *Priestley duality* (see, e.g., [11], [12]) between $\mathbf{PSp}$ and the category

$$\mathbf{DLat}$$

of bounded distributive lattices, usually given by the equivalence functors

$$\mathcal{P} : \mathbf{DLat} \to \mathbf{PSp}^{\mathrm{op}}, \quad \mathcal{D} : \mathbf{PSp} \to \mathbf{DLat}^{\mathrm{op}}$$

defined by

$$\begin{aligned} \mathcal{P}(L) &= \{x \mid x \text{ proper prime ideal on } L\}, & \mathcal{P}(h)(x) &= h^{-1}[x], \\ \mathcal{D}(X) &= \{U \mid U \text{ clopen decreasing subset of } X\}, & \mathcal{D}(f)(U) &= f^{-1}[U], \end{aligned}$$

$\mathcal{P}(L)$ is endowed with a suitable topology and partially ordered by inclusion.

1.3. A Heyting algebra is a (bounded) distributive lattice L possessing a binary operation $\to$ satisfying $a \wedge b \leq c$ iff $a \leq b \to c$; a 2-Heyting algebra has, moreover an operation $\setminus$ such that $a \setminus b \leq c$ iff $a \leq b \vee c$ (note that $\setminus$ is the standard Heyting operation in the dual lattice L^{op}). The homomorphisms also preserve the extra operation(s). The resulting categories will be denoted by

$$\mathbf{Hey} \quad \text{and} \quad \mathbf{2Hey}.$$

It is a well-known fact that a Priestley space X is isomorphic to a $\mathcal{P}(H)$ for a Heyting algebra, resp. 2-Heyting algebra, H iff

whenever $U \subseteq X$ is clopen then also $\uparrow U$ is clopen,
resp. $\uparrow U$ and $\downarrow U$ are clopen.

Priestley spaces with this property will be referred to as *h-spaces*, resp. *$2h$-spaces*. Furthermore, the Priestley maps $f : X \to Y$ corresponding to Heyting (resp. 2-Heyting) homomorphisms, henceforth called *h-maps*, resp. *$2h$-maps*, are known to be characterized by the formula

$$\forall x,\ f(\downarrow x) = \downarrow f(x), \quad \text{resp.} \quad f(\downarrow x) = \downarrow f(x) \quad \text{and} \quad f(\uparrow x) = \uparrow f(x).$$

The resulting subcategories of $\mathbf{PSp}$ will be denoted by

$$\mathbf{HPSp} \quad \text{and} \quad \mathbf{2HPSp}$$

and Priestley duality restricts to the dualities

$$\mathbf{HPSp} \cong \mathbf{Hey}^{\mathrm{op}} \quad \text{and} \quad \mathbf{2HPSp} \cong \mathbf{2Hey}^{\mathrm{op}}.$$

Note that each finite distributive lattice is a 2-Heyting algebra, and each finite poset is a $2h$-space.

1.4. In a finite poset $(P, \leq)$ we will write $x \prec y$ if x is the immediate predecessor of y; the inverse relation (immediate successor) is indicated by $\succ$. The symmetric relation $\succ\!\prec$ on P is defined by

$$x \succ\!\prec y \quad \text{iff} \quad x \prec y \text{ or } x \succ y.$$

A finite poset $(P, \leq)$ is called a *combinatorial tree* if the graph $(P, \succ\!\prec)$ is a tree in the standard sense of combinatorics, that is, if it has no non-trivial cycle. Note the difference between this notion and that of *tree*, as typically used in computer science for connected posets in which for any x there is at most one immediate successor. These trees — to avoid confusion we will sometimes speak of them as CS-trees — are those combinatorial trees that have a top element; see also Section 2.

1.4.1. Let $C \equiv x_0 \succ\!\prec x_1 \succ\!\prec x_2 \succ\!\prec \cdots \succ\!\prec x_{n-1} \succ\!\prec x_n = x_0$ be a cycle in the graph $(P, \succ\!\prec)$. We say that the cycle is *irreducible* if $x_{i-1} \neq x_{i+1}$ for $0 \leq i \leq n-1$.

(All index arithmetic is done $\mod n$. Thus for example, the statement that C is irreducible includes the assertion that $x_1 \neq x_{n-1}$.)

If C is reducible, we may omit from C the point x_i with least index such that $x_{i-1} = x_{i+1}$, identify x_{i-1} with x_{i+1}, re-index the cycle, and repeat the process. The result is either an irreducible cycle or a single point; we refer to it as the *reduced form* of C.

An *upper, resp. lower, turning point in* C is any x_i such that $x_{i-1} \neq x_{i+1}$ and

$$x_{i-1} \prec x_i \succ x_{i+1} \quad \text{resp.} \quad x_{i-1} \succ x_i \prec x_{i+1}.$$

Note that certain types of cycles, for example $x_0 \prec x_1 \prec x_2 \succ x_3 = x_0$, are ruled out by the definition of $\prec$ as the immediate predecessor relation.[1] A cycle must have an even number of turning points, and we refer to a cycle with $2n$ turning points as an *n-cycle*.

1.5. A *configuration* P is a finite poset whose Hasse diagram $(P, \succ\!\prec)$ forms a connected graph. An *embedding* of a configuration P into a Priestley space X is a mapping $m : P \to X$ such that

$$m(x) \leq m(y) \quad \text{iff} \quad x \leq y.$$

The existence, resp. non-existence, of an embedding $m : P \to X$ is indicated by

$$P \hookrightarrow X \quad \text{, resp.} \quad P \not\hookrightarrow X$$

We also say that "X contains, resp. does not contain, the configuration P."

[1] It follows that the minimum length of an irreducible cycle is four; such a cycle forms either a diamond or a 2-crown, both terms to be defined in the sequel.

1.5.1. Let L be a distributive lattice. Consider the Priestley space $\mathcal{P}(L)$. By Lemma 2.5 of [3], for each embedding $m : P \to \mathcal{P}(L)$ there is a *separator*, that is, a mapping $a : P \to L$ such that

$$a(s) \in m(t) \quad \text{iff} \quad t \not\leq s.$$

1.6. The class of Priestley spaces X such that $P \not\hookrightarrow X$ will be denoted by

$$\mathsf{SForb}(P)$$

and its Priestley image in **DLat** by

$$\mathsf{Forb}(P).$$

Our main objective is the question of whether, and when, $\mathsf{Forb}(P)$ is first-order definable, or even *axiomatizable*, that is, describable by a *finite number* of first order sentences, in **DLat**.

The image of $\mathsf{SForb} \cap \mathbf{2HPSp}$ in **2Hey** will be denoted by

$$\mathsf{Forb}_{2H};$$

here we are interested in the question of when it is a variety.

1.7. The coproduct $\coprod_i X_i$ in **PSp** can be represented as the disjoint union

$$\bigcup_{u \in \beta J} X_u,$$

indexed by the set βJ of all ultrafilters on J, where each original space X_i appears as $X_{u(i)}$ with $u(i)$ the principal ultrafilter $\{M \mid i \in M\}$, all the X_u are order independent, and

($*$) *each X_u is the Priestley space dual to the ultraproduct $\prod_u \mathcal{D}(X_i)$* (see [8]).

A configuration P is said to be *coproductive* if it cannot be embedded into a coproduct $\coprod_{i \in J} X_i$ unless it is embeddable into one of the X_i. In general, the X_u indexed by free ultrafilters u can contain new cofigurations — see [4], [5].

By Łoś's Theorem (see, e.g., [9]), a system that can be characterized by first order formulas in a first order theory is closed under ultraproducts. In consequence, using ($*$) we obtain (see also [4])

1.7.1 Proposition. *If the class $\mathsf{Forb}(P)$ is characterized by first order formulas in the theory of bounded distributive lattices then the configuration P is coproductive.*

2. Combinatorial trees

In this section we will prove a simple characteristics of combinatorial trees. The way we do it may not be the shortest possible; however, we will need the lemmas in the present formulation again in the section on varieties of 2-Heyting algebras.

2.1. An *n-crown* in a poset $P = (P, \leq)$ is a subset $\{a_0, a_1, \dots, a_{2n-1}\}$ such that

$$a_i < a_j \text{ if and only if } i \text{ is even and } j = i \pm 1 \mod 2n.$$

A *diamond* in P is a subset $a < b, c < d$ with b, c incomparable.

2.2 Lemma. *In $(P, \leq)$ let*

$$\begin{aligned} &a_0 \prec a_1 \prec \cdots \prec a_r \text{ and} \\ &b_0 \prec b_1 \prec \cdots \prec b_s, \end{aligned}$$

and let $b_0 \not\leq a_r$. Then

1. *$b_i \not\leq a_j$ for any i, j;*
2. *if, moreover, $a_0 = c_0 \prec c_1 \prec \cdots \prec c_t = b_s$, $c_i = a_k$ and $c_j = b_l$ then $i < j$ for some $i > 0$;*
3. *if $a_0 = c_0 \prec c_1 \prec \cdots \prec c_t = b_s$, $a_1 \neq c_1$ and $a_i \leq b_j$ for some $i \geq 1$ and j then P contains a diamond.*

Proof: Those are trivial observations. In (3) observe that a_1, c_1 are necessarily incomparable. □

2.3 Lemma. *A poset P contains a diamond iff $(P, \succ\!\prec)$ contains an irreducible 1-cycle. And any such cycle with a minimum number of points must have the additional feature that two of its elements are related iff no turning point lies strictly between them on the cycle.*

Proof: Given the diamond $a < b$, $c < d$, select from the non-void sets $A_0 = (\downarrow b) \cap (\downarrow c)$ and $A_1 = (\uparrow b) \cap (\uparrow c)$ a maximal $a_0 \in A_0$ and a minimal $a_1 \in A_1$. Choosing arbitrary maximal chains in each of the intervals $[a_0, b]$, $[b, a_1]$, $[a_0, c]$ and $[c, a_1]$ then produces a sequence

$$a_0 \prec a_{01} \prec \cdots \prec a_{0k} \prec a_1 \succ a_{1l} \succ a_{1,l-1} \succ \cdots \succ a_{11} \succ a_0, \tag{1}$$

with $k, l \geq 1$ and $a_{0i} \neq a_{1j}$ for all i, j.

On the other hand, the elements a_{01} and a_{11} are obviously incomparable in the displayed sequence, and hence $a < a_{01}, a_{11} < d$ is a diamond. □

We refer to a configuration which contains no diamond as *diamond-free*. A configuration is diamond-free iff every interval is a chain. Thus in a diamond-free configuration, there is at most one cycle with a given set of turning points. In this context, the *cycle associated with an n-crown* $\{a_0, a_1, \ldots, a_{2n-1}\}$ is the reduced form of the cycle obtained by interpolating between each a_i and a_{i+1} the entire chain $[a_i, a_{i+1}]$.

The associated cycle may have fewer than $2n$ turning points, and may even consist of a single point. This will happen iff there is some point $b \in P$ such that $b \leq a_i$ for all the i odd and $b \geq a_i$ for all the i even. We call a crown *proper* if no such point $b \in P$ exists.

2.4 Lemma. *Every n-crown with $n \geq 3$ is proper. In fact, in a diamond-free configuration the cycle associated with an n-crown, $n \geq 3$, is an irreducible n-cycle, and its turning points also form an n-crown. Furthermore, two distinct elements of such a cycle are related iff no turning point lies strictly between them on the cycle.*

Proof: We leave it to the reader to perform the easy verification that any n-crown with $n \geq 3$ is proper, and remind him that Figure 1 illustrates both proper and improper 2-crowns. Now suppose we are given the n-crown $b_0 < b_1 > b_2 < \cdots < b_{2n-1} > b_{2n} = b_0$, $n \geq 3$, in a diamond-free configuration P. Fill in this crown by successors to obtain

$$\begin{aligned} b_0 =& b_{00} \prec b_{01} \prec \cdots \prec b_{0,l_0} = b_1 = b_{1,l_1} \succ \cdots \succ b_{11} \succ b_{10} = b_2 = b_{20} \prec \\ & b_{21} \prec \cdots \prec b_{2,l_2} = b_3 = b_{3,l_3} \succ \cdots \prec \cdots \succ b_{2n-1,1} \succ b_0. \end{aligned}$$

By 2.2.1, $b_{ij} = b_{kl}$ only if $|i-k| \leq 1$. Denote by a_i

- for i even the last occurrence of $b_{i-1,j'} = b_{i,j}$, and
- for i odd the first occurrence of $b_{i-1,j'} = b_{i,j}$.

By 2.2.2 we have $a_0 < a_1 > a_2 < \cdots < a_{2n-1} > a_0$, and comparing the a_i with the respective b_i we see that they constitute a crown. Finally, by 2.2.1 and 3, the b_{ij} between the a_i, a_{i+1} are incomparable with the b_{kl} between the a_k, a_{k+1} with $k \neq i$. $\square$

2.5 Lemma. *In a diamond-free configuration, the cycle associated with a proper 2-crown is an irreducible 2-cycle whose turning points form a 2-crown. Furthermore, two distinct elements of such a cycle are related iff no turning point lies strictly between them on the cycle.*

Proof: Given proper 2-crown $b_0, b_2 < b_1, b_3$, fill it in by successors as in 2.4 to obtain

$$\begin{aligned} b_0 =& b_{00} \prec b_{01} \prec \cdots \prec b_{0,l_0} = b_1 = b_{1,l_1} \succ \cdots \succ b_{11} \succ b_{10} = b_2 = b_{20} \prec \\ & b_{21} \prec \cdots \prec b_{2,l_2} = b_3 = b_{3,l_3} \succ \cdots \succ b_{31} \succ b_0. \end{aligned}$$

Again, we cannot have $b_{ij} = b_{kl}$ for $|i-k| > 1$, that is, $k = i+2 \bmod 4$, but the reason is different. If, say, $b_{0j} = b_{2l} = c$ we would have $b_0, b_2 < c < b_1, b_3$. The rest is as in 2.4. $\square$

2.6 Proposition. *A finite connected poset $(X,\leq)$ is a combinatorial tree if and only if it contains*

- *no diamond,*
- *no k-crown with $k\geq 3$, and*
- *no proper 2-crown.*

Proof: I. Let $(P,\leq)$ contain a diamond. Then the sequence (1) from 2.3 is a cycle in $(P,\asymp)$.

Let $(P,\leq)$ not contain a diamond and let it contain either a k-crown with $k\geq 3$ or a proper 2-crown. Then the sequence (2.4.2), resp. (2.5.2), from 2.4, resp. 2.5, is a cycle in $(P,\asymp)$.

II. Let

$$\begin{gathered} a_0\prec a_{01}\prec\cdots\prec a_{0,k_0-1}\prec a_1\succ a_{1,k_1-1}\succ\cdots\succ a_{11}\succ a_2\prec \\ a_{21}\prec\cdots\prec a_{2,k_2-1}\prec a_3\succ a_{3,k_3-1}\succ\cdots\prec\cdots\succ a_{m,1}\succ a_0 \end{gathered} \tag{2.6.1}$$

be a cycle in $(P,\asymp)$ with the smallest possible number m of turns. Then indeed $\cdots\succ a_{m,1}\succ a_0$ and $m=2n-1$ is odd since otherwise we could join the segments $a_m\prec\cdots\prec a_0$ and $a_0\prec\cdots\prec a_1$.

If $n=1$ and hence $m=1$, we have a diamond in $(P,\leq)$ by 2.3. If $n\geq 3$ we have an n-crown since if, up to a shift mod $2n$, $a_0<a_{2k-1}$ with $1<2k-1<2n-1$ we would have by 2.4 and 2.5 a cycle in $(P,\asymp)$ with a smaller number of turns.

Thus we are left with the case of a cycle

$$\begin{gathered} a_0\prec a_{01}\prec\cdots\prec a_{0,k_0-1}\prec a_1\succ a_{1,k_1-1}\succ\cdots\succ a_{11}\succ a_2\prec \\ a_{21}\prec\cdots\prec a_{2,k_2-1}\prec a_3\succ a_{3,k_3-1}\succ\cdots\succ a_{3,1}\succ a_0. \end{gathered}$$

We have a 2-crown $a_0,a_2<a_1,a_3$, since any comparability between a_0 and a_2, resp. a_1 and a_3, would create a diamond. Finally, if there is a c with $a_0,a_2<c<a_1,a_3$, it cannot be among both the $a_{01}\prec\cdots\prec a_{0,k_0-1}$ and the $a_{21}\prec\cdots\prec a_{2,k_2-1}$. If it is not in the first, say, we have a diamond $a_0<a_{0j},c<a_1$ for a suitable j. □

Remark. Note that, in contrast with CS-trees, a combinatorial tree can contain a connected subposet that is not a combinatorial tree: a 2-crown can be proper in a $Q\subset P$ without being proper in P.

3. First order formulas for prohibiting combinatorial trees

In this section we will prove that if a configuration P is a combinatorial tree then the class $\mathsf{Forb}(P)$ of distributive lattices corresponding to the Priestley spaces with forbidden P is axiomatizable, that is, it can be characterized among distributive lattices by a finite number of first order formulas. Moreover, we present a recursive procedure producing such formulas.

3.1 Some operations with ideals and filters. Ideals and filters in a bounded distributive lattice are assumed to be non-void but not necessarily proper. Let J_i, resp. F_i, $i = 1, 2, \dots, n$, be ideals, resp. filters, in L. Then

$$\bigvee_{i=1}^{n} J_i = \{x_1 \vee \cdots \vee x_n \mid x_i \in J_i\} \quad \text{resp.} \quad \bigvee_{i=1}^{n} F_i = \{x_1 \wedge \cdots \wedge x_n \mid x_i \in F_i\}$$

is the smallest ideal, resp. filter, containing all the J_i (resp. F_i).

Let J be an ideal and F a filter. Set

$$J \downarrow F = \{c \in L \mid \exists f \in F,\ c \wedge f \in J\},$$
$$J \uparrow F = \{c \in L \mid \exists j \in J,\ c \vee j \in F\}.$$

The following statements are easy to check.

3.1.1. *$J \downarrow F$ is an ideal and $J \subseteq J \downarrow F$.*

3.1.2. *$J \uparrow F$ is a filter and $F \subseteq J \uparrow F$.*

3.1.3. The ideal *$J \downarrow F$ is proper iff the filter $J \uparrow F$ is proper iff $J \cap F = \emptyset$.*

3.1.4. $J' \cap (J \uparrow F) = \emptyset \quad \Rightarrow \quad (J \vee J') \cap F = \emptyset$ *and*
$F' \cap (J \downarrow F) = \emptyset \quad \Rightarrow \quad J \cap (F \vee F') = \emptyset$.

3.2. Let T be a combinatorial tree and let $t_0 \in T$. Set

$$T_i(t_0) = \{t \in T \mid \exists \text{ shortest path } t_0 \asymp \cdots \asymp t_i = t\}.$$

Thus,

$$T_0(t_0) = \{t_0\} \subsetneq T_1(t_0) \subsetneq \cdots \subsetneq T_d(t_0) = T_{d+1}(t_0) = T$$

for some $d = d(t_0)$ determined by the choice of t_0. Note that

(3.2.1) *for each $t \in T_{i+1} \setminus T_i$ there is exactly one $t' \in T_i$ such that $t' \asymp t$.*

3.3. Let L be a bounded distributive lattice. For a mapping $a : T \to L$ define $a' : T \to L$ by setting

$$a'(t) = \bigvee_{s \not\geq t} a(s).$$

Define filters $F_n^a(t)$ and ideals $J_n^a(t)$ by induction as follows.

$$J_0^a(t) = {\downarrow} a'(t), \quad F_0^a(t) = {\uparrow} a(t),$$
$$J_{n+1}^a(t) = \Big(\bigvee_{s \leq t} J_n^a(s)\Big) \downarrow \Big(\bigvee_{s \geq t} F_n^a(s)\Big), \text{ and}$$
$$F_{n+1}^a(t) = \Big(\bigvee_{s \leq t} J_n^a(s)\Big) \uparrow \Big(\bigvee_{s \geq t} F_n^a(s)\Big).$$

3.4 Proposition. *Let T be a tree, $t_0 \in T$ and $a : T \to L$. Suppose that for $d = d(t_0)$ the filter $F^a_{d+1}(t_0)$ is proper. Then $T \hookrightarrow \mathcal{P}(L)$.*

PROOF: For simplicity write T_i for $T_i(t_0)$ and omit the superscripts in J^a_n and F^a_n.

Suppose $F_{d+1}(t_0)$ is proper. Then by 3.1.3,

$$\bigvee_{s\le t_0} J_d(s) \cap \bigvee_{s\ge t_0} F_d(s) = \emptyset$$

and there is a prime ideal $m(t_0)$ such that

$$\bigvee_{s\le t_0} J_d(s) \subseteq m(t_0) \quad \text{and} \quad \bigvee_{s\ge t_0} F_d(s) \subseteq \overline{m}(t_0)$$

where the bar indicates the complementing filter. Suppose we have defined for $t \in T_i$ prime filters $m(t)$ such that

(1) m is monotone, and

(2) $\bigvee_{s\le t} J_{d-i}(s) \subseteq m(t)$ and $\bigvee_{s\ge t} F_{d-i}(s) \subseteq \overline{m}(t)$.

Let $t \in T_{i+1} \setminus T_i$ and $t' \in T_i$ be as in 3.2.1, say $t' < t$. Since $m(t') \cap F_{d-i}(t) \subseteq m(t') \cap (\bigvee_{s\ge t'} F_{d-i}(s)) = \emptyset$ and $F_{d-i}(t) = (\bigvee_{s\le t} J_{d-i-1}(s)) \uparrow (\bigvee_{s\ge t} F_{d-i-1}(s))$ we have by 3.1.4 that

$$(\bigvee_{s\le t} J_{d-i-1}(s) \vee m(t')) \cap \bigvee_{s\ge t} F_{d-i-1}(s) = \emptyset$$

and hence there is a prime filter $m(t)$ such that

$$\bigvee_{s\le t} J_{d-i-1}(s) \subseteq m(t),\ m(t') \subseteq m(t) \ \text{ and } \ (\bigvee_{s\ge t} F_{d-i-1}(s)) \subseteq \overline{m}(t).$$

Similarly if $t' > t$ we obtain from 3.1.4 a prime ideal $m(t)$ such that

$$\bigvee_{s\le t} J_{d-i-1}(s) \subseteq m(t), \quad \text{and} \quad (\bigvee_{s\ge t} F_{d-i-1}(s)) \vee \overline{m}(t') \subseteq \overline{m}(t)$$

and hence $m(t') \supseteq m(t)$. In both cases we have extended m to T_{i+1} so that (1) and (2) are satisfied.

Finally, in particular

$$a'(t) \in J_0(t) \subseteq J_{d-i}(t) \subseteq m(t), \text{ and}$$
$$a(t) \in F_0(t) \subseteq F_{d-i}(t) \subseteq \overline{m}(t)$$

so that $a(t) \notin m(t)$ and if $r \not\leq t$ then $a(r) \in m(t)$. Thus

$$m(s) \subseteq m(t) \quad \Rightarrow \quad s \le t$$

and m is an embedding. $\square$

3.5 Proposition. *Let $T \hookrightarrow \mathcal{P}(L)$ and $t_0 \in T$. Then there is an $a : T \to L$ such that $F^a_{d+1}(t_0)$ is proper.*

Proof: Let $m : T \hookrightarrow \mathcal{P}(L)$ be an embedding and let a be a separator (recall 1.5.1). Thus,

$$J^a_0(t) \subseteq m(t) \quad \text{and} \quad F^a_0(t) \subseteq \overline{m}(t).$$

If x is a prime ideal then

$$x \downarrow \overline{x} = x \quad \text{and} \quad x \uparrow \overline{x} = \overline{x}.$$

(Indeed, if $f \wedge c \in x$ with $f \notin x$ then $c \in x$ by primeness.) Thus, if we know that $J^a_n(t) \subseteq m(t)$ and $F^a_n(t) \subseteq \overline{m}(t)$ we obtain

$$\begin{aligned} J^a_{n+1}(t) &= \Big(\bigvee_{s \le t} J^a_n(s)\Big) \downarrow \Big(\bigvee_{s \ge t} F^a_n(s)\Big) \\ &\subseteq \bigvee_{s \le t} m(s) \downarrow \bigvee_{s \ge t} \overline{m}(s) = m(t) \downarrow \overline{m}(t) = m(t) \end{aligned}$$

and similarly for F and $\overline{m}$ so that in particular all the $J^a_n(t)$ are proper. $\square$

3.6 Theorem. *Let T be a combinatorial tree. Then there is a first order formula $\mathcal{T}$ in the language of bounded distributive lattices such that $T \not\hookrightarrow \mathcal{P}(L)$ iff $L \models \mathcal{T}$.*

Proof: By 3.4 and 3.5, $T \not\hookrightarrow \mathcal{P}(L)$ iff

$$(*) \qquad \text{for each } \; a : T \to L, \; F^a_{d+1}(t_0) \ni 0.$$

Fix $a : T \to L$. We inductively define formulas $\mathcal{A}(n, a, t, c)$ and $\mathcal{B}(n, a, t, c)$ for $t \in T$, $c \in L$, and $n \ge 0$, such that

$$\begin{aligned} c \in J^a_n(t) \quad &\text{iff} \quad L \models \mathcal{A}(n, a, t, c), \\ c \in F^a_n(t) \quad &\text{iff} \quad L \models \mathcal{B}(n, a, t, c). \end{aligned}$$

Here is the definition.

$$\begin{aligned} \mathcal{A}(0, a, t, c) &\equiv \quad c \le \bigvee_{s \not\ge t} a(s), \\ \mathcal{B}(0, a, t, c) &\equiv \quad c \ge a(t), \\ \mathcal{A}(n+1, a, t, c) &\equiv \exists j_s \mathcal{A}(n, a, s, j_s) \exists f_s \mathcal{B}(n, a, s, f_s), \; \bigwedge_{s \ge t} f_s \wedge c \le \bigvee_{s \le t} j_s, \\ \mathcal{B}(n+1, a, t, c) &\equiv \exists j_s \mathcal{A}(n, a, s, j_s) \exists f_s \mathcal{B}(n, a, s, f_s), \; \bigwedge_{s \ge t} f_s \le c \vee \bigvee_{s \le t} j_s. \end{aligned}$$

Then the desired $\mathcal{T}$ can be obtained as

$$\mathcal{T} \;\equiv\; \forall a : T \to L, \mathcal{B}(d+1, a, t_0, 0).$$

$\square$

3.7. From 3.6 and 1.7.1 we immediately obtain

Corollary. *Each combinatorial tree is coproductive.*

4. 2-Heyting varieties obtained by prohibiting a configuration

In this section we are going to prove an incomplete counterpart of the theorem from [3] stating that the class of Heyting algebras determined by prohibiting a configuration P is a variety iff P is a CS-tree.

Here we are interested in combinatorial trees. Instead of Heyting algebras, the structure on the algebraic side of the duality is that of the 2-Heyting algebras. The result is incomplete in that we avoid the class of the configurations containing simultaneously proper and improper 2-crowns. In that case we do not know the general answer.

4.1 The exception $\mathcal{E}$. We will abbreviate the expression

> "with the exception of the diamond-free configurations P containing proper 2-crowns, but each proper 2-crown gives rise to an associated irreducible 2-cycle having only four elements"

by saying

> "with the possible exception $\mathcal{E}$".

4.2. It is a well-known fact that in Priestley duality

- the Priestley maps that are onto correspond exactly to the one-one homomorphisms, and
- the Priestley embeddings correspond exactly to the onto homomorphisms,

and that the same holds in the Heyting and the 2-Heyting restrictions. Further, the products in **Hey** and **2Hey** coincide with those in **DLat**. Consequently, by Birkhoff's Theorem,

4.2.1. *A class of* 2*-Heyting algebras is a variety iff its Priestley dual is closed under*

- *coproducts,*
- 2h*-maps onto, and*
- 2h*-embeddings.*

Furthermore, a prohibition of a configuration is, trivially, inherited by subspaces. Thus,

4.2.2. $\mathsf{Forb}_{2H}(P)$ *is a variety iff it is a quasivariety.*

4.3 Construction. Take a poset P and a mapping $\sigma : S = \{(x, y) \mid x \prec y\} \to \mathbb{N}$. In $n = \{0, 1, \dots, n-1\} > \max_S \sigma(x, y)$ consider the addition modulo n (denoted by $+$) and define a relation $\prec$ on $P \times n$ by setting

$$(x, i) \prec (y, j) \quad \text{if} \quad x \prec y \text{ and } j = i + \sigma(x, y).$$

Set $(x, i) \leq (y, j)$ if $(x, i) = (y, j)$ or there is a sequence $(x, i) = (x_0, i_0) \prec (x_1, i_1) \prec \cdots \prec (x_k, i_k) = (y, j)$. Then obviously $Q = (P \times n, \leq)$ is a poset, and $\prec$ is the associated relation of immediate precedence.

4.3.1 Lemma. *$p = ((x,i) \mapsto x) : P \times n \to P$ is a $2h$-map.*

Proof: Obviously it suffices to check the $2h$-property via the immediate successors and predecessors. If $x \prec y = p(y,i)$ then $x = p(x, i - \sigma(x,y))$; if $x = p(x,i) \prec y$ then $y = p(y, i + \sigma(x,y))$. □

We continue the construction 4.3. Take a natural number r greater than the size of S, and a one-one map $\varphi : S \to \mathbb{N}$. Set $\sigma(x,y) = r^{\varphi(x,y)}$. Then a sum

$$(\Sigma) \qquad \sum_{i=1}^{m} \varepsilon_i \sigma(x_i, y_i) \mod n$$

with $m \leq |S|$ and $\varepsilon_i = \pm 1$ is never zero unless each (x_i, y_i) occurs an even number of times.

4.3.2 Lemma. *Q is diamond-free.*

Proof: Suppose not. Then by 2.3, Q contains a 1-cycle

$$a_0 = a_{00} \prec a_{01} \prec \cdots \prec a_{0,k-1} \prec a_{0k} =$$
$$a_1 = a_{1l} \succ a_{1,l-1} \succ \cdots \succ a_{11} \succ a_{10} = a_0,$$

with $k, l \geq 1$, and no relations holding between these elements except those displayed. In particular, note that $a_{01} \neq a_{11}$. This fact implies that $x_{01} \neq x_{11}$, where we are expressing each a_{ij} in the form (x_{ij}, n_{ij}), for otherwise

$$a_{01} = (x_{01}, n_{01}) = (x_{01}, n_{00} + \sigma(x_{00}, x_{01})) = (x_{11}, n_{00} + \sigma(x_{00}, x_{11})) = a_{11}.$$

The point is that both mentioned values of σ appear just once in the sum

$$\sum_{i=1}^{l} \sigma(x_{i-1}, x_i) - \sum_{j=1}^{k} \sigma(x_{j-1}, x_j),$$

which therefore cannot be 0. But this fact prevents the cycle from closing. □

4.4 Lemma. *Let n be the least positive integer such that $(P, \succ\!\prec)$ contains an irreducible n-cycle. Then the turning points of any n-cycle constitute a proper n-crown.*

Proof: If distinct turning points of an n-cycle C are comparable, the cycle can be shortened by the omission of consecutive points in an obvious way to achieve an irreducible k-cycle with $k < n$. □

4.5 Proposition. *Suppose P is not a combinatorial tree for one of the following reasons. Either P contains a diamond, or P contains an n-crown for $n \geq 3$ but no proper 2-crown, or P contains a proper 2-crown, but every proper 2-crown gives rise to an associated 2-cycle having only four elements. Then there is an onto $2h$-map $p : Q \to P$ such that $P \not\hookrightarrow Q$.*

Proof: Let $p : Q \to P$ be as in 4.3. Suppose $f : P \to Q$ is an order embedding. Then P is diamond-free by Lemma 4.3.2. Let n be the least positive integer such that P contains a proper n-crown. By Lemmas 2.4, 2.5, and 4.4, n is also the least positive integer such that $(P, \rightarrowtail\!\!\!\!\!\prec)$ contains an irreducible n-cycle. Fix such a cycle C, with the provision that it contains at least five elements in case $n = 2$. Then $f(C)$ is not necessarily a cycle in $(Q, \rightarrowtail\!\!\!\!\!\prec)$, of course, for the fact that adjacent elements of C are related by $\rightarrowtail\!\!\!\!\!\prec$ guarantees only that their f-images are related by $<$ or $>$ in Q. But the hypotheses on C do guarantee that there is an irreducible n-cycle in $(Q, \rightarrowtail\!\!\!\!\!\prec)$ associated with $f(C)$, denoted

$$a_0 = a_{00} \prec a_{01} \prec \cdots a_{0,k_0} = a_1 = a_{1,k_1} \succ \cdots \succ a_{11} \succ a_{10} = a_2 = a_{20} \prec$$
$$a_{21} \prec \cdots \prec a_{2,k_2} = a_3 = a_{3,k_3} \succ \cdots \prec \cdots \succ a_{2n-1,1} \succ a_{2n-1,0} = a_0.$$

(This is clear for $n \geq 3$, since the turning points of $f(C)$ constitute an n-crown, but the reader will have no trouble providing the justification for the $n = 2$ case, which depends on the existence of the fifth point in $f(C)$.) Then writing $p(a_{ij})$ as x_{ij} gives the cycle

$$x_{00} \prec x_{01} \prec \cdots x_{0,k_0} = x_1 = x_{1,k_1} \succ \cdots \succ x_{11} \succ x_{10} = x_2 = x_{20} \prec$$
$$x_{21} \prec \cdots \prec x_{2,k_2} = x_3 = x_{3,k_3} \succ \cdots \prec \cdots \succ x_{2n-1,1} \succ x_0$$

in $(P, \rightarrowtail\!\!\!\!\!\prec)$. An argument like the one given in the proof of Lemma 4.3.2 establishes that the latter cycle is irreducible, i.e., $x_{i-1,k_{i-1}-1} \neq x_{i,k_i-1}$ for all i, and because n is minimal, its turning points form a proper crown by Lemma 4.4. The point is that each turning point appears just once in the cycle. But then, again arguing as in the proof of Lemma 4.3.2, the cycle cannot close. This contradiction forces us to abandon our hypothesis that f exists. □

4.6 Lemma. *Let P be a combinatorial tree and let $f : Q \to P$ be a surjective $2h$-map. Then there is an embedding $g : P \hookrightarrow Q$ such that $f(g(x)) = x$ for all $x \in P$.*

Proof: Choose an $x_0 \in P$. Set $P_0 = X_0 = \{x_0\}$ and inductively $P_{k+1} = \{x \mid x \rightarrowtail\!\!\!\!\!\prec y \text{ for some } y \in P_k\}$. Since P is a combinatorial tree, for an $x \in P_{k+1} \setminus P_k$ there is exactly one $y = \psi(x) \in P_k$ such that $x \rightarrowtail\!\!\!\!\!\prec y$ (else there would be distinct paths connecting x with x_0).

Choose a $y_0 = g(x_0)$ in Q such that $f(y_0) = x_0$. Now suppose g is already defined on P_k so that

(1) $f(g(x)) = x$ for all $x \in P_x$, and
(2) g is monotone.

Now for an $x \in P_{k+1} \backslash P_k$ we have either $x \in \downarrow\psi(x) = \downarrow f(g(\psi(x))) = f(\downarrow g(\psi(x)))$ or $x \in \uparrow\psi(x) = f(\uparrow g(\psi(x)))$. Hence we can choose a $y_x \in \downarrow g(\psi(x))$ resp $\uparrow g(\psi(x))$ such that $f(y_x) = x$. Setting $g(x) = y_x$ we have defined g on P_{k+1} such that the (1) and (2) above are satisfied (for (2) use the tree property again).

Thus we inductively obtain a monotone $g : P \to Q$ such that $f(g(x)) = x$; by the last equality, $g(x) \le g(y) \Rightarrow x \le y$ and g is an embedding. □

4.7 Theorem. *With the possible exception $\mathcal{E}$, the following statements are equivalent:*

(1) $\mathsf{Forb}_{2H}(P)$ *is a variety,*
(2) $\mathsf{Forb}_{2H}(P)$ *is a quasivariety,*
(3) P *is a combinatorial tree.*

Proof: The statements (1) and (2) are equivalent by 4.2.2.

Now if P is a combinatorial tree, the Priestley dual of $\mathsf{Forb}_{2H}(P)$ is closed under coproducts by 3.7, and by 4.6 it is closed under onto $2h$-maps.

On the other hand, if P is not a combinatorial tree then with the possible exception $\mathcal{E}$, the corresponding class of spaces is by 4.5 not closed under $2h$-maps onto. □

5. r-crowns with $r \geq 3$ are not coproductive

In case when P is a configuration with top, one knows that P is coproductive (and determines an axiomatizable class of distributive lattices) iff it is a (CS) tree. There are strong indications for the more general

Conjecture. *A general configuration P is coproductive if and only if it is a combinatorial tree.*

This seems to be, however, a much harder task. In this section we present a first step in this direction. (Perhaps it is more accurate to say we present the next step after what we already know concerning configurations with top and diamond – see [5]). Namely, we prove that the r-crown with $r > 2$ is not coproductive.

5.1. Set
$$N = \{2, 3, 4, \dots\}.$$

For $n \in N$ consider the posets
$$X_n = \{1, 2, \dots, r, 1', 2', \dots, r'\} \times \{1, 2, \dots, n\}$$

with the order given by the rule (ij stands for (i, j))

$$\text{for } k < r, \quad ki < l'j \quad \text{iff} \quad l = k \text{ or } k + 1, \quad \text{and} \quad i = j,$$
$$\text{for } k = r, \quad ri < l'j \quad \text{iff} \quad l = r \text{ and } i < j, \quad \text{or} \quad l = 1 \text{ and } i = j.$$

5.2. Let u be a free ultrafilter on N. On the set

$$I = \{(n, j) \mid 1 \le j \le n\}$$

consider the family of subsets ($\sharp M$ is the cardinality of M)

$$F = \{J \subseteq I \mid \exists m,\ \{n \mid \sharp\{j \mid (n, j) \notin J\} \le m\} \in u\}.$$

The F is a filter because $\sharp\{j \mid (n, j) \notin J_i\} \le m_i$, $i = 1, 2$, implies $\sharp\{j \mid (n, j) \notin (J_1 \wedge J_2)\} \le m_1 + m_2$, and F is proper because $\sharp\{j \mid (n, j) \notin \emptyset\} = n$ and hence $\{n \mid \sharp\{j \mid (n, j) \notin \emptyset\} \le m\}$ is finite. Choose an ultrafilter

$$v \supseteq F$$

on I.

5.3. For $J \subseteq I$ set

$$f(J) = \{(n, j) \mid \exists i,\ (n, i) \in J,\ i < j\}.$$

5.3.1 Observation. *If $J_1 \subseteq J_2$ then $f(J_1) \subseteq f(J_2)$.*

5.3.2 Lemma. *If $J \in v$ then $f(J) \in v$.*

Proof: Suppose not. Then there is a $J \in v$ such that $f(J) \notin v$, and hence $I \setminus f(J) \in v$. Replacing J by $J \cap (I \setminus f(J))$ and using 5.3.1 we see that we can choose the $J \in v$ so that

$$J \cap f(J) = \emptyset.$$

Then for any n, $\{j \mid (n, j) \in J\} \le 1$ for otherwise we had some $i < j$ with $(n, i), (n, j) \in J$ so that $(n, j) \in J \cap f(J)$. Then $I \setminus J \in F \subseteq v$ contradicting $J \in v$. □

5.4 Observation. *Let $r \ge 3$. Then none of the X_n contains an r-crown.*

Indeed, once a cycle leaves $\{(r, j), (r', j) \mid j \le n\}$, say in (r, i_0), it returns to (r', i_0) that is not connected with (r, i_0). If it does not leave this set, it contains a 2-crown: take the leftmost edge ij' in the cycle; it has to be succeeded by a $j'k$ with $i < k < j$ and then by a kl' with $k < l$. Then il' is an edge, too.

5.5 Theorem. *None of the r-crowns with $r \ge 3$ is coproductive.*

Proof: We will prove that $\coprod_{n \in N} X_n$ contains an r-crown. The r-crown will be represented as

$$C = \{1, 2, \dots, r, 1', 2', \dots, r'\}$$

with the order given by the rule

$$\begin{aligned} &\text{for}\quad k<r,\quad k<l' \quad\text{iff}\quad l=k,k+1,\\ &\text{for}\quad k=r,\quad r<l' \quad\text{iff}\quad l=r \ \text{ or }\ l=1. \end{aligned}$$

Set $A_n = \mathcal{D}(X_n)$. We will associate with the $p \in C$ prime ideals $m(p)$ in $\prod A_n$ so that $p \leq q$ iff $m(p) \subseteq m(q)$.

Set

$$m(p) = \{a = (a_n)_{n\in N} \mid \{(n,j) \mid pj \notin a_n\} \in v\}.$$

Obviously, $m(p)$ is a decreasing set. If $a, b \in m(p)$ then $\{(n,j) \mid pj \notin (a \vee b)_n = a_n \cup b_n\} = \{(n,j) \mid pj \notin a_n\} \cap \{(n,j) \mid pj \in b_n\} \in v$ and $a \vee b \in m(p)$. If $a \wedge b \in m(p)$ we have $\{(n,j) \mid pj \notin a_n \cap b_n\} = \{(n,j) \mid pj \notin a_n \text{ or } pj \notin b_n\} = \{(n,j) \mid pj \notin a_n\} \cup \{(n,j) \mid pj \notin b_n\} \in v$. As v is a prime filter, either $\{(n,j) \mid pj \notin a_n\} \in v$ or $\{(n,j) \mid pj \notin b_n\} \in v$, that is, either $a \in m(p)$ or $b \in m(p)$. Thus, $m(p)$ is a prime ideal.

Now let $p < q$. First, let $(p,q) \neq (r,r')$. If $a \in m(p)$ we have $\{(n,j) \mid pj \notin a_n\} \in v$; since a_n is decreasing, $pj \notin a_n$ implies $qj \notin a_n$ and $\{(n,j) \mid qj \notin a_n\} \supseteq \{(n,j) \mid pj \notin a_n\}$ and hence it also is in v, and $a \in m(q)$.

Let $(p,q) = (r,r')$ and let $a \in m(r)$. We have $J = \{(n,j) \mid rj \notin a_n\} \in v$. Now if $ri \notin a_n$ then $r'j \notin a_n$ for all $j > i$; thus

$$\{(n,j) \mid r'j \notin a_n\} \supseteq \{(n,j) \mid \exists\, i<j,\ ri \notin a_n\} = f(J) \in v$$

by 5.3.2, and $a \in m(r')$.

Thus, in any case, $p \leq q$ implies that $m(p) \subseteq m(q)$.

Finally, let $p \not\leq q$. Consider the downsets $a_n = \{tj \mid t \not\geq p\}$. We have

$$\begin{aligned} &\{(n,j) \mid pj \notin a_n\} = \{(n,j) \mid p \geq p\} = I \in v \quad\text{and}\\ &\{(n,j) \mid qj \notin a_n\} = \{(n,j) \mid q \geq p\} = \emptyset \notin v \end{aligned}$$

and hence $m(p) \ni a \notin m(q)$ and $m(p) \not\subseteq m(q)$. □

5.6 Notes. 1. The question of the coproductivity of the 2-crown seems to be related to some open problems of number theory. For instance, the 2-crown is not coproductive if the answer to the Erdös problem on Sidon sets is positive.

2. We have seen in Section 4 that the 2-crowns cause problems also in the problem of varieties. But it is not quite the same question: for instance, if P itself is the 2-crown, $\mathsf{Forb}_{2H}(P)$ is not a variety regardless of the 2-crown's coproductivity.

3. Even if the problem of the 2-crown will have been solved, considerable work will need to be done to prove the conjecture presented above. For instance, even in the already mentioned fact on configurations with top and diamond it is not at all easy to remove the requirement about the top.

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