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Internal object actions

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Dedicated to Professor Věra Trnková.

Abstract. We describe the place, among other known categorical constructions, of the internal object actions involved in the categorical notion of semidirect product, and introduce a new notion of representable action providing a common categorical description for the automorphism group of a group, for the algebra of derivations of a Lie algebra, and for the actor of a crossed module.

Keywords: monoidal category, monoidal functor, monoid, action, action of an object, semi-abelian category, semidirect product, groups, Lie algebras, crossed modules, actors

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Introduction

Categorical algebra, understood as a categorical approach to and a categorical generalization of classical algebraic constructions, such as products, epi-mono factorizations, kernels and cokernels, and so on, was initiated by Saunders Mac Lane in his famous "Duality for groups" [M1] at the very earliest stage of development of category theory. Despite its great further achievements, categorical algebra is still full of open questions and simple new concepts to be discovered — especially those that are needed for categorical reformulations and extensions of specific group- and ring-theoretic results. *Semi-abelian* categories ([JMT]) provide a convenient setting for such reformulations, just as abelian categories do in the study of abelian groups and modules. A typical group/ring theoretic result that extends (see [BJ]) to semi-abelian categories is: *Every split epimorphism is a semidirect-product projection.* It involves a new categorical notion of a semidirect product, and in particular a new notion of *internal object action*, which we continue to study in the present paper. Our first aim is to describe it in the light of rather advanced categorical concepts discovered already in the third decade of category theory, but still unfamiliar to most "non-categorical" algebraists. We

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then introduce a new notion of *representable action* providing a common categorical description for the automorphism group of a group, for the algebra of derivations of a Lie algebra, and for the actor of a crossed module (although this last is only briefly mentioned at the end), which does not use any kind of exponentiation. Among other things we show that a purely-categorical observation ("internal = external") can be used to explain the classical observation that the free group on a countable set appears as a normal subgroup in the free group on a two-element set.

We recall all "monoidal-categorical" notions, including the definition of a monoidal category; however we assume the notion of a 2-category to be known (see [KS] for the general theory; however all we need can also be found in the 1998 edition of [M2]).

1. The 2-category of monoidal categories

1.1 Definition. A *monoidal category* is a system $(\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$ in which:

(a) $\mathbf{C}$ is a category;

(b) I is an object in $\mathbf{C}$;

(c) $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor, written as $\otimes(A, B) = A \otimes B$;

(d) $\alpha = (\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C)_{A,B,C \in \mathbf{C}}$, $\lambda = (\lambda_A : A \to I \otimes A)_{A \in \mathbf{C}}$, and $\varrho = (\varrho_A : A \to A \otimes I)_{A \in \mathbf{C}}$ are natural isomorphisms making the following diagrams commute:

$$\begin{tikzcd} A \otimes (I \otimes B) \arrow[r, "\alpha"] & (A \otimes I) \otimes B \\ A \otimes B \arrow[u, "1\otimes\lambda"] \arrow[r, equal] & A \otimes B, \arrow[u, "\varrho\otimes 1"'] \end{tikzcd}$$

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D. \arrow[u, "\alpha\otimes 1"'] \end{tikzcd}$$

Here and below we write just α instead of $\alpha_{A,B,C}$ for short; it is also often useful to write $(\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho) = (\mathbf{C}, I, \otimes) = (\mathbf{C}, \otimes) = \mathbf{C}$. A monoidal category $(\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$ is said to be *strict* if $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for all A, B, C; $I \otimes A = A = A \otimes I$ for all A; and α, λ, and ϱ are the identity morphisms.

1.2 Examples. Monoidal categories are of course "everywhere", but the following five examples are especially important for our purposes.

(a) Any monoid $\mathbf{M} = (\mathbf{M}, e, m)$ can be regarded as a strict monoidal category $(\mathbf{C}, I, \otimes)$, in which $\mathbf{C}$ is the underlying set $\mathbf{M}$ regarded as a discrete category (i.e. a category with no non-identity arrows), $I = e$, and $\otimes = m$.

(b) Any category $\mathbf{X}$ yields the strict monoidal category $\mathrm{End}(\mathbf{X}) = (\mathrm{End}(\mathbf{X}), 1_{\mathbf{X}}, \circ)$ of functors $\mathbf{X} \to \mathbf{X}$, where $1_{\mathbf{X}}$ is the identity functor $\mathbf{X} \to \mathbf{X}$ and $\circ$ is the composition of functors.

(c) If $\mathbf{C}$ is a category with finite products, then $(\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$, in which $I = \mathbf{1}$ is a terminal object in $\mathbf{C}$, $\otimes = \times$ is a (chosen) binary product operation, and α, λ, ϱ arise from the canonical isomorphisms $A \times (B \times C) \cong (A \times B) \times C$, $A \cong \mathbf{1} \times A$, $A \cong A \times \mathbf{1}$ respectively, is a monoidal category. Such a monoidal structure is said to be *cartesian*.

(d) Dually, if $\mathbf{C}$ is a category with finite coproducts, then $(\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$, in which I $= \mathbf{0}$ is an initial object in $\mathbf{C}$, $\otimes = +$ is a (chosen) binary coproduct operation, and α, λ, ϱ arise from the canonical isomorphisms $A + (B + C) \cong (A + B) + C$, $A \cong \mathbf{0} + A$, $A \cong A + \mathbf{0}$ respectively, is a monoidal category.

(e) The category $\mathbf{Ab}$ of abelian groups with $I = \mathbb{Z}$ (= the additive group of integers), $\otimes$ the usual tensor product, and α, λ, ϱ the usual natural isomorphisms, forms a monoidal category. Having this motivating example in mind, the abstract $\otimes$ in Definition 1.1 is often called the *tensor product*.

1.3. Monoidal categories form a 2-category, and there is an evident forgetful 2-functor from the 2-category <u>MonCat</u> of monoidal categories to the 2-category <u>Cat</u> of (ordinary) categories. The 1- and 2-cells in <u>MonCat</u> are called monoidal functors and monoidal natural transformations respectively. We recall:

(a) Let $C = (\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$ and $\mathbf{C}' = (\mathbf{C}', I, \otimes, \alpha, \lambda, \varrho)$ be monoidal categories (we use the prime sign $'$ only for $\mathbf{C}$, although the $I, \otimes$, etc. in $\mathbf{C}$ and in $\mathbf{C}'$ are not, of course, supposed to be the same). A *monoidal functor* $F = (F, \theta, \phi : \mathbf{C} \to \mathbf{C}'$ consists of

(a_1) an ordinary functor $F : \mathbf{C} \to \mathbf{C}'$;

(a_2) a morphism $\theta : I \to F(I)$ in $\mathbf{C}'$;

(a_3) a natural transformation $\phi = (\phi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B))_{A,B \in C}$

making the diagrams

$$\begin{tikzcd}
F(A) \otimes (F(B) \otimes F(C)) \arrow[rr, "\alpha"] \arrow[d, "1\otimes\phi"'] & & (F(A) \otimes F(B)) \otimes F(C) \arrow[d, "\phi\otimes 1"] \\
F(A) \otimes (F(B \otimes C)) \arrow[d, "\phi"'] & & (F(A \otimes B)) \otimes F(C) \arrow[d, "\phi"] \\
F(A \otimes (B \otimes C)) \arrow[rr, "F(\alpha)"'] & & F((A \otimes B) \otimes C),
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes F(A) \arrow[d, "\theta\otimes 1"'] & F(A) \arrow[l, "\lambda"'] \arrow[d, "F(\lambda)"] \\
F(I) \otimes F(A) \arrow[r, "\phi"'] & F(I \otimes A),
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes I \arrow[d, "1\otimes\theta"'] & F(A) \arrow[l, "\varrho"'] \arrow[d, "F(\varrho)"] \\
F(A) \otimes F(I) \arrow[r, "\phi"'] & F(A \otimes I),
\end{tikzcd}$$

commute. A monoidal functor $F = (F, \theta, \phi)$ is said to be *strong* if θ and ϕ are isomorphisms, and *strict* if moreover $F(I) = I$, $F(A) \otimes F(B) = F(A \otimes B)$ for all A and B, and θ and ϕ are the identity morphisms.

(b) Let $F_i = (F_i, \theta_i, \phi_i) : \mathbf{C} \to \mathbf{C'}$ $(i = 1, 2)$ be monoidal functors. A *monoidal natural transformation* $\tau : F_1 \to F_2$ is an ordinary natural transformation $\tau : F_1 \to F_2$ such that the diagrams

$$\begin{tikzcd}
I \arrow[r, "\theta_1"] \arrow[d, equal] & F_1(I) \arrow[d, "\tau"] \\
I \arrow[r, "\theta_2"'] & F_2(I),
\end{tikzcd}$$

$$\begin{tikzcd}
F_1(A) \otimes F_1(B) \arrow[r, "\phi_1"] \arrow[d, "\tau\otimes\tau"'] & F_1(A \otimes B) \arrow[d, "\tau"] \\
F_2(A) \otimes F_2(B) \arrow[r, "\phi_2"'] & F_2(A \otimes B)
\end{tikzcd}$$

commute.

(c) The monoidal natural transformations compose in the same way as the ordinary ones, and therefore we can speak of the category $\underline{\mathrm{MonCat}}(\mathbf{C},\mathbf{C}')$ of monoidal functors from $\mathbf{C}$ to $\mathbf{C}'$, with the monoidal natural transformations as morphisms. Monoidal functors also compose: for $F=(F,\theta,\phi):\mathbf{C}\to\mathbf{C}'$ and $F'=(F',\theta',\phi'):\mathbf{C}'\to\mathbf{C}''$, the composite $F''=(F'',\theta'',\phi''):\mathbf{C}\to\mathbf{C}''$ has $F''=F'F$ (the ordinary composite of functors) with θ'' and $\phi''_{A,B}$ being the composites

$$
\begin{gathered}
I \xrightarrow{\theta'} F'(I) \xrightarrow{F'(\theta)} F'F(I) \text{ and}\\
F'F(A)\otimes F'F(B) \xrightarrow{\phi'_{F(A),F(B)}} F'(F(A)\otimes F(B)) \xrightarrow{F'(\phi_{A,B})} F'F(A\otimes B)
\end{gathered}
$$

respectively. It is this fact that makes $\underline{\mathrm{MonCat}}$ a 2-category equipped with a (2-faithful) forgetful 2-functor into $\underline{\mathrm{Cat}}$.

1.4 Examples. (a) Consider the standard free-forgetful adjunction $(F,G,\eta,\varepsilon):\mathbf{Sets}\to\mathbf{Ab}$ between sets and abelian groups. It turns out that if we make **Sets** and **Ab** monoidal categories (as in 1.2(c) and 1.2(e) respectively), then:

(a_1) The functor $F:\mathbf{Sets}\to\mathbf{Ab}$ equipped with the canonical isomorphisms $\mathbb{Z}\equiv F(\mathbf{1})$ and $F(A)\otimes F(B)\equiv F(A\otimes B)$ $(A,B\in\mathbf{Sets})$ is strong monoidal.

(a_2) The functor $G:\mathbf{Ab}\to\mathbf{Sets}$ equipped with the maps

$$
\begin{aligned}
&\mathbf{1}\to G(\mathbb{Z}),\ \text{that picks up 1 in } \mathbb{Z}, \text{ and}\\
&G(A)\times G(B)\to G(A\otimes B),\ \text{sending } (a,b) \text{ to } a\otimes b\ (A,B\in\mathbf{Ab};\ a\in A,\ b\in B),
\end{aligned}
$$

is a monoidal functor. It is not strong; however the maps above play a fundamental role: they are exactly what is needed to define $\mathbb{Z}$ and $A\otimes B$ by their familiar universal properties.

(a_3) Moreover, as soon as F and G are regarded as monoidal functors, η and ε become monoidal natural transformations.

(b) A monoidal functor $F:\mathbf{C}\to\mathrm{End}(\mathbf{X})$, where $\mathbf{C}$ is an arbitrary monoidal category and $\mathrm{End}(\mathbf{X})$ is as in Example 1.2(b), is also called a $\mathbf{C}$-*action on* $\mathbf{X}$. Equivalently such a $\mathbf{C}$-action can be defined as a triple $(\bullet,\theta,\gamma)$, where $\bullet$ is a functor $\mathbf{C}\times\mathbf{X}\to\mathbf{X}$ written as $(A,X\mapsto A\bullet X$, and $\theta=\theta_X:X\to I\bullet X))_{X\in\mathbf{X}}$ and $\gamma=(\gamma_{A,B,X}:A\bullet(B\bullet X)\to(A\otimes B)\bullet X)_{A,B\in\mathbf{C};X\in\mathbf{X}}$ natural transformations making the diagrams

$$
\begin{tikzcd}
A\bullet(B\bullet(C\bullet X)) \arrow[r, equal] \arrow[d, "1\bullet\gamma"'] & A\bullet(B\bullet(C\bullet X)) \arrow[d, "\gamma"] \\
A\bullet((B\otimes C)\bullet X) \arrow[d, "\gamma"'] & (A\otimes B)\bullet(C\bullet X) \arrow[d, "\gamma"] \\
(A\otimes(B\otimes C))\bullet X \arrow[r, "\alpha\bullet 1"'] & ((A\otimes B)\otimes C)\bullet X,
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\bullet X \arrow[r, equal] \arrow[d, "\theta"'] & A\bullet X \arrow[d, "\lambda\bullet 1"] \\
I\bullet(A\bullet X) \arrow[r, "\gamma"'] & (I\otimes A)\bullet X,
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\bullet X \arrow[r, equal] \arrow[d, "1\bullet\theta"'] & A\bullet X \arrow[d, "\varrho\bullet 1"] \\
A\bullet(I\bullet X) \arrow[r, "\gamma"'] & (A\otimes I)\bullet X,
\end{tikzcd}
$$

commute. The triple $(\bullet, \theta, \gamma)$ corresponding to the monoidal functor $F = (F, \theta, \phi) : \mathbf{C} \to \mathrm{End}(\mathbf{X})$ has of course $A\bullet X = F(A)(X)$, the same θ, and $\gamma_{A,B,X} = (\phi_{A,B})_X : A\bullet(B\bullet X) \to (A\otimes B)\bullet X$. We will also consider strong and strict **C**-actions. Note that for the strong ones (where θ and ϕ are required to be isomorphisms) the second diagram could be omitted; the details are explained in [JK]. (In that paper, and in the writings of many authors, the word "action" means "strong action"; for our purposes we need the wider definition above.)

(c) Let $\underline{\mathbf{1}}$ be the trivial monoid considered as a monoidal category. A monoidal functor from it to an arbitrary monoidal category **C** can be presented as a triple $M = (M, e, m)$, in which M is an object in **C** and $e : I \to M$ and $m : M\otimes M \to M$ are morphisms in **C** making the diagram

$$
\begin{tikzcd}
M\otimes(M\otimes M) \arrow[r, "\alpha"] \arrow[d, "1\otimes m"'] & (M\otimes M)\otimes M \arrow[r, "m\otimes 1"] & M\otimes M \arrow[d, "m"] & M \arrow[l, "(e\otimes 1)\lambda"'] \arrow[dl, equal] \\
M\otimes M \arrow[rr, "m"] & & M & \\
M \arrow[u, "(1\otimes e)\varrho"] \arrow[urr, equal] & & &
\end{tikzcd}
$$

commute. Such a triple is nothing but what is called a *monoid* in $\mathbf{C}$. Moreover, a monoidal natural transformation $\tau : (M_1, e_1, m_1) \to (M_2, e_2, m_2)$ is in effect a morphism $\tau : M_1 \to M_2$ in $\mathbf{C}$ for which $\tau e_1 = e_2$ and $\tau m_1 = m_2(\tau \otimes \tau)$; such a τ is called a *morphism of monoids* in $\mathbf{C}$. So the monoids in $\mathbf{C}$ form a category $\mathrm{Mon}(\mathbf{C})$, and we have a canonical isomorphism of categories

$$\mathrm{Mon}(\mathbf{C}) \cong \underline{\mathrm{MonCat}}(\underline{\mathbf{1}}, \mathbf{C}).$$

The right side here is the value at $\mathbf{C}$ of the representable 2-functor $\underline{\mathrm{MonCat}}(\underline{\mathbf{1}}, -) : \underline{\mathrm{MonCat}} \to \underline{\mathrm{Cat}}$; accordingly $\mathrm{Mon}(\mathbf{C})$, too, is the value at $\mathbf{C}$ of a 2-functor $\mathrm{Mon} : \underline{\mathrm{MonCat}} \to \underline{\mathrm{Cat}}$. For a monoidal functor $F = (F, \theta, \phi) : \mathbf{C} \to \mathbf{C}'$, $\mathrm{Mon}(F) : \mathrm{Mon}(\mathbf{C}) \to \mathrm{Mon}(\mathbf{C}')$ sends (M, e, m) to the composite

$$\underline{\mathbf{1}} \xrightarrow{(M,e,m)} \mathbf{C} \xrightarrow{(F,\theta,\phi)} \mathbf{C}'$$

of monoidal functors, whose explicit value we leave the reader to write down (as we also do for the component at (M, e, m) of $\mathrm{Mon}(\tau) : \mathrm{Mon}(F_1) \to \mathrm{Mon}(F_2)$ for a monoidal natural transformation $\tau : F_1 \to F_2 : \mathbf{C} \to \mathbf{C}'$). There is an evident forgetful functor $U_{\mathbf{C}} : \mathrm{Mon}(\mathbf{C}) \to \mathbf{C}$; by the 2-categorical Yoneda lemma, it is the $\mathbf{C}$-component of a 2-natural transformation from Mon to the forgetful 2-functor $\underline{\mathrm{MonCat}} \to \underline{\mathrm{Cat}}$.

(d) A monoid in $\mathrm{End}(\mathbf{X})$ is the same thing as a *monad* on the category $\mathbf{X}$. For a $\mathbf{C}$-action on $\mathbf{X}$ as in (b) above, the functor $\mathrm{Mon}(F) : \mathrm{Mon}(\mathbf{C}) \to \mathrm{Mon}(\mathrm{End}(\mathbf{X}))$ sends a monoid $M = (M, e, m)$ in $\mathbf{C}$ to the monad $(M \bullet (-), (e \bullet 1)\theta, (m \bullet 1)\gamma)$ on $\mathbf{X}$; and an action of this monad on an object X of $\mathbf{X}$ will also be called *an action of M on X*, or *an M-action on X*; we write $\mathbf{X}^M$ for the category of such actions. Explicitly, an M-action on an object X in $\mathbf{X}$ is a morphism $h : M \bullet X \to X$ in $\mathbf{X}$ making the diagram

$$\begin{tikzcd}
M \bullet (M \bullet X) \arrow[r, "\gamma"] \arrow[d, "1\bullet h"'] & (M \otimes M) \bullet X \arrow[r, "m\bullet 1"] & M \bullet X \arrow[d, "h"'] & X \arrow[l, "(e\bullet 1)\theta"'] \arrow[dl, equal] \\
M \bullet X \arrow[rr, "h"'] & & X &
\end{tikzcd}$$

commute. Various examples will be considered in the next section.

2. Definition and examples of representable monoid actions

2.1 Definition. We will say that an action of a monoidal category $\mathbf{C}$ on a category $\mathbf{X}$ has *representable monoid actions* on an object X in $\mathbf{X}$, if the functor

$$\mathrm{Act}(-, X) : (\mathrm{Mon}(\mathbf{C}))^{\mathbf{op}} \to \mathbf{Sets}$$

carrying a monoid in $\mathbf{C}$ to the set of its actions on X is representable; if this is the case, we will write $\mathrm{Act}(-, X) \equiv \mathrm{Mon}(\mathbf{C})(-, [X])$, thus denoting the representing monoid by $[X]$.

2.2 Examples. (a) Let us call a strong action of $\mathbf{C}$ on $\mathbf{X}$ *right closed* if for every object X in $\mathbf{X}$, the functor $(-) \bullet X : \mathbf{C} \to \mathbf{X}$ has a right adjoint, which we will denote by $\underline{\mathbf{X}}(X, -) : \mathbf{X} \to \mathbf{C}$. Using the fact that this makes $\underline{\mathbf{X}}$ a $\mathbf{C}$-category with the underlying category $\mathbf{X}$ (see [JK], where this fact is recalled in detail), it is easy to show that $\mathbf{X}$ then has representable monoid actions with $[X]$ above being the monoid $\underline{\mathbf{X}}(X, X)$.

(b) Any monoidal category $\mathbf{C} = (\mathbf{C}, I, \otimes, \alpha, \lambda, \varrho)$ has a canonical (strong) action on itself, given by taking $\mathbf{X} = \mathbf{C}$, $\bullet = \otimes$, $\theta = \lambda$, and $\gamma = \alpha$. Assuming this action to be right closed is the same as assuming $\mathbf{C}$ to be right closed, or just *closed* if $\mathbf{C}$ was a *symmetric* or *braided* monoidal category (see the 1998 edition of [M2]). In particular this applies to any *cartesian closed* category (for instance to **Sets**) considered as a monoidal category as in Example 1.2(c), and to the category of abelian groups considered as a monoidal category as in Example 1.2(e). In these cases $[X]$ is the usual exponent X^X and the usual endomorphism group $\mathrm{Hom}(X, X)$ respectively. On the other hand, if $\mathbf{C}$ is a non-trivial category with finite coproducts considered as a monoidal category as in Example 1.2(d), then it cannot be right closed, but its canonical action satisfies Definition 2.1 with $[X] = X$.

For further details and further examples, see the following table:

Monoidal category $\mathbf{C}$	Action of $\mathbf{C}$ on $\mathbf{X}$	Monoid M in $\mathbf{C}$	M-action on an object in $\mathbf{X}$	$[X]$
An ordinary monoid $\mathbf{M}$ (1.2(a))	An ordinary M-action on a set $\mathbf{X}$	e, the identity element of $\mathbf{M}$	Every object has a unique action of e	e
$\mathrm{End}(\mathbf{X})$ (1.2(b))	The (strict) evaluation action of $\mathrm{End}(\mathbf{X})$ on $\mathbf{X}$ defined by $A \bullet X = A(X)$	A monad T on $\mathbf{X}$ (2.2(b))	A T-algebra	$\langle X, X\rangle$, the right Kan extension of X (considered as a functor from the category 1 to $\mathbf{X}$) along itself, provided that Kan extension does exist
A category $\mathbf{C}$ with finite products regarded as a monoidal category with $\otimes = \times$ (1.2(c))	$\mathbf{C}$ canonically acting on itself; so that $A \bullet X = A \times X$	An internal monoid (= monoid object) M in $\mathbf{C}$ in the usual sense	An internal M-action in $\mathbf{C}$ (= M-object in $\mathbf{C}$) in the usual sense	X^X provided $\mathbf{C}$ is cartesian closed
As above, but with $\mathbf{C} = \mathbf{Sets}$	As above, but with $\mathbf{X} = \mathbf{C} = \mathbf{Sets}$	An ordinary monoid M	An ordinary M-action in **Sets** (= M-set)	X^X

A category **C** with finite coproducts regarded as a monoidal category with $\otimes = +$ (1.2(d))	**C** canonically acting on itself; so that $A \otimes X = A + X$	Every object M in **C** has a unique monoid structure given by the unique morphism $\mathbf{0} \to M$ and the codiagonal $M + M \to M$	An object in **C** equipped with a morphism from M to it	X
Ab, the category of abelian groups with $\otimes$ the ordinary tensor product (1.2(e))	**C** canonically acting on itself; so that $A \bullet X = A \otimes X$	A ring M (with 1)	An M-module	The ring $\mathrm{Hom}(X, X)$
Any monoidal category **C**	**C** trivially acting (i.e. $A \bullet X = X$) on any category **X**	A monoid M in **C** in the sense of 1.4(c)	Every object has a unique action of M	A terminal object in $\mathrm{Mon}(\mathbf{C})$, provided it exists
Sets regarded as a monoidal category with $\otimes = \times$ (1.2(c))	**X** a category with coproducts, and $A \bullet X =$ the coproduct of X with itself "A times", with the evident remaining structure	An ordinary monoid M	An ("external") M-action on an object in **X** in the usual sense	$\mathrm{End}(X)$, the monoid of endomorphisms of X

3. The object actions

3.1 Theorem. *If* $\mathbf{C} = (\mathbf{C}, +)$ *is as in Example* 1.2(d), *and* $\mathbf{C}' = (\mathbf{C}', I, \otimes, \alpha, \lambda, \varrho)$ *is an arbitrary monoidal category, then the* 2*-functor* $\mathrm{Mon}(-) \cong \underline{\mathrm{MonCat}}(\underline{\mathbf{1}}, -)$ *induces an isomorphism of categories*

$$\underline{\mathrm{MonCat}}(\mathbf{C}, \mathbf{C}') \cong \underline{\mathrm{Cat}}(\mathrm{Mon}(\mathbf{C}), \mathrm{Mon}(\mathbf{C}')) = \underline{\mathrm{Cat}}(\mathbf{C}, \mathrm{Mon}(\mathbf{C}')),$$

where $\mathrm{Mon}(\mathbf{C})$ *is identified with the category* $\mathbf{C}$, *since* (*see the table in* 2.2) *every object in* $\mathbf{C}$ *has a unique monoid structure.*

Proof: Let us write $(F, \theta, \phi) \mapsto (A \mapsto (M_A, e_A, m_A))$ for the functor $\underline{\mathrm{MonCat}}(\mathbf{C}, \mathbf{C}') \to \underline{\mathrm{Cat}}(\mathrm{Mon}(\mathbf{C}), \mathrm{Mon}(\mathbf{C}'))$ induced by $\mathrm{Mon}(-)$. Then $M_A = F(A)$, the collection of e_A's can be identified with θ, and the collection of m_A's is "almost" the same as ϕ; in fact ϕ and $m = (m_A)_{A \in C}$ can be displayed as

$$
\begin{tikzcd}
\mathbf{C}\times\mathbf{C} \arrow[d, "F\times F"'] & \mathbf{C} \arrow[l, "+"'] \arrow[d, "F"] \\
\mathbf{C}'\times\mathbf{C}' \arrow[r, "\otimes"'] \arrow[ur, phantom, "\phi"] & \mathbf{C}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}\times\mathbf{C} \arrow[r, "\Delta"] \arrow[d, "F\times F"'] & \mathbf{C} \arrow[d, "F"] \\
\mathbf{C}'\times\mathbf{C}' \arrow[r, "\otimes"'] \arrow[ur, phantom, "m"] & \mathbf{C}'
\end{tikzcd}
$$

where $\Delta : \mathbf{C} \to \mathbf{C}\times\mathbf{C}$ is the diagonal functor, and they correspond as mates (see [KS]) under the adjunction $+ \dashv \Delta$. The unit $1 \to \Delta + : \mathbf{C}\times\mathbf{C} \to \mathbf{C}\times\mathbf{C}$ has for its (A,B)-component the coproduct injections $(i : A \to A+B,\ j : B \to A+B)$, while the counit $+\Delta \to 1 : \mathbf{C} \to \mathbf{C}$ has for its A-component the codiagonal $[1,1] : A + A \to A$. Accordingly m_A is given in terms of ϕ as the composite

$$F(A)\otimes F(A) \xrightarrow{\phi_{A,A}} F(A+A) \xrightarrow{F([1,1])} F(A),$$

while $\phi_{A,B}$ is given in terms of m as the composite

$$F(A)\otimes F(B) \xrightarrow{F(i)\otimes F(j)} F(A+B)\otimes F(A+B) \xrightarrow{m_{A+B}} F(A+B).$$

It is then a routine calculation to show that the conditions to be satisfied by ϕ are equivalent to those to be satisfied by m whenever they are mates as above. This proves that $(F,\theta,\phi) \mapsto (A \mapsto (M_A, e_A, m_A))$ is bijective on objects; that it is also bijective on morphisms then follows easily, completing the proof. $\square$

3.2. Theorem 3.1 helps to construct monoidal functors with domain $(\mathbf{C},+)$ and, in particular the actions of $(\mathbf{C},+)$. It tells us that to give a $(\mathbf{C},+)$-action on a category $\mathbf{X}$ is the same as to give a functor from $\mathbf{C}$ to the category of monads on $\mathbf{X}$. On the other hand we know that monads arise from adjunctions, and so to give a $(\mathbf{C},+)$-action on $\mathbf{X}$ all we need is to give an adjunction with domain $\mathbf{X}$ for every object B in $\mathbf{C}$, functorially in B.

When $\mathbf{X} = \mathbf{C}$ we have the canonical action of $(\mathbf{C},+)$ on $\mathbf{C}$ as in 2.2(b); the corresponding functor from $\mathbf{C}$ to the category of monads on $\mathbf{C}$ sends an object B of $\mathbf{C}$, with its unique monoid structure, to the monad $B+(-)$ on $\mathbf{C}$, whose unit by 1.4(d) has its component $X \to B+X$ the coproduct injection and whose multiplication has by 1.4(d) the component

$$B+(B+X) \xrightarrow{\alpha} (B+B)+X \xrightarrow{[1,1]+1} B+X.$$

As in the table above, an algebra for this monad is an object X along with a morphism $B \to X$; that is, an object of the co-slice category $(B \downarrow \mathbf{C})$. The monad is given by the adjunction between the forgetful functor $(B \downarrow \mathbf{C}) \to \mathbf{C}$ and its left adjoint sending X to $(B \to B + X)$.

Our central concern is with a sub-monad $B \flat (-)$ of $B+(-)$, which exists when $\mathbf{C}$ is a pointed category with finite limits and finite coproducts; it is given by the adjunction[4]

$$(F, G, \eta, \varepsilon) : \mathbf{C} \to \mathrm{Pt}(B),$$

where:

(a) $\mathbf{C}$ is a pointed category with finite limits and finite coproducts;

(b) $\mathrm{Pt}(B)$ is the category of pointed objects in $(C \downarrow B)$, i.e. the category of triples (A, p, s) in which $p : A \to B$ and $s : B \to A$ are morphisms in $\mathbf{C}$ with $ps = 1$, a morphism $(A, p, s) \to (A', p', s')$ in $\mathrm{Pt}(B)$ being a morphism $f : A \to A'$ in $\mathbf{C}$ with $p'f = p$ and $fs = s'$;

(c) $F(X) = (B + X, \pi_{B,X}, \iota_{B,X})$, where $\pi_{B,X} = [1, 0] : B + X \to B$ is induced by the identity morphism of B and the zero morphism $X \to B$, and $\iota_{B,X} : B \to B+X$ is the coproduct injection;

(d) $G(A, p, s) = \mathrm{Ker}(p)$, the kernel of p;

(e) $\eta_X : X \to GF(X) = \mathrm{Ker}(\pi_{B,X} : B + X \to B) = B \flat X$ is the morphism induced by the coproduct injection $X \to B + X$;

(f) $\varepsilon_{(A,p,s)} : FG(A, p, s) = (B + \mathrm{Ker}(p), \pi_{B,\mathrm{Ker}(p)}, \iota_{B,\mathrm{Ker}(p)}) \to (A, p, s)$ is the morphism induced by $s : B \to A$ and the canonical morphism $\mathrm{Ker}(p) \to A$.

We denote the corresponding monad GF on $\mathbf{C}$ by $B \flat (-)$, as in (e) above. As an object $B \flat X$ is given by the kernel

$$B \flat X \xrightarrow{\kappa_{B,X}} B + X$$

of the morphism $[1, 0] : B + X \to B$ (for technical reasons we use both B and X as indices for κ, but not, say, for η). In fact the $\kappa_{B,X}$ constitute a monad-map $\kappa_{B,-}$ from $B \flat (-)$ to $B + (-)$. The unit condition requires the composite

$$B \xrightarrow{\eta_X} B \flat X \xrightarrow{\kappa_{B,X}} B + X$$

to be the coproduct-injection; and that this is so is the content of (e) above. The multiplication $\mu = G\varepsilon F$ for the monad $B \flat (-)$ has for its X-component the unique morphism μ_X making commutative the diagram

[4]This adjunction, also used in [BJ], arises from the fibration of points originally studied by D. Bourn in [B] and in several other papers (see [BB]).

$$
\begin{tikzcd}
B \flat (B \flat X) \arrow[rr, "\mu_X"] \arrow[d, "\kappa_{B,B\flat X}"'] & & B \flat X \arrow[dd, "\kappa_{B,X}"] \\
B + (B \flat X) \arrow[d, "1+\kappa_{B,X}"'] & & \\
B + (B + X) \arrow[r, "\alpha"'] & (B + B) + X \arrow[r, "{[1,1]+1}"'] & B + X;
\end{tikzcd}
$$

but this is also the remaining condition for $\kappa_{B,-} : B \flat (-) \to B + (-)$ to be a monad-map.

For each morphism $f : B \to B'$ in $\mathbf{C}$, it is clear that $f+(-) : B+(-) \to B'+(-)$ is a monad-map. Such an f also induces a morphism $f \flat (-)$ making commutative the diagram

$$
\begin{tikzcd}
B \flat (-) \arrow[r, "\kappa_{B,-}"] \arrow[d, "f\flat(-)"'] & B + (-) \arrow[d, "f+(-)"] \\
B' \flat (-) \arrow[r, "\kappa_{B',-}"'] & B' + (-);
\end{tikzcd}
$$

and since the top leg here is a monad map, so too is the bottom leg. But the monad map $\kappa_{B',-}$ is a monomorphism; it follows that $f \flat (-)$ too is a monad-map. So, $B \mapsto B \flat (-)$ is a functor from $\mathbf{C}$ to the category of monads on $\mathbf{C}$, which therefore corresponds by Theorem 3.1 to an action $\flat : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ of $(\mathbf{C}, +)$ on $\mathbf{C}$. We can express $\gamma_{A,B,X} : A \flat (B \flat X) \to (A + B) \flat X$ in terms of μ by the results in the proof of Theorem 3.1; but, since $\kappa_{B,-} : B \flat (-) \to B + (-)$ is a monomorphic monad-map, $\gamma_{A,B,X}$ is the unique morphism making commutative the diagram

$$
\begin{tikzcd}
A \flat (B \flat X) \arrow[r, "\gamma_{A,B,X}"] \arrow[d, "\kappa_{A,B\flat X}"'] & (A + B) \flat X \arrow[dd, "\kappa_{A+B,X}"] \\
A + (B \flat X) \arrow[d, "1+\kappa_{B,X}"'] & \\
A + (B + X) \arrow[r, "\alpha"'] & (A + B) + X.
\end{tikzcd}
$$

3.3. With $(\mathbf{C}, +)$ acting on $\mathbf{C}$ as above by the functor $\flat : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, we have as in Section 1.4 the notion of an action of $B \in \mathbf{C} = \mathrm{Mon}(\mathbf{C})$ on $X \in \mathbf{C}$; such an action consists, by 1.4(d), of a morphism $h : B \flat X \to X$ making commutative the diagram

$$\begin{tikzcd} B \flat (B \flat X) \arrow[r, "\gamma_{B,B,X}"] \arrow[d, "1 \flat h"'] & (B+B) \flat X \arrow[r, "{[1,1] \flat 1}"] & B \flat X \arrow[d, "h"] & X \arrow[l, "\eta_X"'] \arrow[dl, equal] \\ B \flat X \arrow[rr, "h"'] & & X & \end{tikzcd}.$$

Such an action will be called *an action on X of the object B of* $\mathbf{C}$. It is to such actions that our title "Internal object actions" refers. We again write $\mathrm{Act}(B, X)$ for the set of all actions of B on X, and write $\mathbf{C}^B$ for the category of B-actions — which is also the category $\mathbf{C}^{B \flat (-)}$ of algebras for the monad $B \flat (-)$.

3.4. According to 3.3, a B-action is the same as an algebra structure for the monad determined by the adjunction $(F, G, \eta, \varepsilon) : \mathbf{C} \to \mathrm{Pt}(B)$. Therefore, assuming in addition that $\mathbf{C}$ has coequalizers, we obtain (see [M2, Chapter VI, Section 7, Exercise 4]) the comparison adjunction

$$(F', G', \eta', \varepsilon') : \mathbf{C}^B \to \mathrm{Pt}(B),$$

in which:

(a) $F'(X, h)$, written as $(B \ltimes (X, h), \pi'_{(X,h)}, \iota'_{(X,h)})$, is defined together with a morphism $\sigma_{(X,h)} : B + X \to B \ltimes (X, h)$ via the coequalizer diagram

$$\begin{tikzcd} B + (B \flat X) \arrow[r, shift left, "{[\iota_1, \kappa_{B,X}]}"] \arrow[r, shift right, "1+h"'] & B + X \arrow[r, "\sigma_{(X,h)}"] & B \ltimes (X, h); \end{tikzcd}$$

ι_1 is the coproduct injection and $\kappa_{B,X}$ the canonical morphism $B \flat X \to B + X$, with $\iota'_{(X,h)} = \sigma_{(X,h)} \iota_{B,X}$, and with $\pi'_{(X,h)}$ uniquely determined by $\pi'_{(X,h)} \sigma_{(X,h)} = \pi_{B,X}$.

(b) $G'(A, p, s) = (\mathrm{Ker}(p), h)$, where $h = G(\varepsilon_{(A,p,s)})$ is the unique morphism making the diagram

$$\begin{tikzcd} B \flat \mathrm{Ker}(p) \arrow[r, "\kappa_{B,\mathrm{Ker}(p)}"] \arrow[d, "h"'] & B + \mathrm{Ker}(p) \arrow[d, "{[s, k_p]}"] \\ \mathrm{Ker}(p) \arrow[r, "k_p"'] & A \end{tikzcd}$$

commute; here k_p is the canonical morphism defining the kernel of p.
(c) $\eta'_{(X,h)} : (X,h) \to G'F'(X,h) = (\mathrm{Ker}(\pi'_{(X,h)} : B \ltimes (X,h) \to B), h')$, where h' makes the diagram

$$\begin{tikzcd} B \flat \mathrm{Ker}(\pi'_{(X,h)}) \arrow[r, "\kappa_{B,\mathrm{Ker}(\pi'_{(X,h)})}"] \arrow[d, "h'"'] & B + \mathrm{Ker}(\pi'_{(X,h)}) \arrow[d, "{[\iota'_{(X,h)}, k_{\pi'_{(X,h)}}]}"] \\ \mathrm{Ker}(\pi'_{(X,h)}) \arrow[r, "k_{\pi'_{(X,h)}}"'] & B \ltimes (X,h) \end{tikzcd}$$

commute, is the composite of the second coproduct injection $X \to B + X$ and $\sigma_{(X,h)} : B + X \to B \ltimes (X,h)$.
(d) $\varepsilon'_{(A,p,s)} : F'G'(A,p,s) = (B \ltimes (\mathrm{Ker}(p),h), \pi'_{(\mathrm{Ker}(p),h)}, \iota'_{(\mathrm{Ker}(p),h)}) \to (A,p,s)$ (with h as in (b) above), is the unique morphism $B \ltimes (\mathrm{Ker}(p),h) \to A$ making the diagram

$$\begin{tikzcd} B + \mathrm{Ker}(p) \arrow[r, "\sigma_{(X,h)}"] \arrow[dr, "{[s,k_p]}"'] & B \ltimes (\mathrm{Ker}(p),h) \arrow[d] \\ & A \end{tikzcd}$$

commute.

Let us also recall that the object $B \ltimes (X,h)$, defined as in (a), is the *semidirect product* of B and (X,h) in the sense of [BJ], and that whenever the ground category $\mathbf{C}$ is *semi-abelian* in the sense of [JMT], the adjunction $(F,G,\eta,\varepsilon) : \mathbf{C} \to \mathrm{Pt}(B)$ is monadic (see [BJ]), and hence the adjunction $(F',G',\eta',\varepsilon') : \mathbf{C}^B \to \mathrm{Pt}(B)$ is an equivalence.

Whether this last adjunction is equivalence or not, we have the functor $G' : \mathrm{Pt}(B) \to \mathbf{C}^B$ given as in (b) above. Now write $\mathbf{SplEpi}(\mathbf{C})$ for the category of *split epimorphisms* of $\mathbf{C}$: an object in it is a system (that is, a diagram) (B,A,p,s) where B and A are objects of $\mathbf{C}$ while $p : A \to B$ and $s : B \to A$ are morphisms satisfying $ps = 1$, and a morphism between such objects is a morphism of diagrams. In other words, an object of $\mathbf{SplEpi}(\mathbf{C})$ is an object B of $\mathbf{C}$ along with an object (A,p,s) of $\mathrm{Pt}(B)$. So G' gives rise to a functor $\check{G}$ from $\mathbf{SplEpi}(\mathbf{C})$ to the category $\mathbf{Act}(\mathbf{C})$ of *object actions* in $\mathbf{C}$, an object of which is a triple (B,X,h), where B and X are objects of $\mathbf{C}$ and $h : B \flat X \to X$ is an action of B on X, and a morphism $(B,X,h) \to (B',X',h')$ of which consists of morphisms $B \to B'$ and $X \to X'$ making the evident diagram commutative. Of course the functor $\check{G} : \mathbf{SplEpi}(\mathbf{C}) \to \mathbf{Act}(\mathbf{C})$ is also an equivalence whenever the adjunction $F' \dashv G'$ is so (for each B), and in particular whenever $\mathbf{C}$ is semi-abelian. In any case, it follows from (a) above that $\check{G}$ has the left adjoint sending (B, X, h) to $(B \ltimes (X, h), \pi'_{(X,h)}, \iota'_{(X,h)})$.

3.5 Definition. We will say that $\mathbf{C}$ *has representable object actions* if the action of $\mathbf{C}$ on itself considered in 3.3 has representable monoid actions in the sense of Definition 2.1, i.e. if the functor

$$\mathrm{Act}(-, X) : \mathbf{C}^{\mathbf{op}} \to \mathbf{Sets}$$

(in the situation of 3.3) is representable for each $X \in \mathbf{C}$.

Some interesting cases will be described in the following sections; at the moment let us just mention a "trivial" one:

Observation. Under the assumptions of Section 3, the category $\mathbf{C}$ acts on itself trivially (in the sense of the table in Section 2) whenever for every two objects B and X in $\mathbf{C}$, the canonical morphism $\eta_X : X \to B \flat X$ is an isomorphism. For instance this is the case if the canonical morphism $B + X \to B \times X$ is a monomorphism for all B and X. Therefore $[X] = \mathbf{0}$ for every X in $\mathbf{C}$ in the following cases:

(a) when $\mathbf{C}$ has an enrichment in the category of abelian monoids, and hence in particular when $\mathbf{C}$ is additive;

(b) when $\mathbf{C}^{\mathbf{op}}$ is protomodular in the sense of D. Bourn [B].

4. Internal group actions

4.1. The following well known theorem plays a crucial role in the homological algebra of groups.

Theorem. *The category* **SplEpi(Groups)** *of split epimorphisms in the category* **Groups** *(of groups) is equivalent to the category* **ActGroups** *of (classical) group actions in* **Groups**, *i.e. of triples (B, X, f), where B and X are groups and $f : B \to \mathrm{Aut}(X)$ is a group homomorphism. Under this equivalence:*

(a) *a system (A, B, p, s) corresponds to the triple $(B, \mathrm{Ker}(p), f)$ in which f is defined by $f(b)(x) = s(b)+x-s(b)$ (here and below we are using additive notation for groups);*

(b) *a triple (B, X, f) corresponds to the system (A, B, p, s) in which A is the (classical) semidirect product of B and (X, f) with the canonical p and s; that is $A = B \times X$ as a set, with*

$$(b_1, x_1) + (b_2, x_2) = (b_1 + b_2, x_1 + f(b_1)(x_2)),\ p(b, x) = b,\ s(b) = (b, 0).$$

Since all the structures and constructions here involve only finite products, they are invariant under the Yoneda embeddings, so that we have:

Conclusion. The theorem above is *Yoneda invariant*, and so the same correspondence expressed in the language of generalized elements determines an equivalence between:

- the category **SplEpi**(**Groups**(**E**)) of split epimorphisms in the category **Groups**(**E**) of internal groups in a category **E** with finite limits,
- and the category **ActGroups**(**E**) of internal group actions in **Groups**(**E**).

4.2. The category **Groups**(**E**) above is semi-abelian in the sense of [JMT] whenever it is exact and has finite coproducts. Therefore comparing the constructions of Section 3 for $\mathbf{C} = \mathbf{Groups}(\mathbf{E})$ (which we suppose to be exact and to have finite coproducts) with those of 4.1, we conclude that there are canonical category equivalences

$$\begin{bmatrix} \text{the category of} \\ B\text{-actions in } \mathbf{C} \\ \text{defined as in 4.1} \end{bmatrix} \sim \mathrm{Pt}(B) \sim \begin{bmatrix} \text{the category of} \\ B\text{-actions in } \mathbf{C} \\ \text{defined as in 3.3} \end{bmatrix}$$

for any object B in $\mathbf{C}$. Let us describe the equivalence between the two kinds of B-actions explicitly:

Since we deal with internal groups in **E** (not just ordinary groups in **Sets**), a B-action in the sense of 4.1 has to be given as a morphism $f : B \times X \to X$ in **E** (not as anything like a "group homomorphism" $B \to \mathrm{Aut}(X)$, unless **E** was cartesian closed — as in 4.4 below) satisfying the usual conditions adapted from 1.4(d). The corresponding object (A, p, s) in $\mathrm{Pt}(B)$ is as in 4.1(b), with all formulas rewritten for generalized elements, and accordingly $f(b_1)(x_2)$ replaced by $f(b_1, x_2)$ — or, better, written just as $b_1 x_2$. Since $p : A \to B$, considered as a morphism in **E**, is nothing but the first projection $B \times X \to B$, its kernel can be identified with X, embedded by the morphism $\langle 0, 1\rangle : X \to B \times X$ (= the morphism induced by the zero morphism $X \to B$ and the identity morphism $X \to X$). After that 3.4(b) tells us that the desired B-action of the other kind (i.e. in the sense of 3.3) has $h : B \flat X \to X$ defined as the unique morphism making the diagram

$$\begin{tikzcd} B \flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "h"'] & B + X \arrow[d, "{[\langle 1,0\rangle, \langle 0,1\rangle]}"] \\ X \arrow[r, "\langle 0,1\rangle"'] & A \end{tikzcd}$$
(has $B \times X$ as the underlying object in **E**)

commute. Note that although in general $A \neq B \times X$ in **Groups**(**E**), the morphisms involved here do belong to **Groups**(**E**); of course the right hand vertical arrow is not the same as the canonical morphism from the coproduct of B and X to their product — unless the action $f : B \times X \to X$ is trivial, i.e. coincides with the second projection. Accordingly, the second projection $B \times X \to X$ in general is not a morphism from A to X in **Groups**(**E**); however, as we see from the diagram above, the morphism h can be calculated as the commposite $B \flat X \to B + X \to B \times X \to X$ in **E**.

In the special case of ordinary groups (i.e. for **E** = **Sets**), any element in $B \flat X$ is a product of words in $B + X$ of the form $(b, x, -b)$, for which, applying the map $B + X \to A$ above, we obtain

$$(b, 0) + (0, x) + (-b, 0) = (b, bx) + (-b, 0) = (0, bx),$$

and so $h(b, x, -b) = bx$ — as was also mentioned in [BJ].

4.3. The equivalence of two kinds of B-actions in 4.2 induces of course an isomorphism of the two corresponding monads on **Groups**(**E**); let us recall briefly what they are in the case **E** = **Sets**.

(a) The first one (that of 4.1) is as in the last row of the table in 2.2: the underlying endofunctor **Groups** $\to$ **Groups** of that monad carries each group to its coproduct with itself "B times" (= the B-copower).

(b) The second one is as in Section 3; in particular its underlying endofunctor is $B \flat (-)$: **Groups** $\to$ **Groups**.

There is a remarkable conclusion:

Since the endofunctors mentioned in (a) and (b) are isomorphic, the group $B \flat X$ (where B and X are arbitrary groups) is always isomorphic to the coproduct of "B copies" of X. This isomorphism can easily be recovered from our previous results (or directly): as we have already mentioned, every word in $B \flat X$ can be uniquely presented as a product of words of the form $(b, x, -b)$, and such a word should correspond to x with the index b in the coproduct above. Let us take $B = X = \mathbb{Z}$, the additive group of integers; then $B + X$ is the free group on two generators, and this isomorphism will

> *present* (*via the short exact sequence* $B \flat X \to B + X \to B$) *the free group on infinitely many generators as a normal subgroup of the free group on two generators.*

The possibility of such a presentation is of course known from any elementary course in group theory, but now we have a categorical explanation of it!

4.4. If **E** is cartesian closed, then there is a straightforward way to define the internal automorphism group $\mathrm{Aut}(X)$ of a group X in **E**, and then, just as for **E** = **Sets**, there is a canonical bijection between the actions $B \times X \to X$ and the internal group homomorphisms $B \to \mathrm{Aut}(X)$. Thus from the equivalence of two kinds of B-actions in 4.2 we obtain:

Theorem. *Let* **E** *be a cartesian closed category, for which the category* **Groups**(**E**) *is exact and has finite coproducts. Then* **Groups**(**E**) *has representable object actions in the sense of Definition* 3.5 *with* $[X] = \mathrm{Aut}(X)$ *for each* X *in* **Groups**(**E**).

4.5 Remark. Of course if the category **E** is exact, then so is **Groups**(**E**). However there are important examples of $\mathbf{E} = \mathbf{Cat}_n$ (= the category of n-categories; $n = 1, 2, \ldots$) and of other categorical structures, where **E** is not exact but satisfies all the assumptions of Theorem 4.4.

5. Internal Lie algebra actions

5.1. Lie algebras turn out to be "as good as groups". The well-known "Lie version" of Theorem 4.1 is

Theorem. *Let* R *be a commutative ring* (*with* 1). *The category* $\mathbf{SplEpi}(\mathbf{Lie}_R)$ *of split epimorphisms in the category* $\mathbf{Lie}_R$ *of Lie* R*-algebras is equivalent to the category* $\mathbf{ActLie}_R$ *of triples* (B, X, f), *where* B *and* X *are Lie* R*-algebras and* $f : B \to \mathrm{Der}(X)$ *is a Lie algebra homomorphism. Here* $\mathrm{Der}(X)$ *denotes the Lie* R*-algebra of derivations on* X*; such a derivation is an* R*-module endomorphism* d *of* X *with* $d(xy) = d(x)y + xd(y)$, *and two derivations* d *and* e *are composed via* $(de)(x) = d(e(x)) - e(d(x))$. *Under this equivalence:*

(a) *a system* (A, B, p, s) *corresponds to the triple* $(B, \mathrm{Ker}(p), f)$ *in which* f *is defined by* $f(b)(x) = s(b)x$;

(b) *a triple* (B, X, f) *corresponds to the system* (A, B, p, s), *in which* A *is the* (*classical*) *semidirect product of* B *and* (X, f) *with the canonical* p *and* s*; that is* $A = B \times X$ *as an* R*-module with*

$$(b_1, x_1)(b_2, x_2) = (b_1 b_2, f(b_1)(x_2) - f(b_2)(x_1) + x_1 x_2).$$

Furthermore, we can copy the conclusion concerning internal actions made in 4.1; but again, the actions are to be presented as morphisms $f : B \times X \to X$ in **E** satisfying suitable equations. We could also make R internal or external, or even use two rings, one internal and another external.

5.2. Omitting the "Lie copy" of Section 4.2 and further details, let us just formulate the result analogous to Theorem 4.4:

Theorem. *Let* **E** *be a cartesian closed category, for which the category* $\mathbf{Lie}_R(\mathbf{E})$ *is exact and has finite coproducts. Then* $\mathbf{Lie}_R(\mathbf{E})$ *has representable object actions in the sense of Definition* 3.5 *with* $[X] = \mathrm{Der}(X)$ *for each* X *in* $\mathbf{Lie}_R(\mathbf{E})$.

Note also that $\mathbf{E} = \mathbf{Cat}_n$ is again an important example.

6. Split extension classifier

6.1. We return to the case of a general semi-abelian category $\mathbf{C}$. By a *split extension of B by X* we mean a diagram (A, p, s, k) where (A, p, s) is an object in $\mathrm{Pt}(B)$ and $k : X \to A$ is a kernel of $p : A \to B$. In fact these extensions form a category, a morphism from (A, p, s, k) to (A', p', s', k') being a morphism $\alpha : A \to A'$ with $p'\alpha = p$, $\alpha s = s'$, and $\alpha k = k'$, as in

$$\begin{tikzcd} X \arrow[r, "k"'] \arrow[d, equal] & A \arrow[r, "p"'] \arrow[d, "\alpha"'] & B \arrow[l, bend right, "s"'] \arrow[d, equal] \\ X \arrow[r, "k'"'] & A' \arrow[r, "p'"'] & B. \arrow[l, bend right, "s'"'] \end{tikzcd}$$

By the short five lemma, every such α is invertible; we write $\mathbf{SplExt}(B, X)$ for the set of isomorphism classes of such extensions.

From the equivalence $(F', G', \eta', \varepsilon') : \mathbf{C}^B \to \mathrm{Pt}(B)$ of 3.4, we easily obtain a bijection $\mathrm{SplExt}(B, X) \cong \mathrm{Act}(B, X)$; it sends (A, p, s, k) to the action $h : B \flat X \to X$, where h is the unique morphism for which $kh : B \flat X \to A$ equals the composite of $\kappa_{B,X} : B \flat X \to B + X$ and $[s, k] : B + X \to A$; and its inverse sends an action $h : B \flat X \to X$ to the diagram $(B \ltimes (X, h), \pi'_{(X,h)}, \iota'_{(X,h)}, k)$, where $B \ltimes (X, h)$, $\pi'_{(X,h)}$ and $\iota'_{(X,h)}$ are as in 3.4(a), while k is the (chosen) kernel of $\pi'_{(X,h)}$.

$\mathrm{Act}(B, X)$ is, for a fixed X, a contravariant functor of B: from a morphism $f : C \to B$ and an action $h : B \flat X \to X$ of B on X we get the action

$$C \flat X \xrightarrow{f \flat X} B \flat X \xrightarrow{h} X$$

of C on X. It is now easy to verify that:

6.2 Theorem. *The bijection* $\mathrm{SplExt}(B, X) \cong \mathrm{Act}(B, X)$ *extends to an isomorphism* $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$ *of functors* $\mathbf{C}^{\mathbf{op}} \to \mathbf{Sets}$, *where we make* $\mathrm{SplExt}(-, X)$ *a functor as follows: for $f : C \to B$, the function* $\mathrm{SplExt}(f, X) : \mathrm{SplExt}(B, X) \to \mathrm{SplExt}(C, X)$ *takes the diagram (A, p, s, k) to (A', p', s', k'), where*

$$\begin{tikzcd} A' \arrow[r, "p'"] \arrow[d, "g"'] & C \arrow[d, "f"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

is a pullback, while $s' : C \to A'$ is the unique morphism having $p's' = 1_C$ and $gs' = sf$, and $k' : X \to A'$ is the unique morphism having $p'k' = 0$ and $gk' = k$.

6.3. As follows from Theorem 6.2, when $\mathbf{C}$ is semi-abelian, it has representable object actions if and only if the functor $\mathrm{SplitExt}(-, X)$ is representable. Since a functor $\mathbf{C}^{\mathbf{op}} \to \mathbf{Sets}$ is representable if and only its category of elements has a terminal object, we obtain

Theorem. *A semi-abelian category* $\mathbf{C}$ *has representable object actions if and only if for every object* X *in* $\mathbf{C}$*, there exists an action* $[X] \flat X \to X$ *satisfying the following universal property:*

For every object B *in* $\mathbf{C}$ *and every split extension* (A, p, s, k) *of* B *with the kernel* X*, there is a unique morphism* $B \to [X]$ *such that there exists a morphism* $A \to [X] \ltimes X$ *making commutative all "squares" formed by corresponding arrows in the diagram*

$$\begin{tikzcd}
X \arrow[r, "k"'] \arrow[d, equal] & A \arrow[r, "p"'] \arrow[d] & B \arrow[l, bend right, "s"'] \arrow[d] \\
X \arrow[r, "k_X"'] & {[X] \ltimes X} \arrow[r, "p_X"'] & {[X]} \arrow[l, bend right, "s_X"']
\end{tikzcd}$$

where p_X*,* s_X*,* k_X *are the appropriate canonical morphisms that arise from the adjunctions* $\mathbf{C} \to \mathrm{Pt}([X])$ *and* $\mathbf{C}^{[X]} \to \mathrm{Pt}([X])$*.*

According to this universal property, we could call $[X]$ the *split extension classifier for* X, or call the bottom row in the diagram above the *generic split extension with the kernel* X.

7. Actors

Actors of crossed modules were introduced by K. Norrie [N], who also uses older work of J.H.C. Whitehead [W] and A.S.-T. Lue [L]. Norrie's actor $A(X)$ of a crossed module X is an example of our $[X]$; according to our approach its existence follows from Theorem 4.4. Note that Norrie's actors are defined via an explicit technical construction — not using any universal property; however Norrie shows that they satisfy a certain desired property, which is weaker than the universal property that defines our $[X]$. The same can be said about the actors of crossed modules of Lie algebras in the sense of J.M. Casas and M. Ladra [CL]. Yet, our $[X]$ is not really an abstract-categorical version of an actor; it is a rather special concept that exists for groups, Lie algebras, crossed modules (and various categorical structures internal to these), but not for associative algebras and other similar cases.

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