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On tilting and cotilting-type modules

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Dedicated to Vlastimil Dlab on the occasion of his 70^{th} birthday.

Abstract. We use modules of finite length to compare various generalizations of the classical tilting and cotilting modules introduced by Brenner and Butler [BrBu].

Keywords: tilting and cotilting modules, quivers and Auslander-Reiten quivers

Classification: 16G70, 16E70

Introduction

The first part of the announcement of the Tilting Tagung "Twenty years of tilting theory" says the following: "Tilting modules were born about twenty years ago in the context of finite dimensional algebras. Since then, tilting theory has spread in many different directions, ... "[1]

In this note we describe the answer to two questions by S. Bazzoni, on the so called n-tilting and n-cotilting modules studied in her papers [Ba2] and [Ba3]. Roughly speaking, the first question (i.e. Question A) deals with the gap between the modules which satisfy a word-by-word generalization (to modules of higher projective (resp. injective) dimension) of one of the two equivalent definitions of finitely presented tilting (resp. cotilting) modules of projective (resp. injective) dimension ≤ 1 (Theorem 4; Corollaries 6 and 7). The second question (i.e. Question B) deals with the gap between a cotilting-type module U of injective dimension > 1, and its injective envelope $E(U)$. As we shall see, $E(U)$ may be large enough so that the factor module $E(U)/U$ admits selfextensions (Theorem 5 and Proposition 8).

We stress the fact that all useful modules, with a tilting-type (resp. cotilting-type) behaviour constructed in the sequel, are actually injective (resp. projective) modules of finite length.

This paper is organized as follows. In Section 1 we fix the notation, and we recall some definitions of modules used in the sequel, and belonging to the large tilting and cotilting worlds. In Section 2 we collect all the proofs.

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[1](see `www.mathematik.uni-muenchen.de/~tilting`).

1. Definitions and notation

Throughout the paper, R denotes a ring and R-Mod denotes the class of all left R-modules. Moreover, if $M \in R$-Mod and λ is a cardinal, then $M^{(\lambda)}$ (resp. M^{λ}) denotes the direct sums (resp. products) of λ copies of M, and $\operatorname{Gen} M$ (resp. $\operatorname{Cogen} M$) denotes the class of all modules generated (resp. cogenerated) by M. Finally, $\operatorname{Add} M$ (resp. $\operatorname{Prod} M$) denotes the class of all modules isomorphic to summands of direct sums (resp. products) of copies of M, and $E(M)$ denotes the injective envelope of M.

Keeping the above notation and following the terminology of [Ba2], we say that an R-module T is an *n-tilting module* if the following conditions hold.

(T1) The projective dimension of T is at most n.
(T2) $\operatorname{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ.
(T3) There exists a long exact sequence of the form

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_n \to 0,$$

where $T_i \in \operatorname{Add} T$ for every $i = 0, \ldots, n$.

Moreover, we say that T is a *partial n-tilting module*, if T satisfies conditions (T1) and (T2). Dually, we say that an R-module U is an *n-cotilting module* if the following conditions hold.

(C1) The injective dimension of U is at most n.
(C2) $\operatorname{Ext}^i_R(U^{\lambda}, U) = 0$ for every $i \geq 1$ and every cardinal λ.
(C3) There exists a long exact sequence of the form

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to E \to 0,$$

where E is an injective cogenerator of R-Mod and $U_i \in \operatorname{Prod} U$ for every $i = 0, \ldots, n$.

Moreover, we say that U is a *partial n-cotilting module*, if U satisfies conditions (C1) and (C2). As in [Ba2], for every $M \in R$-Mod, we denote by $M^{\perp_\infty}$ and ${}^{\perp_\infty}M$ the following classes of modules:

$$M^{\perp_\infty} = \left\{ X \in R\text{-}\operatorname{Mod} \mid \operatorname{Ext}^i_R(M, X) = 0 \text{ for every } i \geq 1 \right\},$$
$${}^{\perp_\infty}M = \left\{ X \in R\text{-}\operatorname{Mod} \mid \operatorname{Ext}^i_R(X, M) = 0 \text{ for every } i \geq 1 \right\}.$$

We know from [C1, Theorem 3] that a finitely presented R-module T is a 1-tilting module iff T is a partial 1-tilting module and $\operatorname{Ker}\operatorname{Hom}_R(T, -) \cap \operatorname{Ker}\operatorname{Ext}^1_R(T, -) = 0$.

On the other hand, by [AnTT, Proposition 2.3], [CDT1, Proposition 1.7] and [CDT2], an R-module U is a 1-cotilting module iff U is a partial 1-cotilting module and $\operatorname{Ker}\operatorname{Hom}_R(-, U) \cap \operatorname{Ker}\operatorname{Ext}^1_R(-, U) = 0$.

A remark of [Ba2] points out that the following condition (T3') (resp. (C3')) holds for every n-tilting R-module T (resp. n-cotilting R-module U):

(T3') $\operatorname{Ker}\operatorname{Hom}_R(T,-)\cap T^{\perp_\infty}=0$.

(resp. (C3') $\operatorname{Ker}\operatorname{Hom}_R(-,U)\cap {}^{\perp_\infty}U=0$.)

A preliminary version of [Ba2] suggested the following question:

Question A: *Are there partial n-tilting* (*resp. n-cotilting*) *modules satisfying condition* (T3') (*resp.* (C3')), *but not condition* (T3) (*resp.* (C3))*?*

Our answer (Theorem 4) to Question A, mentioned in [Ba2, Example 1], is positive even for $n=2$ and for non faithful modules. Surprisingly enough, it actually occurs that a faithful summand of a 2-tilting (resp. 2-cotilting) module may satisfy condition (T3') (resp. (C3')) without being an n-tilting (resp. n-cotilting) module for any n (Corollary 7).

In addition to this, condition (T3') may hold for two summands of an n-tilting module T with quite different properties. For instance, exactly one of them may be both a faithful module and a maximal summand of T (Corollary 7).

A new important result by S. Bazzoni [Ba1, Theorem 2.8] states that 1-cotilting modules are pure-injective (see e.g. several characterizations of pure-injective modules in [JL, Theorem 7.1], [Hu, Theorems 1 and 5] and [R2, Sections 3 and 4]). Moreover, by [Ba3, Theorem 4.2 and Proposition 4.4], the pure-injectivity on an n-cotilting module U is equivalent to a property of its injective envelope $E(U)$. Since any 1-cotilting module C has the property that $E(C)/C$ is injective, it is natural to investigate the gap between n-cotilting modules U and their injective envelopes $E(U)$. In particular, the following question arises:

Question B: *Are there n-cotilting modules U such that the modules $E(U)/U$ are not partial* $(n-1)$*-cotilting modules?*

We shall see that for any $n\geq 2$ Question B has a positive answer obtained by means of projective modules (Theorem 5 and Proposition 8).

In the next sections K denotes an algebraically closed field, and all useful rings are K-algebras A of finite representation type, given by quivers and defined according to [R1]. Moreover, we always identify indecomposable left A-modules and their isomorphic classes. In particular, we describe the structure of these modules by means of some pictures, where the vertices of some quiver index the simple composition factors of these modules. If A admits only finitely many simple modules up to isomorphism, we say that an A-module M of finite length is *sincere* [AuReS, p. 317], if every simple A-module appears as a composition factor of M. Under the same hypotheses on M, we say that M is *multiplicity-free* [HR], if M is the direct sum of pairwise non-isomorphic indecomposable modules. Moreover, if $M\simeq\bigoplus_{i=1}^m M_i^{d_i}$, where $d_i>0$ for every i and $M_1,\dots,M_m$ are indecomposable with $M_k\not\simeq M_j$, for $i\neq j$, then we denote m by $\delta(M)$.

For unexplained representation-theoretic terminology, we refer to [AF], [AuReS] and [R1].

For new and old generalizations of tilting and cotilting modules, equivalences, dualities, approximations etc., see e.g. the following papers [AnT], [Ba1], [BK], [C2], [CbF], [D], [G], [ReR] (presented in both lectures and talks at the Venice Algebra Conference 2002), and all the references therein.

2. Proofs

In the next lemmas we collect some tricks used to construct more or less large modules of finite length needed in the sequel.

Lemma 1. *Let A be a finite dimensional K-algebra of finite representation type and of finite global dimension m. Then the following facts hold.*

(i) *The regular module ${}_AA$ is an n-cotilting module for some $n \leq m$.*
(ii) *The minimal injective cogenerator ${}_AD = \mathrm{Hom}_K(A_A, K)$ is an n-tilting module for some $n \leq m$.*

Proof: (i) By our assumptions on A, it suffices to note that the projective module ${}_AA$ is product-complete [KS, Theorem 4.1].

(ii) This is an immediate consequence of the fact that ${}_AD$ is an injective module over a left noetherian algebra. $\square$

Keeping all the notation of the introduction, we show that only reasonably large partial n-tilting (resp. n-cotilting) modules may satisfy condition (T3') (resp. (C3')).

Lemma 2. *Let R be a left perfect ring, and let M be a left R-module of finite length satisfying one of the following conditions:*

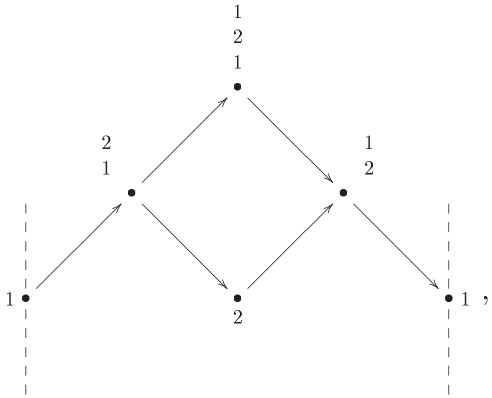
(a) $\mathrm{Ker}\,\mathrm{Hom}_R(M, -) \cap M^{\perp_\infty} = 0$;
(b) $\mathrm{Ker}\,\mathrm{Hom}_R(-, M) \cap {}^{\perp_\infty}M = 0$.

Then M is sincere.

Proof: Let S be a simple R-module, let P be the projective cover of S, and let E be the injective envelope of S. Since $P \in {}^{\perp_\infty}M$ and $E \in M^{\perp_\infty}$, we have either $\mathrm{Hom}_R(P, M) \neq 0$ or $\mathrm{Hom}_R(M, E) \neq 0$. Hence S is isomorphic to a composition factor of M. Consequently, M is sincere, as claimed. $\square$

The following example shows that the modules satisfying the hypotheses of Lemma 2 are not necessarily faithful.

Example 3. Let A be the K-algebra given by the quiver $\underset{1}{\bullet} \underset{b}{\overset{a}{\rightleftarrows}} \underset{2}{\bullet}$ with relation $ab = 0$, and let T and U be the modules $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}2\\1\end{smallmatrix}$ respectively. Since the Auslander-Reiten quiver is of the form



we have $\operatorname{Ext}^2_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, 2\right) \simeq \operatorname{Ext}^1_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, 1\right) \neq 0$ and $\operatorname{Ext}^2_A\left(2, \begin{smallmatrix}2\\1\end{smallmatrix}\right) \simeq \operatorname{Ext}^1_A\left(1, \begin{smallmatrix}2\\1\end{smallmatrix}\right) \neq 0$. Moreover, the simple module 2 is the unique indecomposable module belonging to $\operatorname{Ker}\operatorname{Hom}_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, -\right)$ (resp. $\operatorname{Ker}\operatorname{Hom}_A\left(-, \begin{smallmatrix}2\\1\end{smallmatrix}\right)$). Consequently, T (resp. U) satisfies condition (a) (resp. (b)) of Lemma 2.

By Lemma 2, the next result gives a positive answer to Question A by means very small modules.

Theorem 4. *There are a representation-finite K-algebra A of global dimension two, and indecomposable not faithful A-modules T and U with the following properties:*

(i) *T (resp. U) is a summand of a 2-tilting (resp. 2-cotilting) module;*
(ii) *T (resp. U) satisfies condition* (T3') (*resp.* (C3')), *but no proper submodule of T (resp. U) has this property.*

Proof: Let A, T and U be as in Example 3. Then the global dimension of A is equal to two, and T (resp. U) is a summand of the 2-tilting injective module $\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$ (resp. 2-cotilting projective module $\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$). By Lemmas 1 and 2, these observations and the remarks in Example 3 imply that (i) and (ii) hold. □

The next result gives a positive answer to Question B, by means of a 2-cotilting projective module admitting a reasonably large injective envelope.

Theorem 5. *There are a K-algebra A and a 2-cotilting A-module U with the following properties.*

(i) *$E(U)/U$ is semisimple.*
(ii) *Every simple A-module of injective dimension at most one is isomorphic to a summand of $E(U)/U$.*
(iii) $\operatorname{Ext}^1_A(E(U)/U, E(U)/U) \neq 0$.

Proof: Let A be the K-algebra given by the quiver

$$\begin{tikzcd}
 & \overset{1}{\bullet} \arrow[dl, "a"'] \arrow[dr, "c"] & \\
2\,\bullet \arrow[dr, "b"'] & & \bullet\,3 \arrow[dl, "d"] \\
 & \underset{4}{\bullet} &
\end{tikzcd}$$

with relations $ba = 0$ and $dc = 0$.

Then the following Auslander-Reiten quiver indicates that the global dimension of A is equal to two.

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}2\\4\end{smallmatrix} \arrow[dr] & & 3 \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
4 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\,3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[ur] \arrow[dr] & & 1 \\
 & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] & & 2 \arrow[ur] & & \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Hence, we deduce from Lemma 1 that the regular module ${}_AA = 4 \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\,3\end{smallmatrix}$ is a 2-cotilting module. Since $E(A)/A \cong 2 \oplus 3 \oplus 2 \oplus 3 \oplus 1$, it immediately follows that ${}_AA$ satisfies conditions (i), (ii) and (iii). $\square$

It is now easy to see that even faithful, and very special, partial 2-tilting (resp. 2-cotilting) modules satisfy condition (T3') (resp. (C3')), without being n-tilting (resp. n-cotilting) modules for any n.

Corollary 6. *Let A be a K-algebra of finite representation type and of global dimension two, and let ${}_AD = \mathrm{Hom}_K(A_A, K)$. Assume T and U are faithful A-modules with the following properties:*

(i) *T is a summand of D, $\mathrm{Gen}\, T = \mathrm{Gen}\, D$, and T satisfies condition* (T3');

(ii) *U is a summand of A, $\mathrm{Cogen}\, U = \mathrm{Cogen}\, A$, and U satisfies condition* (C3').

Then T (resp. U) is not necessarily a 2-tilting (resp. 2-cotilting) module. Moreover, T (resp. U) does not necessarily generate (resp. cogenerate) the modules belonging to $T^{\perp_\infty}$ (resp. ${}^{\perp_\infty}U$).

Proof: Let A be the algebra used to prove Theorem 5, and let T and U denote the following faithful modules:

$$T = \begin{smallmatrix}2\ 3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}, \qquad U = \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\ 3\end{smallmatrix}.$$

Since ${}_AD = T \oplus 1$ and ${}_AA = U \oplus 4$, we clearly have $\operatorname{Gen} T = \operatorname{Gen} D$ and $\operatorname{Cogen} U = \operatorname{Cogen} A$. On the other hand, $\operatorname{Ker}\operatorname{Hom}_A(T,-)$ and $\operatorname{Ker}\operatorname{Hom}_A(-,U)$ contain the indecomposable modules $4, \begin{smallmatrix}2\\4\end{smallmatrix}, \begin{smallmatrix}3\\4\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, 1$ respectively. To end the proof, we first note that

(1) $\operatorname{Ext}^1_A\left(\begin{smallmatrix}2\ 3\\4\end{smallmatrix}, 4\right) \neq 0$ and $\operatorname{Ext}^1_A\left(1, \begin{smallmatrix}1\\2\ 3\end{smallmatrix}\right) \neq 0$;

(2) $$\operatorname{Ext}^2_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\4\end{smallmatrix}\right) \cong \begin{array}{l} \diagup\ \operatorname{Ext}^1_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, 3\right) \neq 0 \\ \diagdown\ \operatorname{Ext}^1_A\left(3, \begin{smallmatrix}2\\4\end{smallmatrix}\right) \neq 0 \end{array};$$

(3) $$\operatorname{Ext}^2_A\left(\begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}3\\4\end{smallmatrix}\right) \cong \begin{array}{l} \diagup\ \operatorname{Ext}^1_A\left(\begin{smallmatrix}1\\3\end{smallmatrix}, 2\right) \neq 0 \\ \diagdown\ \operatorname{Ext}^1_A\left(2, \begin{smallmatrix}3\\4\end{smallmatrix}\right) \neq 0 \end{array}.$$

By the previous remarks, the conclusion that (i) and (ii) hold is an immediate consequence of (1), (2) and (3). Moreover, we clearly have

(4) $\begin{smallmatrix}1\\2\ 3\end{smallmatrix} \notin \operatorname{Gen} T, \qquad \begin{smallmatrix}2\ 3\\4\end{smallmatrix} \notin \operatorname{Cogen} U$;

(5) $\operatorname{Ext}^1_A\left(T, \begin{smallmatrix}1\\2\ 3\end{smallmatrix}\right) = 0, \qquad \operatorname{Ext}^1_A\left(\begin{smallmatrix}2\ 3\\4\end{smallmatrix}, U\right) = 0.$

Since the injective (resp. projective) dimension of $\begin{smallmatrix}1\\2\ 3\end{smallmatrix}$ (resp. $\begin{smallmatrix}2\ 3\\4\end{smallmatrix}$) is equal to one, we deduce from (4) and (5) that

(6) $\begin{smallmatrix}1\\2\ 3\end{smallmatrix} \in T^{\perp_\infty} \setminus \operatorname{Gen} T$ and $\begin{smallmatrix}2\ 3\\4\end{smallmatrix} \in {}^{\perp_\infty}U \setminus \operatorname{Cogen} U$.

Consequently, by (6) and [Ba2, Theorem 3.14], T (resp. U) is not an n-tilting (resp. n-cotilting) module for any n. The corollary is proved. □

Note that the partial n-tilting A-modules T (satisfying condition (T3'), without being n-tilting modules), constructed up to now, are actually maximal summands of multiplicity-free n-tilting modules. Hence these modules T are *almost complete tilting modules* (in the sense of [CoHU], [HU] and [BSo]). However, the next corollary shows that the modules with the above property are not necessarily almost complete tilting modules.

Corollary 7. *Let A be a K-algebra of finite representation type and of global dimension two. Assume T, T', U and U' are multiplicity-free A-modules with the following properties:*

(a) *T and T' (resp. U and U') are injective (resp. projective) modules satisfying condition* (T3') (*resp* (C3'))*, but no proper summand of these modules has this property;*

(b) *T and T' (resp. U and U') are not n-tilting (resp. n-cotilting) modules for any n;*

(c) $\delta(T) = \delta(U) = \delta({}_AA) - 1$.

Then we may have $\delta(T') < \delta(T)$ and $\delta(U') < \delta(U)$.

Proof: Let A be the algebra used to prove Theorem 5 and Corollary 6, and let T, T', U, U' be the following modules:

$$T = \begin{matrix}2\,3\\4\end{matrix} \oplus \begin{matrix}1\\3\end{matrix} \oplus \begin{matrix}1\\2\end{matrix}; \qquad T' = \begin{matrix}2\,3\\4\end{matrix} \oplus 1,$$

$$U = \begin{matrix}2\\4\end{matrix} \oplus \begin{matrix}3\\4\end{matrix} \oplus \begin{matrix}1\\2\,3\end{matrix}, \qquad U' = \begin{matrix}1\\2\,3\end{matrix} \oplus 4.$$

Since T' and U' are not faithful, the proof of Corollary 6 guarantees that (b) holds. On the other hand, we have $\delta({}_AA) = 4$, $\delta(T) = \delta(U) = 3$ and $\delta(T') = \delta(U') = 2$. Consequently, also (c) holds, and we have $\delta(T') < \delta(T)$ and $\delta(U') < \delta(U)$, as claimed. Since no indecomposable A-module is sincere, we deduce from Lemma 2 that

(1) no proper summand of T' (resp. U') satisfies condition (T3') (resp. (C3')).

We next observe that the indecomposable modules M such that $\mathrm{Hom}_A(T', M) = 0$ and $\mathrm{Ext}^1_A(T', M) = 0$ (resp. $\mathrm{Hom}_A(M, U') = 0$ and $\mathrm{Ext}^1_A(M, U') = 0$) are $\begin{smallmatrix}2\\4\end{smallmatrix}$ and $\begin{smallmatrix}3\\4\end{smallmatrix}$ (resp. $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix}$). Let $\{2,3\} = \{i,j\}$; then we have $\mathrm{Ext}^2_A\left(1, \begin{smallmatrix}i\\4\end{smallmatrix}\right) \simeq \mathrm{Ext}^1_A(1, j) \neq 0$ and $\mathrm{Ext}^2_A\left(\begin{smallmatrix}1\\i\end{smallmatrix}, 4\right) \simeq \mathrm{Ext}^1_A(j, 4) \neq 0$. These remarks and the definition of T' and U' guarantee that

(2) T' (resp. U') satisfies condition (T3') (resp. (C3')).

Finally, let X, Y, V, W denote the following modules:

$$X = \begin{matrix}2\,3\\4\end{matrix} \oplus \begin{matrix}1\\2\end{matrix}, \quad Y = \begin{matrix}2\,3\\4\end{matrix} \oplus \begin{matrix}1\\3\end{matrix}, \quad V = \begin{matrix}1\\2\,3\end{matrix} \oplus \begin{matrix}2\\4\end{matrix}, \quad W = \begin{matrix}1\\2\,3\end{matrix} \oplus \begin{matrix}3\\4\end{matrix}.$$

We first note that

(3) $\mathrm{Hom}_A\left(X, \begin{smallmatrix}3\\4\end{smallmatrix}\right) = 0$, $\mathrm{Hom}_A\left(Y, \begin{smallmatrix}2\\4\end{smallmatrix}\right) = 0$, $\mathrm{Hom}_A\left(\begin{smallmatrix}1\\3\end{smallmatrix}, V\right) = 0$ and $\mathrm{Hom}_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, W\right) = 0$.

Moreover, by [AuReS, Proposition 4.6], and by dimension shifting, we obtain

(4) $\mathrm{Ext}^i_A\left(X, \begin{smallmatrix}3\\4\end{smallmatrix}\right) = 0$, $\mathrm{Ext}^i_A\left(Y, \begin{smallmatrix}2\\4\end{smallmatrix}\right) = 0$, $\mathrm{Ext}^i_A\left(\begin{smallmatrix}1\\3\end{smallmatrix}, V\right) = 0$ and $\mathrm{Ext}^i_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, W\right) = 0$ for $i = 1, 2$.

By Lemma 2, we deduce from (3) and (4) that

(5) No proper summand of T (resp. U) satisfies condition (T3') (resp. (C3')).

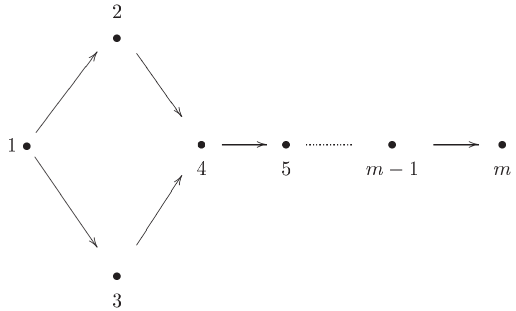
Putting (1), (2) and (5) together, we conclude that also property (a) holds. The proof is finished. $\square$

The next statement indicates that the gap between an n-cotilting module and its injective envelope may by reasonably large.

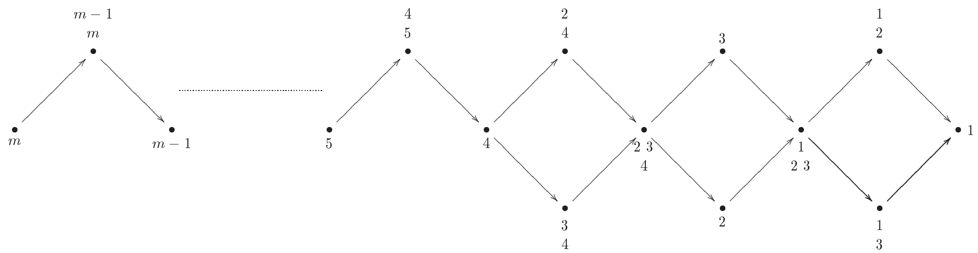
Proposition 8. *For any positive integer n, there are a K-algebra A and an n-cotilting A-module U such that*

$$\operatorname{Ext}_A^{n-1}\left(\frac{E(U)}{U}, \frac{E(U)}{U}\right) \neq 0.$$

Proof: By Theorem 5, it suffices to assume that $n \geq 3$. Next, let $m = n + 2$ and let A be the K-algebra given by the following quiver, with m vertices and m arrows $\alpha_1, \ldots, \alpha_m$ such that $\alpha_i \alpha_j = 0$ for every $i, j \in \{1, \ldots, m\}$.



Then it is easy to see that A has finite representation type. Moreover, the Auslander-Reiten quiver is of the form



Hence the global dimension of A (and the injective dimension of ${}_AA$) is equal to n. Consequently, we deduce from Lemma 1 that $U = {}_AA$ is an n-cotilting module. Moreover, the definition of A implies that $E(U)/U$ is a semisimple module of the form

(1) $E(U)/U \simeq 1 \oplus 2 \oplus 3 \oplus (m-1)$.

On the other hand, we obviously have

(2) $$\mathrm{Ext}_A^{n-1}(1, m-1) \simeq \mathrm{Ext}_A^{n-2}(1, m-2) \simeq \dots \\ \dots \simeq \mathrm{Ext}_A^3(1,5) \simeq \mathrm{Ext}_A^2(1,4) \simeq \mathrm{Ext}_A^1(1, 2\oplus 3) \neq 0.$$

Comparing (1) and (2), we obtain $\mathrm{Ext}_A^{n-1}\left(\frac{E(U)}{U}, \frac{E(U)}{U}\right) \neq 0$, as desired. □

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