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Ordinary selfdistributive rings

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Abstract. Left selfdistributive rings (i.e., $xyz = xyxz$) which are semidirect sums of boolean rings and rings nilpotent of index at most 3 are studied.

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1. Introduction

The present short note is an immediate continuation of [3] and the reader is referred to [3] as concerns terminology, notation, prerequisities, comments, further references, etc.

2. Preliminaries

In what follows, all rings are associative, possibly non–commutative and with or without unity.

If R is a ring, then $\mathrm{Id}(R)$ is the set of idempotent elements of R and $\mathrm{Nl}(R)$ that of nilpotent elements of R. The ring R will be called *id-generated*, if R is generated by the set $\mathrm{Id}(R)$ (as a ring). If A is a subset of R, then $(0 : A)_\mathrm{l} = \{r \in R \mid rA = 0\}$ and $(0 : A)_\mathrm{r} = \{r \in R \mid Ar = 0\}$. The subset A will be called *reduced*, if $A \cap \mathrm{Nl}(R) \subseteq \{0\}$.

A ring R is called *left selfdistributive* (an LD-ring) if it satisfies the equation $xyz = xyxz$. An id-generated LD-ring will be called an ILD-*ring* in the sequel.

2.1 Proposition ([2]). *Let R be an* LD*-ring, $I = \mathrm{Id}(R)$ and $N = \mathrm{Nl}(R)$. Then:*

- (i) $a^3 = a^n \in I$ *for all* $a \in R$ *and* $n \geq 4$;
- (ii) $2abc = 0$ *for all* $a, b, c \in R$ *(i.e.,* $2R^3 = 0$*)*;
- (iii) N *is an ideal of* R, $N^3 = 0$ *and* R/N *is a boolean ring;*
- (iv) $RN \cup NR \subseteq (0 : R)_\mathrm{l} \subseteq N$ *and* $NR \subseteq (0 : R)_\mathrm{r} \subseteq N$;
- (v) $S = R/(0 : R)_\mathrm{l}$ *is a commutative ring satisfying* $2S^2 = 0$ *and* $w^2 = w^3$, $w \in S$;
- (vi) $(0 : I)_\mathrm{l} = N$;

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(vii) *if R contains a right unity, then R is a boolean ring;*
(viii) $(0:R)_{\mathrm{r}}R \subseteq (0:R)_{\mathrm{l}} \cap (0:R)_{\mathrm{r}}$*;*
(ix) *if $R = R^2$, then $(0:R)_{\mathrm{r}} \subseteq (0:R)_{\mathrm{l}}$.*

2.2 Corollary. *A ring R is an* LD*-ring if and only if R satisfies the equations $xyz = yxz$ and $xyz = x^2yz$ (or $(x - x^2)yz = 0$).*

2.3 Lemma. *Let A be a generator set of an* LD*-ring R. Denote by K the ideal generated by $\{a - a^2 \mid a \in A\}$ and by L the left ideal generated by $\{ab - ba \mid a, b \in A\}$. Then $K + L = \mathrm{Nl}(R)$.*

Proof: Clearly, $J = K + L \subseteq \mathrm{Nl}(R)$ and J is an ideal. On the other hand, the factor-ring $S = R/J$ is generated by a set of pair-wise commuting idempotents. Consequently, S is a boolean ring and $\mathrm{Nl}(R) \subseteq K$. □

2.4 Remark. According to [2, 2.6], a subdirectly irreducible LD-ring is either nilpotent of index at most 3 or a two-element field or is isomorphic to the semigroup ring $\mathbb{Z}_2(T)$, T being a two-element semigroup of left units.

2.5 Lemma. *Let R be an* LD*-ring. Then:*

(i) $(a - a^2)^2 = a^2 - a^3 = (a - a^3)^2$ *for every $a \in R$;*
(ii) $(a - a^2)^3 = (a - a^3)^2 = (a^2 - a^3)^2 = 0$ *for every $a \in R$;*
(iii) $(a - a^2)bc = 0 = (a - a^3)bc$ *for all $a, b, c \in R$;*
(iv) $(a^2 - a^3)b = 0$ *for all $a, b \in R$.*

Proof: Use 2.1(i) and 2.2. □

2.6 Lemma. *The following conditions are equivalent for an* LD*-ring R:*

(i) $\mathrm{Nl}(R) \subseteq (0:R)_{\mathrm{l}}$*;*
(ii) $\mathrm{Nl}(R) = (0:R)_{\mathrm{l}}$*;*
(iii) $(a + a^2)b = 0$ *for all $a, b \in R$;*
(iv) $(a + a^3)b = 0$ *for all $a, b \in R$.*

Proof: Clearly, (i) is equivalent to (ii) and (ii) implies (iii) by 2.5(ii).

If (iii) is true, then $0 = (a + a^3)b = (a + a^2)b + (a^2 + a^3)b = (a + a^2)b$, since $a^2b = a^3b$.

Finally, if (iv) is true, and $a \in \mathrm{Nl}(R)$, then $a^3 = 0$ by 2.1(iii), and hence $ab = 0$ for every $b \in R$. □

An LD-ring satisfying the equations $2x = 0 = (x + x^2)y$ will be called an SILD-*ring*.

2.7 Proposition. *Let R be an* LD*-ring.*

(i) *If $R = R^2$, then R is an* SILD*-ring.*
(ii) *If R is an* SILD*-ring, then $\mathrm{Nl}(R) = (0:R)_{\mathrm{l}}$ (i.e., $\mathrm{Nl}(R)R = 0$) and $\mathrm{Nl}(R)^2 = 0$.*

Proof: (i) Firstly, $2R = 0$ follows from 2.1(ii). Next, by 2.2, $(a + a^2)bc = 0$ for all $a, b, c \in R$, and hence $(a + a^2)R = 0$.

(ii) Use 2.6. □

2.8 Corollary ([3]). *Every* ILD-*ring is an* SILD-*ring.*

2.9 Remark. It follows easily from [2, 2.6] that, up to isomorphism, the only subdirectly irreducible SILD-rings are the two-element field $\mathbb{Z}_2$, the semigroup ring $\mathbb{Z}_2(T)$ and the zero-multiplication ring $\mathbb{Z}_2^0$ defined on $\mathbb{Z}_2(+)$. The field $\mathbb{Z}_2$ has a unity, the ring $\mathbb{Z}_2(T)$ has just two left unities and the ring $\mathbb{Z}_2^0$ is isomorphic to a subring of $\mathbb{Z}_2(T)$.

2.10 Proposition. *Let an* LD-*ring R contain a left unity e. Then R is an* ILD-*ring.*

Proof: Denote by S the subring of R generated by the set $\mathrm{Id}(R)$. Clearly, R is an SILD-ring and, if $a \in \mathrm{Nl}(R)$, then $(a + e)^2 = a^2 + ae + ea + e = a + e$ (use also 2.1(vi)), $a + e \in \mathrm{Id}(R)$ and $a = (a + e) + e$. It follows that $\mathrm{Nl}(R) \subseteq S$. Finally, if $b \in R$, then $b = b^2 + (b + b^2)$ and we have $b^2 \in \mathrm{Id}(R) \subseteq S$ and $b + b^2 \in \mathrm{Nl}(R) \subseteq \mathrm{Id}(R)$. □

2.11 Theorem. *The following conditions are equivalent for an* LD-*ring R:*

(i) *R is an* SILD-*ring;*
(ii) *R is a subring of an* LD-*ring with left unity;*
(iii) *R is a subring of an* ILD-*ring;*
(iv) *R is a subring of an* LD-*ring S with $S = S^2$.*

Proof: (i) implies (ii) by 2.9, (ii) implies (iii) by 2.10, (iii) implies (iv) trivially and (iv) implies (i) by 2.8. □

3. Semidirect decompositions of LD-rings

3.1 Proposition ([3]). *Let R be an* LD-*ring.*

(i) *A subset A of R is a maximal reduced left ideal of R if and only if A is a maximal set of commuting idempotents.*
(ii) *If A is a maximal reduced left ideal of R, then $(0 : A)_l = \mathrm{Nl}(R)$, $A + \mathrm{Nl}(R)$ is an ideal and A contains every reduced right ideal of R.*

3.2 Lemma ([3]). *The following conditions are equivalent for a subset A of an* LD-*ring R:*

(i) *A is a reduced subring of R and every reduced left ideal is in $A + N$, where $N = \mathrm{Nl}(R)$;*
(ii) *A is a set of commuting idempotents and $R = A + N$;*
(iii) *A is a maximal reduced left ideal of R and $R = A + N$.*

Let R be an LD-ring. We will say that R is ordinary if $R = A + \mathrm{Nl}(R)$ for a (maximal) reduced left ideal A of R (see Lemma 3.2). Any such left ideal A will be called *critical*.

3.3 Theorem ([3])**.** *Let R be an LD-ring and $S = R/\mathrm{Nl}(R)$. Then R is ordinary in each of the following cases:*

(i) *R is countable;*
(ii) *the (boolean) factor-ring S is countable;*
(iii) *S is a ring direct sum of copies of the two-element field $\mathbb{Z}_2$;*
(iv) *R possesses a left (or right) unity;*
(v) *S possesses a unity;*
(vi) *the subring of R generated by the set $\mathrm{Id}(R)$ is ordinary.*

3.4 Remark. An example of a non-ordinary LD-ring R is given in [4] (see also [3, 5.2]). The ring R enjoys the properties $R = R^2$, $2R = 0$, $\mathrm{Nl}(R)^2 = 0$ and $\mathrm{Nl}(R)R = 0$.

3.5 Example ([2])**.** Let A denote the set of sequences $\alpha = (\alpha(0), \alpha(1), \dots) \in \mathbb{Z}_2^\omega$ such that at least one of the sets $\mathrm{supp}(\alpha) = \{i \mid i < \omega, \alpha(i) \neq 0\}$ and $\omega \setminus \mathrm{supp}(\alpha)$ is finite. Then A is a boolean ring with unit and R is an LD-ring, where $R = A \times B$, $B = \{\alpha \mid |\mathrm{supp}(\alpha)| < \omega\}$, $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac, ad)$ for all $a, c \in A$ and $b, d \in B$. Moreover $I = \{(b,b) \mid b \in B\}$ is a maximal reduced left ideal of R and I is not critical.

3.6 Example ([3])**.** Let A be an uncountable set and let S' denote the set of ordered pairs (F, f), where $F \subseteq A$, $|F| \leq 2$ and $f \in F$. Put $S = S' \cup \{o\}$, $o \notin S'$, and define a multiplication on S by $(F,f)(G,g) = (F \cup G, g)$ if $|F \cup G| \leq 2$, $(F,f)(G,g) = o$ otherwise and $\alpha o = o = o\alpha$ for every $\alpha \in S$. Then S becomes an idempotent semigroup satisfying $xyz = yxz$, o is an absorbing element of S and the corresponding contracted semigroup ring R of S over the two-element field $\mathbb{Z}_2$ is a non-ordinary LD-ring.

3.7 Remark. (i) The class of ordinary LD-rings is closed under homomorphic images.

(ii) According to [2], every subdirectly irreducible LD-ring is ordinary. Consequently, every LD-ring is a subring of an ordinary LD-ring.

Proof: (i) Obvious.

(ii) The assertion follows immediately from [2, 2.6]. □

3.8 Corollary. *Every LD-ring is a subring of an ordinary LD-ring.*

3.9 Proposition. *Let R be an LD-ring with a left unity e. Then R is ordinary and Re is a critical left ideal of R.*

Proof: For every $a \in R$, we have $a = ae + (a + ae)$, where $ae \in Re$ and $a + ae \in \mathrm{Nl}(R)$. Since $Re \subseteq \mathrm{Id}(R)$, Re is a reduced left ideal which is critical. □

3.10 Theorem (cf. 2.11)**.** *The following conditions are equivalent for an* LD-*ring:*

(i) *R is an ordinary* SILD-*ring;*
(ii) *R is an ideal of an* LD-*ring S with left unity such that $S/R \simeq \mathbb{Z}_2$;*
(iii) *R is a right ideal of an* LD-*ring with left unity.*

PROOF: (i) implies (ii). We have $R = A + N$, where $N = \mathrm{Nl}(R)$ and A is a (maximal) reduced left ideal of R. Then $A \subseteq \mathrm{Id}(R)$, $A \cap N = 0$ and we put $f(a + w) = a$ for all $a \in A$ and $w \in N$. Now, $d(a + w)f(b + v) = ab = f(ab + av) = f(a + w(b + w))$ and $f(a + w) + a + w = 2a + w = w \in N = (0 : R)_{\mathrm{l}}$ (2.7(ii)). It follows that f is a (ring) endomorphism of R such that $\mathrm{Ker}(f) = N$, $\mathrm{Im}(f) \subseteq \mathrm{Id}(R)$, $f(R)$ is a boolean ring, $f^2 = f$ and $f(x) + x \in (0 : R)_{\mathrm{l}}$ for every $x \in R$.

Put $S = R \times \mathbb{Z}_2$ and define an addition and a multiplication on S by $(a, i) + (b, j) = (a + b, i + j)$ and $(a, i)(b, j) = (ab + jf(a) + ib, ij)$ for all $a, b \in R$ and $i, j \in \mathbb{Z}_2$. One verifies readily that S becomes an LD-ring and that the element (0,1) is a left unity of S.

(ii) implies (iii). This implication is trivial.

(iii) implies (i). It follows from 2.10 that R is an SILD-ring. By 3.7(i), R is ordinary. □

4. Construction of ordinary selfdistributive rings

4.1 Construction (cf. [1])**.** Let A be a boolean ring and M an (associative) A-algebra nilpotent of index at most 3 and such that $AM^2 = 0$. Put $R = A \times M$ and define an addition and a multiplication on R by $(a, u) + (b, v) = (a + b, u + v)$ and $(a, u)(b, v) = (ab, av + uv)$. Then $R = \mathbb{R}(A, M)$ becomes an ordinary LD-ring, $\mathrm{Id}(R) = \{(a, u) \mid au = u\}$, $\mathrm{Nl}(R) = \{(0, u)\} \simeq M$ (as A-algebras), $A_1 = \{(a, 0)\} \simeq A$ (as rings), $A \simeq R/\mathrm{Nl}(R)$ and $R = A_1 + \mathrm{Nl}(R)$. Put $M_1 = (0 : M)_{M,\mathrm{r}} = \{u \in M \mid Mu = 0\}$; clearly, M_1 is an ideal of M and $AM_1 \subseteq M_1$.

(i) Let B be an ideal of A and $\rho : B \to M_1$ an A-module homomorphism. Then the set $\{(b, \rho(b)) \mid b \in B\}$ is a reduced left ideal of R and every reduced left ideal of R is of this type.

(ii) Every maximal reduced left ideal of R is critical if and only if the A-module ${}_AM_1$ is injective with respect to all imbeddings $B \subseteq A$, B being an ideal of A.

4.2 Theorem. (i) *If A is a boolean ring and M an A-algebra with $M^3 = 0 = AM^2$, then $R = \mathbb{R}(A, M)$ is an ordinary* LD-*ring.*

(ii) *Critical left ideals of R are just left ideals of the form $\{(a, \phi(a)) \mid a \in A\}$, $\phi :_A A \to_A M_1$ being a module homomorphism, $M_1 = (0 : M)_{M,\mathrm{r}}$.*
(iii) *Every maximal reduced left ideal of R is critical if and only if the A-module ${}_AM_1$ is injective with respect to all imbeddings $B \subseteq A, B$ being an ideal of A.*
(iv) *The A-algebras M and* $\mathrm{Nl}(R)$ *are isomorphic and the boolean rings A and $R/\mathrm{Nl}(R)$ are isomorphic.*

(v) *R is generated by* $\mathrm{Id}(R)$ *(as a ring) if and only if $AM = M$.*

Proof: See 4.1. □

4.3 Proposition. *$\mathbb{R}(A,M) \simeq \mathbb{R}(B,N)$ if and only if there exist ring isomorphisms $\rho : A \to B$ and $\lambda : M \to N$ such that $\lambda(au) = \rho(a)\lambda(u)$ for all $a \in A$ and $u \in M$.*

Proof: An easy exercise. □

4.4 Theorem. *Let R be an ordinary* LD*-ring, $N = \mathrm{Nl}(R)$ and $S = R/N$. Then:*

(i) *for every critical left ideal A, the mapping $a \to \overline{a} = a + N$, $a \in A$, is a ring isomorphism of A onto S and N becomes both an A-algebra and S-algebra;*
(ii) *$AN^2 = 0 = N^3$ and $SN^2 = 0 = N^3$;*
(iii) *$R \simeq \mathbb{R}(A,N)$ and $R \simeq \mathbb{R}(S,N)$;*
(iv) *every maximal reduced left ideal of R is critical if and only if the A-module (S-module, respectively) ${}_A(0 : N)_{N,\mathrm{r}}$ (${}_S(0 : N_{N,\mathrm{r}})$, resp.) is injective with respect to all imbeddings $B \subseteq A$ ($T \subseteq S$, resp.), B being an ideal of A (T an ideal of S, resp.);*
(v) *R is generated by* $\mathrm{Id}(R)$ *if and only if $AN = N$ (or $SN = N$).*

Proof: Combine 2.1 and 4.2. □

4.5 Corollary. *There exists a one-to-one correspondence between ordinary* LD*-rings and ordered pairs (A,M), A being a boolean ring and M an A-algebra with $M^3 = 0 = AM^2$.*

4.6 Remark. Let A be a boolean ring with unit. An A-module M is injective with respect to all the inclusions $B \subseteq A$, B being an ideal of A, if and only if the submodule $M_1 = AM$ is an injective A-module. Notice also that M_1 is a unitary module.

5. Free left selfdistributive rings

5.1 Construction. Let X be a non-empty set and let E be the set of ordered triples (U,u,i), where U is a finite subset of X, $u \in X$ and either $i = 0$ or $|U| = 1$ and $i = 1$. Now, define a multiplication on E by $(U,u,i)(V,v,j) = (U \cup V \cup \{u\}, v, k)$, where $k = 0$ for $U \cup V \neq \{u\}$ and $k = 1$ for $U \cup V = \{u\}$. Then E becomes a free left permutable LD-semigroup freely generated by the set $\{(\emptyset, x, 0) \mid x \in X\}$.

5.2 Construction. Let F be a free left permutable LD-semigroup freely generated by a non-empty set X and let $S = \mathbb{Z}[F]$ be the corresponding semigroup ring of S over the ring $\mathbb{Z}$ of integers. Clearly, S is a left permutable ring. Now, put $G = F \setminus \{x, xy \mid x, y \in X\}$, denote by I the ideal of S generated by $2G$ (notice that G is a subsemigroup of F) and by R the corresponding factor-ring S/I; let $\phi : S \to R$ be the natural projection. Clearly, $\phi \,|\, F$ is injective and we will identify F with its image $\phi(F)$. Now, it is easy to check that R is a free LD-ring over X. Every element of R can be written in a unique way as a sum

$$\sum_{x\in X} n_x + \sum_{x,y\in X} n_{(x,y)} + \sum_{g\in G} k_g$$

where $n_x, n_{(x,y)} \in \mathbb{Z}$, $k_g \in \mathbb{Z}_2 = \{0,1\}$ and only finitely many of these coefficients are non-zero. It follows from 3.3(i) and [3, 5.2] that R is ordinary if and only if X is countable.

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