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Finitely generated almost universal varieties of 0-lattices

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Dedicated to Professor Věra Trnková on her 70th birthday.

Abstract. A concrete category $\mathbb{K}$ is (algebraically) *universal* if any category of algebras has a full embedding into $\mathbb{K}$, and $\mathbb{K}$ is *almost universal* if there is a class $\mathcal{C}$ of $\mathbb{K}$-objects such that all non-constant homomorphisms between them form a universal category. The main result of this paper fully characterizes the finitely generated varieties of 0-lattices which are almost universal.

Keywords: (algebraically) universal category, finite-to-finite universal category, almost universal category, 0-lattice, variety of 0-lattices

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According to the results collected in the book [12] by A. Pultr and V. Trnková, (algebraic) universality of a concrete category $\mathbb{K}$ is equivalent to the existence of a full embedding $F : \mathbb{G} \longrightarrow \mathbb{K}$ of the category $\mathbb{G}$ of all graphs and all their compatible mappings into $\mathbb{K}$. Categorical structure of any universal category $\mathbb{K}$ is quite rich. For instance, for every monoid M, any such $\mathbb{K}$ has a proper class of pairwise non-isomorphic objects whose endomorphism monoids are all isomorphic to M, and arbitrarily large sets of such $\mathbb{K}$-objects with no morphisms between their distinct members. We say that a concrete category $\mathbb{K}$ is *finite-to-finite universal* if some full embedding $F : \mathbb{G} \longrightarrow \mathbb{K}$ assigns a finite $\mathbb{K}$-object FG to every finite graph G.

A concrete category $\mathbb{K}$ is *almost universal* if some faithful functor $F : \mathbb{G} \longrightarrow \mathbb{K}$ is *almost full*, in the sense that the non-constant $\mathbb{K}$-morphisms between $\mathbb{K}$-objects from the image of F form a category that is (algebraically) universal. The notion of almost universality is often suitable for classification of categorical properties of varieties of algebras having singleton subalgebras (such as varieties of lattices or monoids), for it says that the singleton algebras alone cause the failure of universality. This view is not entirely formal. For instance, no variety of monoids is universal because of the existence of the constant maps whose value is a monoid's unit element; yet the almost universal varieties of monoids were characterized in [6] by the same structural conditions as the universal varieties of semigroups in [5].

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Almost universality of a variety $\mathbb{K}$ of algebras can also be regarded as a special case of 'relative' universality. To illustrate this view, we begin by calling a class $\mathcal{Z}$ of $\mathbb{K}$-morphisms an *ideal of* $\mathbb{K}$ if for any $\mathbb{K}$-morphisms $f : A \longrightarrow C$ and $g : C \longrightarrow B$, the composite $g \circ f$ belongs to $\mathcal{Z}$ whenever f or g does. A nonvoid ideal $\mathcal{Z}$ of $\mathbb{K}$ may thus consist of all constant maps, or of all morphisms whose image belongs to a proper subvariety of $\mathbb{K}$, or to a union of proper subvarieties of $\mathbb{K}$. A faithful functor $F : \mathbb{L} \longrightarrow \mathbb{K}$ is *$\mathcal{Z}$-relatively full* if, first, $F\lambda \notin \mathcal{Z}$ for every $\mathbb{L}$-morphism $\lambda : a \longrightarrow b$ and, secondly, for any $\mathbb{K}$-morphisms $\kappa : Fa \longrightarrow Fb$ with $\kappa \notin \mathcal{F}$ there is an $\mathbb{L}$-morphism $\lambda : a \longrightarrow b$ for which $\kappa = F\lambda$. The variety $\mathbb{K}$ is *$\mathcal{Z}$-relatively universal* if there is a $\mathcal{Z}$-relatively full embedding $F : \mathbb{G} \longrightarrow \mathbb{K}$ of the category $\mathbb{G}$ of graphs into $\mathbb{K}$. Thus, for the empty ideal $\mathcal{Z} = \emptyset$, the $\mathcal{Z}$-universality of $\mathbb{K}$ is just its universality. If the ideal $\mathcal{Z}$ of a $\mathcal{Z}$-relatively universal variety $\mathbb{K}$ consists of all homomorphisms whose image belongs to some given subvariety $\mathbb{V}$ of $\mathbb{K}$, we say that $\mathbb{K}$ is *$\mathbb{V}$-relatively universal*. Hence the almost universality of a variety $\mathbb{K}$ is nothing else but the $\mathbb{T}$-relative universality for the trivial subvariety $\mathbb{T}$ of $\mathbb{K}$. Finally, if $\mathbb{K}$ is $\mathcal{Z}$-relatively universal for the ideal $\mathcal{Z}$ formed by all homomorphisms whose image belongs to the union of all proper subvarieties of $\mathbb{K}$, then $\mathbb{K}$ is *weakly var-universal*.

Observe that if $\mathbb{K}$ is almost universal due to the existence of a $\mathbb{T}$-relatively full functor $F : \mathbb{G} \longrightarrow \mathbb{K}$ having the additional property that the image $\mathrm{Im}(F\lambda)$ generates the variety $\mathbb{K}$ for every $\mathbb{G}$-morphism λ, then $\mathbb{K}$ is both $\mathbb{V}$-relatively universal for any proper subvariety $\mathbb{V}$ of $\mathbb{K}$ and also weakly var-universal.

Now we turn to the varieties of lattices.

Because of the idempotence of lattice operations and the consequent existence of constant lattice homomorphisms, the variety of all lattices and all their homomorphisms is not universal. On the other hand, the variety of all $(0,1)$-lattices (that is, lattices with the least element 0 and the greatest element 1) and all their $(0,1)$-homomorphisms is universal, as shown already in [4]. The ensuing extensive search for universal varieties of $(0,1)$-lattices was completed by the following result of [3].

Theorem 01 ([3]). *For a variety $\mathbb{V}$ of $(0,1)$-lattices, the following claims are equivalent:*

(1) *$\mathbb{V}$ is [finite-to-finite] universal;*
(2) *for every [finite] monoid M, the variety $\mathbb{V}$ contains a [finite] $(0,1)$-lattice L whose $(0,1)$-endomorphisms form a monoid isomorphic to M;*
(3) *$\mathbb{V}$ contains a [finite] $(0,1)$-lattice with no prime ideal;*
(4) *$\mathbb{V}$ contains a finitely generated [finite] non-distributive simple $(0,1)$-lattice.*

Thus, for instance, the variety of $(0,1)$-lattices generated by the five-element modular lattice $M_3 = \{0, a, b, c, 1\}$ with $0 < a, b, c < 1$ is finite-to-finite universal, while the $(0,1)$-lattices from the variety generated by the five element non-modular lattice N_5 in which $0 < a < b < 1$ and $0 < c < 1$ cannot represent all monoids by their $(0,1)$-endomorphisms. From [11], we recall an even stronger reason for the non-universality of the variety of distributive $(0,1)$-lattices: their $(0,1)$-endomorphisms determine the lattice itself up to an isomorphism or an anti-isomorphism.

Theorem 01 shows that augmenting bounded lattices by the two nullary operations 0 and 1 not only removes the constant homomorphisms, but also produces 'quite small' finite-to-finite universal varieties. The main result of this paper shows that the addition of the nullary operation 0 has a similar effect on finite-to-finite almost universality in the case of finitely generated varieties of 0-lattices.

Before stating our main result, we recall a few more facts indicating its wider context.

When viewed informally, unlike adding enough nullaries to obtain universality by removing the constant maps, the notion of almost universality disregards the constant maps altogether. Almost universal non-modular varieties exist, see [7], and even though the lattice variety generated by the modular lattice M_3 is not almost universal, the larger variety generated by the modular lattice $M_{3,3} = \{0, a, b, c, d, e, f, 1\}$ determined by $0 < a, b, c < d$ and $c < e, f$ and $d, e, f < 1$ is, see [8]. In [10] it was shown that the variety of 0-lattices generated by M_3 is almost universal, and this brought out the question of whether or not a variant of Theorem 01 quoted above holds also for varieties of 0-lattices. Our main result, Theorem 0 below, is a partial positive answer to this question.

Theorem 0. *For any finitely generated variety $\mathbb{V}$ of 0-lattices, these claims are equivalent:*

(1) $\mathbb{V}$ *is almost universal;*
(2) $\mathbb{V}$ *is finite-to-finite almost universal;*
(3) $\mathbb{V}$ *contains a non-distributive simple lattice;*
(4) $\mathbb{V}$ *contains a lattice having at least three elements and no prime ideal.*

The almost universality of a variety $\mathbb{K}$ of 0-lattices means, of course, that disregarding the constant $\mathbb{K}$-morphisms with the value 0 gives rise to a universal full subcategory.

Note the formal similarity between Theorem 0 and Theorem 01. For instance, M_3 generates an almost universal 0-variety while N_5 does not. Since the monoid of 0-endomorphisms of a distributive 0-lattice determines the lattice up to an isomorphism [13], the 0-variety generated by M_3 is a minimal almost universal 0-variety.

Since Theorem 0 is narrower in scope than Theorem 01, the question below is quite natural.

Problem 1. Does Theorem 0 extend also to all varieties of 0-lattices that are not finitely generated?

Since Theorem 0 also describes all finite-to-finite almost universal varieties of 0-lattices, it may become a suitable source of examples for questions about

Q-universality. According to Sapir [14], a quasivariety $\mathbb{Q}$ of algebras of a finite similarity type is *Q-universal* if the inclusion-ordered lattice $L(\mathbb{Q})$ of its subquasivarieties has the property that for any quasivariety $\mathbb{R}$ of algebras of a finite type, the lattice $L(\mathbb{R})$ is a quotient lattice of a sublattice of $L(\mathbb{Q})$. Just as with categorical universality, numerous instances of Q-universal varieties exist and are well documented by Adams and Dziobiak in [1] and elsewhere. Of particular interest was the result by Dziobiak [2] characterizing the Q-universal varieties of modular lattices as those which contain the variety generated by the lattice $M_{3,3}$.

Finite-to-finite universality and Q-universality are linked together by the remarkable Adams-Dziobiak Theorem [1]. It says that any finite-to-finite universal quasivariety of algebras of a finite type must be Q-universal (the converse implication is false). To further improve their result, Adams and Dziobiak asked whether a weaker form of categorical universality (such as finite-to-finite almost universality) would still imply Q-universality. Motivated by this question, in [9] we gave an example showing that the categorical hypothesis of finite-to-finite universality cannot be weakened to its natural extreme of weak var-relative finite-to-finite universality. Together with the fact that the lattice variety generated by $M_{3,3}$ is finite-to-finite almost universal while that generated by M_3 is not almost universal [8], this brought up the question asking whether the variety of 0-lattices generated by M_3 is Q-universal. In [10] it is shown that this is the case, and this leads to the following question.

Problem 2. Are all the varieties Theorem 0 speaks about also Q-universal?

Our proof of Theorem 0 modifies constructions from [3] to a form suitable for varieties of 0-lattices, and this determines the structure of the present paper. In particular, there are two parallel sections dealing with 'tall' and 'short' simple lattices, and their identical results are then jointly applied in the final section which completes the proof of Theorem 0.

1. The standing hypothesis and some notation

For sets X and A, let X^A denote the Cartesian power of X, that is, the set of all functions from A to X. If X is a lattice then the lattice operations on X^A are defined componentwise, of course.

Throughout the paper, we assume that

> K is a finitely generated non-distributive simple lattice whose every non-constant endomorphism preserves the bounds 0 and 1.

In particular, any finite non-distributive simple lattice satisfies this hypothesis. It is clear that K and all its Cartesian powers are $(0,1)$-lattices, and that for every $a \in A$ the projection $p_a : K^A \to K$ defined by $p_a(\phi) = \phi(a)$ is a lattice $(0,1)$-homomorphism.

We shall identify any natural number $n \geq 0$ with the set $\{0, 1, \ldots, n-1\}$. For any finite $n \geq 1$ and each $i \in n+1$ we define the element $\chi_i \in K^n$ by

$$\chi_i(j) = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j \geq i. \end{cases}$$

Then χ_n is the least element of K^n and χ_0 is the largest element of K^n. For any given $x \in K$, let $x^* \in K^n$ denote the constant mapping with the value x, and let $n - m$ denote the arithmetic difference for each $m \in n$. We define

$$x^*_{j,m} = (\chi_j \wedge x^*) \vee \chi_{j+m} \quad \text{for all } m \in n \text{ and } j \leq n - m.$$

Next, let I denote the $(0,1)$-sublattice of K^n consisting of all non-decreasing functions (that is, functions satisfying $\phi(i) \leq \phi(j)$ for all $i, j \in n$ with $i \leq j$). For $m \in n$, we define

$$I_m = \{\phi \in I \mid \forall i \in n - m \quad \phi(i) = 0 \text{ or } \phi(i+m) = 1\}.$$

Thus I_0 is the chain $\chi_n < \chi_{n-1} < \cdots < \chi_0$, for instance, and I_m is a $(0,1)$-sublattice of $I_{m'}$ for $m \leq m' \in n$.

For any given sublattice A of I, let $I_m A$ denote the sublattice of K^n generated by the set $I_m \cup A$.

For any $(0,1)$-lattices L_1 and L_2, the sublattice of $L_1 \times L_2$ consisting of all pairs (ϕ, ψ) with $\phi = 0$ or $\psi = 1$ will be denoted as L_1/L_2. Informally, the lattice L_1/L_2 is obtained by placing a copy of L_1 atop a copy of L_2 so that the least element $0 \in L_1$ is amalgamated with the greatest element $1 \in L_2$.

2. Tall simple lattices

In this section we assume that the simple finitely generated non-distributive lattice K whose all non-constant 0-endomorphisms preserve 0 and 1 has a chain $0 < a < b < c < 1$, and select and fix such a chain. It is easily seen that $|K| > 7$, and hence K has a set D^1 of generators such that $0, 1 \notin D^1$ and $|D^1| \geq 5$. We select and fix such a set D^1 and denote $D = D^1 \setminus \{b\}$. Then $|D| \geq 4$ and $D \cup \{b\}$ generates K. As in [3], we set $m = |D| + 2$ and $n = 4m + 7$.

We need to recall other notions and constructions from [3]. Let Δ be the set of all binary relations $\delta \subseteq D \times \{1, 2, \ldots, m\}$ such that for every $d \in D$ there exists $k \in \{1, 2, \ldots, m\}$ with $(d, k) \in \delta$ and there exists no $k \in \{1, 2, \ldots, m\}$ such that $(d, k), (d', k) \in \delta$ for distinct $d, d' \in D$. We may thus think of any $\delta \in \Delta$ as a relation whose opposite δ^{-1} is a partial map of $\{1, 2, \ldots, m\}$ onto D.

For $\delta \in \Delta$, denote $\delta(D) = \{k \in \{1, 2, \ldots, m\} \mid \exists\, d \in D \text{ with } (d, k) \in \delta\}$. Let $A(\delta)$ be the $(0,1)$-sublattice of K^n generated by the set

$$\{d^*_{4k,3} \mid (d, k) \in \delta\} \cup \{b^*, \beta\}$$

where $\beta(0) = a$, $\beta(i) = b$ for $i = 1, 2, \ldots, n-2$ and $\beta(n-1) = c$. Then $I_2A(\delta) \subseteq I$ and, according to Lemma 3.1 of [3], the sublattice $I_2A(\delta)$ of K^n has the property that $\phi(2) = b$ for every $\phi \in I_2A(\delta)$ with $\mathrm{Im}(\phi) \cap \{0, 1\} = \emptyset$.

For $\delta, \varepsilon \in \Delta$, let $L_{\delta,\varepsilon}$ denote the sublattice of $I_2A(\delta) \times I_2A(\varepsilon)$ consisting of all pairs (ϕ, ψ) such that

$$(\phi, \psi) \leq (\beta, \beta) \quad \text{or} \quad (\phi, \psi) \geq (\beta, \beta) \quad \text{or} \quad \phi = \beta \quad \text{or} \quad \psi = \beta.$$

Now we describe several properties of the lattices $I_2A(\delta)$ and $L_{\delta,\varepsilon}$.

Proposition 2.1. *Let $\delta, \varepsilon \in \Delta$. Then*

(1) *If there exists a lattice $(0, 1)$-homomorphism $f : I_2A(\delta) \to I_2A(\varepsilon)$ then $\delta(D) \subseteq \varepsilon(D)$.*

(2) *If $f : K \to I_2A(\delta)$ is a one-to-one lattice homomorphism, then there exist i and j such that $0 \leq j < i \leq n \neq i - j$, $f(0) = \chi_i$ and $f(1) = \chi_j$.*

(3) *If $f : K \to L_{\delta,\varepsilon}$ is a one-to-one lattice homomorphism, then there exist i and j such that $0 \leq j < i \leq n \neq i - j$ and either $f(0) = (\chi_i, \beta)$ and $f(1) = (\chi_j, \beta)$ or else $f(0) = (\beta, \chi_i)$ and $f(1) = (\beta, \chi_j)$.*

(4) *If $(\phi, \psi) \in L_{\delta,\varepsilon}$ is such that $(\chi_i, \beta) \wedge (\phi, \psi) = (\chi_n, \chi_n)$ or $(\beta, \chi_i) \wedge (\phi, \psi) = (\chi_n, \chi_n)$ for some $i < n$, then $(\phi, \psi) = (\chi_n, \chi_n)$.*

(5) *If $f : L_{\delta,\varepsilon} \to L_{\delta',\varepsilon'}$ is a lattice 0-homomorphism for some $\delta', \varepsilon' \in \Delta$, then either f is the constant mapping or f is a lattice $(0, 1)$-homomorphism satisfying*

$$f\{(\chi_n, \beta), (\beta, \chi_n)\} = \{(\chi_n, \beta), (\beta, \chi_n)\} \quad \text{and}$$
$$f\{(\chi_0, \beta), (\beta, \chi_0)\} = \{(\chi_0, \beta), (\beta, \chi_0)\}.$$

(6) *For $\delta', \varepsilon' \in \Delta$ such that $\delta(D) \not\subseteq \delta'(D) \cup \varepsilon'(D)$, any lattice 0-homomorphism $f : L_{\delta,\varepsilon} \to L_{\delta',\varepsilon'}$ is constant.*

(7) *For $\delta', \varepsilon' \in \Delta$ such that $\delta \subseteq \delta'$, $\varepsilon \subseteq \varepsilon'$, $\delta(D) \not\subseteq \varepsilon'(D)$, and $\varepsilon(D) \not\subseteq \delta'(D)$, the lattice $L_{\delta,\varepsilon}$ is a $(0, 1)$-sublattice of $L_{\delta',\varepsilon'}$ and the only $(0, 1)$-homomorphism from $L_{\delta,\varepsilon}$ to $L_{\delta',\varepsilon'}$ is the inclusion mapping.*

(8) *For $\delta', \varepsilon' \in \Delta$ such that $\delta \subseteq \delta'$, $\varepsilon \subseteq \varepsilon'$, $\delta(D) \not\subseteq \varepsilon'(D)$, and $\varepsilon(D) \not\subseteq \delta'(D)$, if $f : L_{\delta,\varepsilon}/L_{\delta,\varepsilon} \to L_{\delta',\varepsilon'}$ is a lattice $(0, 1)$-homomorphism then $f(0, 1) \in \{0, 1\}$.*

Proof: Claim (1) is Statement 3.2 in [3].

We prove (2). Let $f : K \to I_2A(\delta)$ be a one-to-one lattice homomorphism. For the sake of brevity, the restriction of the i-th projection $p_i : K^n \to P$ to the sublattice $I_2A(\delta) \subseteq K^n$ will be also denoted as p_i.

Since $I_2A(\delta)$ is a subdirect power of K and because f is one-to-one, the composite $p_i \circ f : K \to K$ is non-constant for some $i \in n$. But then $p_i \circ f(0) = 0$ and $p_i \circ f(1) = 1$ for any such i, by the standing hypothesis.

Denote $i_1 = \min\{i \in n \mid p_i \circ f \text{ is not constant}\}$, $i_2 = \max\{i \in n \mid p_i \circ f$ is not constant$\}$. Thus $i_1 \le i_2$, $f(0)(i) = p_i \circ f(0) = 0$ and $f(1)(i) = p_i \circ f(1) = 1$ for $i \in \{i_1, i_2\}$. From $I_2A(\delta) \subseteq I$ it follows that $f(0)(i) = 0$ for all $i \le i_2$ and $f(1)(i) = 1$ for all $i \ge i_1$. Since $p_i \circ f$ is the constant mapping for all i with $i < i_1$ or $i > i_2$ it follows that $f(0) = \chi_{i_2+1}$ and $f(1) = \chi_{i_1}$. Write $i = i_2+1$ and $j = i_1$. Then $0 \le j < i \le n$, and it remains to show that $i - j < n$. To see this, suppose that $j = 0$ and $i = n$. Then $f : K \longrightarrow I_2A(\delta)$ is a $(0,1)$-homomorphism, and hence $p_i \circ f$ is a $(0,1)$-endomorphism of K for every $i \in n$, and $p_i \circ f$ is one-to-one because K is simple. Thus if $x \in K \setminus \{0,1\}$ then $f(x)(i) = p_i \circ f(x) \in K \setminus \{0,1\}$ for every $i \in n$. The property of $I_2A(\delta)$ from [3] quoted just below the definition of $I_2A(\delta)$ then gives $f(x)(2) = b$ for all $x \in K \setminus \{0,1\}$. Therefore $\mathrm{Im}(p_2 \circ f) = \{0, b, 1\}$, contradicting the fact that K is simple. Thus $i - j < n$, and (2) holds.

To prove (3), let $\pi_1 : L_{\delta,\varepsilon} \longrightarrow I_2A(\delta)$ and $\pi_2 : L_{\delta,\varepsilon} \longrightarrow I_2A(\varepsilon)$ be the restrictions of the respective projections to the sublattice $L_{\delta,\varepsilon}$ of $I_2A(\delta) \times I_2A(\epsilon)$.

Assume that $f : K \longrightarrow L_{\delta,\varepsilon}$ is a one-to-one lattice homomorphism.

Then $\pi_1 \circ f$ or $\pi_2 \circ f$ is not constant and thus $\pi_1 \circ f$ or $\pi_2 \circ f$ is a one-to-one lattice homomorphism because K is simple. Assume that $\pi_1 \circ f$ is a one-to-one lattice homomorphism. By (2), for some $0 \le j < i \le n \ne i-j$ we have $\pi_1 \circ f(0) = \chi_i$ and $\pi_1 \circ f(1) = \chi_j$. Observe that χ_k is incomparable to $\beta = (a, b, \dots, b, c)$ for each $k \in \{1, 2, \dots, n-1\}$, so that the condition defining $L_{\delta,\epsilon}$ implies that $f(1) = (\chi_j, \beta)$ if $j > 0$, and $f(0) = (\chi_i, \beta)$ if $i < n$. To complete the proof, next suppose that $f(0) = (\chi_n, \lambda)$ for some $\lambda \ne \beta$. Since $\chi_n < \beta$ and $\lambda \ne \beta$, the definition of $L_{\delta,\epsilon}$ implies that $\lambda < \beta$. We also have $f(1) = (\chi_j, \kappa)$ for some $0 < j < n$ and $\kappa = \pi_2 \circ f(1) \ge \pi_2 \circ f(0) = \lambda$. If $\kappa = \lambda$ then $\lambda < \beta$ implies that $\chi_j \le \beta$, a contradiction because $j > 0$. Suppose that $\kappa > \lambda$. Then (2) implies that $\kappa = \chi_k$ for some $k < n$. But then $f(1) = (\chi_j, \chi_k)$ with $j, k < n$, and this element belongs to $L_{\delta,\epsilon}$ only when $j = 0$. This contradicts (2), and hence $f(0) = (\chi_n, \beta)$, after all. A dual argument applies when $j = 0$, and the remainder of (3) follows by symmetry.

Let $(\phi, \psi) \in L_{\delta,\varepsilon}$ satisfy $(\chi_i, \beta) \wedge (\phi, \psi) = (\chi_n, \chi_n)$ for some $i < n$. Then $\chi_i(n-1) = 1$, $\chi_n(n-1) = 0$, and thus $\phi(n-1) = 0$, and from $I_2A(\delta) \subseteq I$ we conclude that $\phi = \chi_n$. The definition of $L_{\delta,\varepsilon}$ yields $\psi \le \beta$, and $\psi \wedge \beta = \chi_n$ implies that $\psi = \chi_n$. If $(\beta, \chi_i) \wedge (\phi, \psi) = (\chi_n, \chi_n)$ for some $i < n$ then a symmetric argument shows that $(\phi, \psi) = (\chi_n, \chi_n)$. The proof of (4) is complete.

To prove (5), assume that $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ is any lattice 0-homomorphism, and denote

$$S = \{(\chi_i, \beta) \mid 0 < i \le n\} \cup \{(\beta, \chi_i) \mid 0 < i \le n\}.$$

Suppose first that $f(\chi_n, \beta) \notin S$. For each $i \in n$ and all $x \in K$ set $g_i(x) = (x^*_{i,1}, \beta)$. Then $g_i : K \longrightarrow L_{\delta,\epsilon}$ is a one-to-one lattice homomorphism such that $g_i(0) = (\chi_{i+1}, \beta)$, $g_i(1) = (\chi_i, \beta)$. We have $f \circ g_{n-1}(0) = f(\chi_n, \beta) \notin S$ and hence, by (3), the composite $f \circ g_{n-1}$ must be constant. But then $f(\chi_{n-1}, \beta) =$

$f \circ g_{n-2}(0) = f \circ g_{n-1}(1) = f(\chi_n, \beta) \notin S$, and the repeated use of (3) yields $f(\chi_0, \beta) = f(\chi_n, \beta)$.

Next we show that $f(\chi_0, \beta) = f(\chi_n, \beta)$ only when f is constant. From $(\beta, \chi_n) \wedge (\chi_n, \beta) = (\chi_n, \chi_n)$ and $(\chi_0, \beta) > (\beta, \chi_n)$ it follows that $f(\beta, \chi_n) = f(\chi_n, \chi_n) = (\chi_n, \chi_n)$ because f preserves the zero (χ_n, χ_n). Using injective homomorphisms $h_i : K \longrightarrow L_{\delta,\epsilon}$ given by $h_i(x) = (\beta, x^*_{i,1})$ for $i \in n$ and (3), we find that $f(\beta, \chi_0) = (\chi_n, \chi_n)$. But then from $(\chi_n, \beta) < (\beta, \chi_0)$ and $(\beta, \chi_0) \vee (\chi_0, \beta) = (\chi_0, \chi_0)$ it follows that f is the constant mapping with the value (χ_n, χ_n). By symmetry, this is also true when $f(\beta, \chi_n) \notin S$. Therefore $f(\chi_n, \beta), f(\beta, \chi_n) \in S$ for any non-constant f.

Suppose that f is non-constant, and that $f(\chi_n, \beta) = (\chi_i, \beta)$ for some $i < n$. Then $(\chi_n, \beta) \wedge (\beta, \chi_n) = (\chi_n, \chi_n)$ and (4) imply that $f(\beta, \chi_n) = (\chi_n, \chi_n)$ and, since $(\chi_n, \chi_n) \notin S$, this is impossible. The same reasoning shows that $f(\chi_n, \beta) \neq (\beta, \chi_i)$ for $i < n$ and, by symmetry, $f(\beta, \chi_n) \neq (\chi_i, \beta), (\beta, \chi_i)$ for $i < n$ as well; hence $f(\chi_n, \beta), f(\beta, \chi_n) \in \{(\chi_n, \beta), (\beta, \chi_n)\}$. And $f(\chi_n, \beta) \neq f(\beta, \chi_n)$ because $(\chi_n, \beta) \wedge (\beta, \chi_n) = (\chi_n, \chi_n)$ and f is a 0-homomorphism. Therefore $f\{(\chi_n, \beta), (\beta, \chi_n)\} = \{(\chi_n, \beta), (\beta, \chi_n)\}$.

Consider the case of $f(\chi_n, \beta) = (\chi_n, \beta)$. Then $f(\beta, \chi_n) = (\beta, \chi_n)$ and, using (3) and the mappings g_i and h_i defined earlier, we find that $f(\chi_0, \beta) = (\chi_i, \beta)$ and $f(\beta, \chi_0) = (\beta, \chi_j)$ for some $i, j \leq n$. From $(\chi_0, \beta) > (\beta, \chi_n)$ it follows that $(\chi_i, \beta) > (\beta, \chi_n)$ and hence $i = 0$. Analogously, $(\beta, \chi_0) > (\chi_n, \beta)$ yields $j = 0$ and thus $f(\chi_0, \chi_0) = (\chi_0, \chi_0)$. Therefore f is a lattice $(0,1)$-homomorphism and $f\{(\chi_0, \beta), (\beta, \chi_0)\} = \{(\beta, \chi_0), (\chi_0, \beta)\}$. If $f(\chi_n, \beta) = (\beta, \chi_n)$ then, in the same manner, we find that $f(\beta, \chi_n) = (\chi_n, \beta)$, $f(\chi_0, \beta) = (\beta, \chi_0)$, $f(\beta, \chi_0) = (\chi_0, \beta)$, $f(\chi_0, \chi_0) = (\chi_0, \chi_0)$. Thus f is again a lattice $(0,1)$-homomorphism satisfying $f\{(\chi_0, \beta), (\beta, \chi_0)\} = \{(\beta, \chi_0), (\chi_0, \beta)\}$. The proof of (5) is complete.

Assume that $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ is a lattice 0-homomorphism for some $\delta', \varepsilon' \in \Delta$ with $\delta(D) \not\subseteq \delta'(D) \cup \varepsilon'(D)$. If non-constant, then, by (5), f is a $(0,1)$-lattice homomorphism with $f\{(\chi_n, \beta), (\beta, \chi_n)\} = \{(\beta, \chi_n), (\chi_n, \beta)\}$ and $f\{(\chi_0, \beta), (\beta, \chi_0)\} = \{(\beta, \chi_0), (\chi_0, \beta)\}$. Suppose that $f(\chi_n, \beta) = (\chi_n, \beta)$. Then, using (3) and (5), we obtain $f(\chi_0, \beta) = (\chi_0, \beta)$. The interval A from (χ_n, β) to (χ_0, β) in $L_{\delta,\varepsilon}$ is a sublattice of $L_{\delta,\varepsilon}$ isomorphic to $I_2 A(\delta)$. The interval A' from (χ_n, β) to (χ_0, β) in $L_{\delta',\varepsilon'}$ is a sublattice of $L_{\delta',\varepsilon'}$ isomorphic to $I_2(A(\delta'))$. We have $f(A) \subseteq A'$, and thus the domain-range restriction g of f is a lattice $(0,1)$-homomorphism from A to A'. Then $\delta(D) \subseteq \delta'(D)$, by (1), and this contradicts $\delta(D) \not\subseteq \delta'(D) \cup \varepsilon'(D)$. Thus $f(\chi_n, \beta) \neq (\chi_n, \beta)$. If $f(\chi_n, \beta) = (\beta, \chi_n)$ then, in the same manner, we obtain that $\delta(D) \subseteq \varepsilon'(D)$ and this is again a contradiction. Therefore $f(\chi_n, \beta) \notin \{(\chi_n, \beta), (\beta, \chi_n)\}$ and f is the constant mapping with the value (χ_n, χ_n). The proof of (6) is complete.

Claim (7) is proved in [3] as Statement 3.4.

For (8), let $f : L_{\delta,\varepsilon}/L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ be a lattice $(0,1)$-homomorphism. The sublattice A of $L_{\delta,\varepsilon}/L_{\delta,\varepsilon}$ consisting of pairs $(0, \psi)$ is isomorphic to $L_{\delta,\varepsilon}$; let g be the restriction of f to A. Clearly, g is a lattice 0-homomorphism from A to $L_{\delta',\varepsilon'}$. By Statement 3.5 in [3], the element $f(0,1) = g(0,1)$ must be comparable to $(\beta,\beta) \in L_{\delta',\varepsilon'}$. If $f(0,1) = g(0,1) \leq (\beta,\beta)$ then, by (5), g is the constant mapping with the value 0 and hence $f(0,1) = g(0,1) = 0$. If $f(0,1) = g(0,1) \geq (\beta,\beta)$ then, by (5) again, g is a lattice $(0,1)$-homomorphism and $f(0,1) = g(0,1) = 1$. The proof is complete. □

Now we apply Proposition 2.1 to assemble the building blocks of subsequent constructions.

Definition. In any finitely generated simple lattice K of length at least three, we select a generating set D[1] so that $|D| \geq 4$. Recalling that $m = |D| + 2$, we choose distinct relations $\delta, \epsilon, \nu, \nu' \subseteq D \times \{1, \dots, m\}$ from Δ so that $|\delta(D)| = |\varepsilon(D)| = |\nu(D)| = |\nu'(D)| = |D|$, $\delta(D), \varepsilon(D), \nu(D), \nu'(D) \subseteq \{1, 2, \dots, |D| + 1\}$, and $\delta'(D) = \delta(D) \cup \{|D| + 2\}$, $\varepsilon'(D) = \varepsilon(D) \cup \{|D| + 2\}$. The six relations then satisfy $\delta \subset \delta'$, $\varepsilon \subset \varepsilon'$, $\delta(D) \nsubseteq \varepsilon'(D)$, $\varepsilon(D) \nsubseteq \delta'(D)$, $\delta(D) \nsubseteq \nu(D) \cup \nu'(D)$, $\varepsilon(D) \nsubseteq \nu(D) \cup \nu'(D)$, $\nu(D) \nsubseteq \delta'(D) \cup \varepsilon'(D)$, and $\nu'(D) \nsubseteq \nu(D)$.

Using Proposition 2.1(1), (5), (6), (7), and (8), we obtain

Statement 2.2. *The lattices $L_0 = L_{\delta,\epsilon}$, $L_1 = L_{\delta',\epsilon}$, $L_2 = L_{\epsilon',\delta}$, $L_3 = L_{\delta',\epsilon'}$ and $L_{\nu,\nu'}$ and their sublattices have the following properties:*

(1) *L_1 and L_2 are distinct proper $(0,1)$-sublattices of L_3 and L_0 is a proper $(0,1)$-sublattice of $L_1 \cap L_2$;*
(2) *for $i, j \in 4$, any lattice 0-homomorphism $L_i \longrightarrow L_j$ is either constant or one of the $(0,1)$-inclusions from (1);*
(3) *if $f : L_0/L_0 \longrightarrow L_3$ is a $(0,1)$-homomorphism then $f(0,1) \in \{0,1\}$;*
(4) *only constant 0-homomorphisms exist between any lattice L_i with $i \in 4$ and the lattice $L_{\nu,\nu'}$;*
(5) *there is no lattice $(0,1)$-homomorphism from $I_2A(\delta)$ or $I_2A(\varepsilon)$ to $I_2A(\nu)$ or $I_2A(\nu')$;*
(6) *the only lattice $(0,1)$-endomorphism of $L_{\nu,\nu'}$ is the identity map.*

3. Short simple lattices

This section deals with simple lattices of length at most three. We begin by restating Propositions 4.1 and 4.2 from [3].

Proposition 3.1 ([3]). *Let L be a finite lattice of length at most 3 that has no prime ideal. Then the variety $\mathrm{Var}(L)$ generated by L contains one of the four lattices in Figure 1. Each lattice K of Figure 1 is simple, and has a complemented pair $\{b, d\}$.*

Proposition 3.1 allows us to restrict our attention to the short simple lattices of Figure 1. In this section, the symbol K will denote any one of these four lattices.

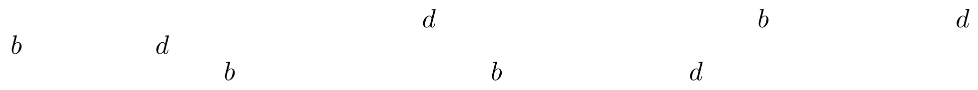

FIGURE 1. The four short lattices

Let Δ be the set of all non-empty subsets of the set $\{0,1,2,3\}$. Set $n=6$, and for $\delta \in \Delta$, let $B(\delta)$ denote the $(0,1)$-sublattice of K^6 generated by the set

$$\{d^*_{j,2} \mid j \in 5\} \cup \{d^*_{j,3} \mid j \in \delta\} \cup \{b^*\},$$

where b and d are the elements shown in Figure 1. Clearly $B(\delta) \subseteq I$ and, since $\{b,d\} \subset K$ is a complemented pair, for every $\alpha \in B(\delta)$ and every $i \in 6$ we have $\alpha(i) \in \{0,b,d,1\}$. The sublattice $I_1B(\delta)$ of K^6 generated by the set $I_1 \cup B(\delta)$ is contained in I. We need another property of the lattice $I_1B(\delta)$; it is an easy consequence of Lemma 2.5 in [3].

Observation. *If $\phi \in I_1B(\delta)$ satisfies $\phi(i) \in K \setminus \{0,1\}$ for every $i \in 6$, then $\phi(1) \in \{b,d\}$.*

PROOF: Indeed, Lemma 2.5 in [3] applied to $I_1B(\delta)$ says that for every $\phi \in I_1B(\delta)$ there exist $\alpha_0 \le \alpha_1 \le \dots \le \alpha_4$ in $B(\delta)$ such that $\phi(j) \le \alpha_j(j) \le \alpha_j(j+1) \le \phi(j+1)$ for $j=0,\dots,4$. So let $\phi \in I_1B(\delta)$ satisfy $0 < \phi(i) < 1$ for all $i \in 6$. Then $0 < \alpha_j(j) \le \alpha_j(j+1) < 1$, and $\alpha_j(j) = \alpha_j(j+1) \in \{b,d\}$ for all $j = 0,\dots,4$ follows because b and d are incomparable. Suppose that $\alpha_0(0) = b$. Then $b = \alpha_0(0) \le \phi(1) \le \alpha_1(1) < 1$, and $\alpha_1(1) = b$ follows because $\alpha_1(1) \in \{b,d\}$ and the elements b,d are incomparable. But then $b = \alpha_0(0) \le \phi(1) \le \alpha_1(1) = b$, and hence $\phi(1) = b$. The case of $\alpha_0(0) = d$ is similar. $\square$

For $\delta, \varepsilon \in \Delta$, let $L_{\delta,\varepsilon}$ be the sublattice of $I_1B(\delta) \times I_1B(\varepsilon)$ consisting of pairs (ϕ,ψ) such that

$$(\phi,\psi) \le (b^*,b^*) \quad \text{or} \quad (\phi,\psi) \ge (b^*,b^*) \quad \text{or} \quad \phi = b^* \quad \text{or} \quad \psi = b^*.$$

Below is a result analogous to Proposition 2.1.

Proposition 3.2. *Let $\delta, \varepsilon \in \Delta$. Then*

(1) *If there exists a lattice $(0,1)$-homomorphism $f : I_1B(\delta) \to I_1B(\varepsilon)$ then $\delta \subseteq \varepsilon$.*
(2) *If $f : K \to I_1B(\delta)$ is a one-to-one lattice homomorphism, then there exist i and j such that $0 \le j < i \le 6 \ne i-j$, $f(0) = \chi_i$ and $f(1) = \chi_j$.*

(3) *If* $f : K \longrightarrow L_{\delta,\varepsilon}$ *is a one-to-one lattice homomorphism, then there exist* i *and* j *such that* $0 \leq j < i \leq 6 \neq i - j$ *and either* $f(0) = (\chi_i, b^*)$ *and* $f(1) = (\chi_j, b^*)$ *or else* $f(0) = (b^*, \chi_i)$ *and* $f(1) = (b^*, \chi_j)$.

(4) *If* $(\phi, \psi) \in L_{\delta,\varepsilon}$ *is such that* $(\chi_i, b^*) \wedge (\phi, \psi) = (\chi_6, \chi_6)$ *or* $(b^*, \chi_i) \wedge (\phi, \psi) = (\chi_6, \chi_6)$ *for some* $i < 6$, *then* $(\phi, \psi) = (\chi_6, \chi_6)$.

(5) *If* $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ *is a lattice* 0*-homomorphism for some* $\delta', \varepsilon' \in \Delta$, *then either* f *is the constant mapping or* f *is a lattice* $(0,1)$*-homomorphism satisfying*

$$f\{(\chi_6, b^*), (b^*, \chi_6)\} = \{(\chi_6, b^*), (b^*, \chi_6)\} \text{ and}$$
$$f\{(\chi_0, b^*), (b^*, \chi_0)\} = \{(\chi_0, b^*), (b^*, \chi_0)\}.$$

(6) *For* $\delta', \varepsilon' \in \Delta$ *such that* $\delta \not\subseteq \delta' \cup \varepsilon'$, *any lattice* 0*-homomorphism* $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ *is constant.*

(7) *For* $\delta', \varepsilon' \in \Delta$ *such that* $\delta \subseteq \delta'$, $\varepsilon \subseteq \varepsilon'$, $\delta \not\subseteq \varepsilon'$, *and* $\varepsilon \not\subseteq \delta'$, *the lattice* $L_{\delta,\varepsilon}$ *is a* $(0,1)$*-sublattice of* $L_{\delta',\varepsilon'}$ *and the only* $(0,1)$*-homomorphism from* $L_{\delta,\varepsilon}$ *to* $L_{\delta',\varepsilon'}$ *is the inclusion mapping.*

(8) *For* $\delta', \varepsilon' \in \Delta$ *such that* $\delta \subseteq \delta'$, $\varepsilon \subseteq \varepsilon'$, $\delta \not\subseteq \varepsilon'$, *and* $\varepsilon \not\subseteq \delta'$, *if* $f : L_{\delta,\varepsilon}/L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ *is a lattice* $(0,1)$*-homomorphism then* $f(0,1) \in \{0,1\}$.

Proof: Claim (1) is Statement 4.5 in [3].

We prove (2). Let $f : K \longrightarrow I_1B(\delta)$ be a one-to-one lattice homomorphism. For the sake of brevity, the restriction of the i-th projection $p_i : K^6 \longrightarrow P$ to the sublattice $I_1B(\delta) \subseteq K^6$ will be also denoted as p_i. Since $I_1B(\delta)$ is a subdirect power of K and because f is one-to-one, for some $i \in 6$ the composite $p_i \circ f : K \longrightarrow K$ is non-constant. Since K is finite and simple, the composite $p_i \circ f$ is an automorphism of K, and hence $p_i \circ f(0) = 0$ and $p_i \circ f(1) = 1$ whenever $p_i \circ f$ is non-constant.

Denote $i_1 = \min\{i \in 6 \mid p_i \circ f \text{ is not constant}\}$, $i_2 = \max\{i \in 6 \mid p_i \circ f \text{ is not constant}\}$. Thus $i_1 \leq i_2$, $f(0)(i) = p_i \circ f(0) = 0$ and $f(1)(i) = p_i \circ f(1) = 1$ for $i \in \{i_1, i_2\}$. From $I_1B(\delta) \subseteq I$ it follows that $f(0)(i) = 0$ for all $i \leq i_2$ and $f(1)(i) = 1$ for all $i \geq i_1$. Since $p_i \circ f$ is the constant mapping for all i with $i < i_1$ or $i > i_2$ it follows that $f(0) = \chi_{i_2+1}$ and $f(1) = \chi_{i_1}$. Write $i = i_2 + 1$ and $j = i_1$. Then $0 \leq j < i \leq 6$, and it remains to show that $i - j < 6$. If not, that is, if $j = 0$ and $i = 6$, then $f : K \longrightarrow I_1B(\delta)$ is a $(0,1)$-homomorphism, and hence for every $i \in 6$, the composite $p_i \circ f$ is an automorphism of the finite simple lattice K. Thus $f(x)(i) = p_i \circ f(x) \in K \setminus \{0,1\}$ for each $x \in K \setminus \{0,1\}$ and all $i \in 6$. But then $f(x)(1) \in \{b, d\}$ for each $x \in K \setminus \{0,1\}$, by the Observation following the definition of $I_1B(\delta)$. It follows that $p_1 \circ f$ is a $(0,1)$-homomorphism from K to the distributive sublattice $\{0, b, d, 1\}$ of K, a contradiction completing the proof of (2).

To prove (3), let $\pi_1 : L_{\delta,\varepsilon} \longrightarrow I_1B(\delta)$ and $\pi_2 : L_{\delta,\varepsilon} \longrightarrow I_1B(\varepsilon)$ be the restrictions of the respective projections to the sublattice $L_{\delta,\varepsilon}$ of $I_1B(\delta) \times I_1B(\epsilon)$. Assume that $f : K \longrightarrow L_{\delta,\varepsilon}$ is a one-to-one lattice homomorphism.

Then $\pi_1 \circ f$ or $\pi_2 \circ f$ is not constant and thus $\pi_1 \circ f$ or $\pi_2 \circ f$ is a one-to-one lattice homomorphism because K is simple. With no loss of generality we may assume that $\pi_1 \circ f$ is a one-to-one lattice homomorphism. By (2), for some $0 \le j < i \le 6 \ne i - j$ we have $\pi_1 \circ f(0) = \chi_i$ and $\pi_1 \circ f(1) = \chi_j$. Observe that χ_k is incomparable to b^* for each $k \in \{1, 2, \dots, 5\}$, so that the definition of $L_{\delta,\epsilon}$ implies that $f(1) = (\chi_j, b^*)$ if $j > 0$, and $f(0) = (\chi_i, b^*)$ if $i < 6$. To continue the proof, next assume that $f(0) = (\chi_6, \lambda)$ for some $\lambda \ne b^*$. Since $\chi_6 < b^* \ne \lambda$, the definition of $L_{\delta,\epsilon}$ implies that $\lambda < b^*$. We also have $f(1) = (\chi_j, \kappa)$ for some $0 < j < 6$ and $\kappa = \pi_2 \circ f(1) \ge \pi_2 \circ f(0) = \lambda$. If $\kappa = \lambda$ then $\lambda < b^*$ implies that $\chi_j \le b^*$, a contradiction. Suppose that $\kappa > \lambda$. Then (2) implies that $\kappa = \chi_k$ for some $k < 6$. But $f(1) = (\chi_j, \chi_k)$ with $j, k < 6$ belongs to $L_{\delta,\epsilon}$ only when $j = 0$, and this contradicts (2). Thus $f(0) = (\chi_6, b^*)$, after all. A dual argument applies when $j = 0$, and the remainder of (3) follows by symmetry.

Assume that $(\phi, \psi) \in L_{\delta,\varepsilon}$ is such that $(\chi_i, b^*) \wedge (\phi, \psi) = (\chi_6, \chi_6)$ for some $i < 6$. Then $\chi_i(5) = 1$, $\chi_n(5) = 0$. Thus $\phi(5) = 0$, and from $I_1 B(\delta) \subseteq I$ we conclude that $\phi = \chi_6$. The definition of $L_{\delta,\varepsilon}$ yields $\psi \le b^*$, and $\psi \wedge b^* = \chi_6$ implies that $\psi = \chi_6$. If $(b^*, \chi_i) \wedge (\phi, \psi) = (\chi_6, \chi_6)$ for some $i < 6$ then a symmetric argument shows that $(\phi, \psi) = (\chi_6, \chi_6)$. This proves (4).

To prove (5) assume that $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ is a lattice 0-homomorphism, and denote

$$S = \{(\chi_i, b^*) \mid 0 < i \le 6\} \cup \{(b^*, \chi_i) \mid 0 < i \le 6\}.$$

Suppose first that $f(\chi_6, b^*) \notin S$. For each $i \in 6$ and all $x \in K$ set $g_i(x) = (x^*_{i,1}, b^*)$. Then $g_i : K \longrightarrow L_{\delta,\epsilon}$ is a one-to-one lattice homomorphism such that $g_i(0) = (\chi_{i+1}, b^*)$, $g_i(1) = (\chi_i, b^*)$. We have $f \circ g_5(0) = f(\chi_6, b^*) \notin S$ and hence, by (3), the composite $f \circ g_5$ must be constant. But then $f(\chi_5, b^*) = f \circ g_4(0) = f \circ g_5(1) = f(\chi_6, b^*) \notin S$, and the repeated use of (3) and g_i with $i < 5$ yields $f(\chi_0, b^*) = f(\chi_6, b^*)$.

Next we show that $f(\chi_0, b^*) = f(\chi_6, b^*)$ only when f is constant. From $(b^*, \chi_6) \wedge (\chi_6, b^*) = (\chi_6, \chi_6)$ and $(\chi_0, b^*) > (b^*, \chi_6)$ it follows that $f(b^*, \chi_6) = f(\chi_6, \chi_6) = (\chi_6, \chi_6)$ because f preserves the zero (χ_6, χ_6). Using injective homomorphisms $h_i : K \longrightarrow L_{\delta,\epsilon}$ given by $h_i(x) = (b^*, x^*_{i,1})$ for $i \in 6$ and (3), we find that $f(b^*, \chi_0) = (\chi_6, \chi_6)$. But then from $(b^*, \chi_0) > (\chi_6, b^*)$ and $(b^*, \chi_0) \vee (\chi_0, b^*) = (\chi_0, \chi_0)$ it follows that f is the constant mapping with the value (χ_6, χ_6). By symmetry, this is also true when $f(b^*, \chi_6) \notin S$. Altogether, $f(\chi_6, b^*), f(b^*, \chi_6) \in S$ for any non-constant f.

Suppose that f is non-constant. Let $f(\chi_6, b^*) = (\chi_i, b^*)$ for some $i < 6$. Then $(\chi_6, b^*) \wedge (b^*, \chi_6) = (\chi_6, \chi_6)$ and (4) imply that $f(b^*, \chi_6) = (\chi_6, \chi_6)$ and, since $(\chi_6, \chi_6) \notin S$, this is impossible. The same reasoning shows that $f(\chi_6, b^*) \ne (b^*, \chi_i)$ for $i < 6$ and, by symmetry, $f(b^*, \chi_6) \ne (\chi_i, b^*), (b^*, \chi_i)$ for $i < 6$ as well; hence $f(\chi_6, b^*), f(b^*, \chi_6) \in \{(\chi_6, b^*), (b^*, \chi_6)\}$. And $f(\chi_6, b^*) \ne f(b^*, \chi_6)$ because $(\chi_6, b^*) \wedge (b^*, \chi_6) = (\chi_6, \chi_6)$ and f is a 0-homomorphism. Therefore $f\{(\chi_6, b^*), (b^*, \chi_6)\} = \{(\chi_6, b^*), (b^*, \chi_6)\}$.

Now suppose that $f(\chi_6, b^*) = (\chi_6, b^*)$. Then $f(b^*, \chi_6) = (b^*, \chi_6)$ and, using (3) and the mappings g_i and h_i defined earlier, we find that $f(\chi_0, b^*) = (\chi_i, b^*)$ and $f(b^*, \chi_0) = (b^*, \chi_j)$ for some $i, j \leq 6$. From $(\chi_0, b^*) > (b^*, \chi_6)$ it follows that $(\chi_i, b^*) > (b^*, \chi_6)$ and hence $i = 0$. Analogously $(b^*, \chi_0) > (\chi_6, b^*)$ yields $j = 0$ and thus $f(\chi_0, \chi_0) = (\chi_0, \chi_0)$. Therefore f is a lattice $(0,1)$-homomorphism and $f\{(\chi_0, b^*), (b^*, \chi_0)\} = \{(b^*, \chi_0), (\chi_0, b^*)\}$. If $f(\chi_6, b^*) = (b^*, \chi_6)$ then, in the same manner, we find that $f(b^*, \chi_6) = (\chi_6, b^*)$, $f(\chi_0, b^*) = (b^*, \chi_0)$, $f(b^*, \chi_0) = (\chi_0, b^*)$, $f(\chi_0, \chi_0) = (\chi_0, \chi_0)$. Thus f is again a lattice $(0,1)$-homomorphism satisfying $f\{(\chi_0, b^*), (b^*, \chi_0)\} = \{(b^*, \chi_0), (\chi_0, b^*)\}$. The proof of (5) is complete.

For (6), assume that $f : L_{\delta,\varepsilon} \longrightarrow L_{\delta',\varepsilon'}$ is a lattice 0-homomorphism for some $\delta', \varepsilon' \in \Delta$ with $\delta \nsubseteq \delta' \cup \varepsilon'$. When non-constant then, by (5), f is a $(0,1)$-lattice homomorphism satisfying $f\{(\chi_6, b^*), (b^*, \chi_6)\} = \{(b^*, \chi_6), (\chi_6, b^*)\}$ and $f\{(\chi_0, b^*), (b^*, \chi_0)\} = \{(b^*, \chi_0), (\chi_0, b^*)\}$. Assume that $f(\chi_6, b^*) = (\chi_6, b^*)$. Then, using (3) and (5), we conclude that $f(\chi_0, b^*) = (\chi_0, b^*)$. The interval A between (χ_6, b^*) and (χ_0, b^*) in $L_{\delta,\varepsilon}$ is a sublattice of $L_{\delta,\varepsilon}$ isomorphic to $I_1B(\delta)$, and the interval A' between (χ_6, b^*) and (χ_0, b^*) in $L_{\delta',\varepsilon'}$ is a sublattice of $L_{\delta',\varepsilon'}$ isomorphic to $I_1B(\delta')$. We have $f(A) \subseteq A'$, and thus the domain-range restriction g of f is a lattice $(0,1)$-homomorphism from A into A'. Then $\delta \subseteq \delta'$ by (1), and this contradicts $\delta \nsubseteq \delta' \cup \varepsilon'$. Thus $f(\chi_6, b^*) \neq (\chi_6, b^*)$. If $f(\chi_6, b^*) = (b^*, \chi_6)$ then, in the same manner, we obtain that $\delta \subseteq \varepsilon'$, and this again is a contradiction. Therefore $f(\chi_6, b^*) \notin \{(\chi_6, b^*), (b^*, \chi_6)\}$ and f is the constant mapping with the value (χ_6, χ_6), by (5). The proof of (6) is complete.

Claims (7) and (8) were proved in [3] as Statements 4.6 and 4.7, respectively. □

Definition. For any K in Figure 1, we choose $\delta = \{0\}$, $\varepsilon = \{1\}$, $\delta' = \{0,2\}$, $\varepsilon' = \{1,2\}$, $\nu = \{3\}$, $\nu' = \{2,3\}$ in Δ, and denote $L_0 = L_{\delta,\epsilon}$, $L_1 = L_{\delta',\epsilon}$, $L_2 = L_{\epsilon',\delta}$ and $L_3 = L_{\delta',\epsilon'}$.

It is then clear that $\delta \subset \delta'$, $\varepsilon \subset \varepsilon'$, $\delta \nsubseteq \varepsilon'$, $\varepsilon \nsubseteq \delta'$, $\delta \nsubseteq \nu \cup \nu'$, $\varepsilon \nsubseteq \nu \cup \nu'$, $\nu \nsubseteq \delta' \cup \varepsilon'$, and $\nu \neq \nu'$. Claims (1), (5), (6), (7), and (8) of Proposition 3.2 then imply

Statement 3.3. *The lattices just defined and their sublattices have these properties:*

(1) *L_1 and L_2 are distinct proper $(0,1)$-sublattices of L_3 and L_0 is a proper $(0,1)$-sublattice of $L_1 \cap L_2$;*
(2) *for $i, j \in 4$, any lattice 0-homomorphism $L_i \longrightarrow L_j$ is either constant or one of the $(0,1)$-inclusions from (1);*
(3) *if $f : L_0/L_0 \longrightarrow L_3$ is a $(0,1)$-homomorphism then $f(0,1) \in \{0,1\}$;*
(4) *only constant 0-homomorphisms exist between any lattice L_i with $i \in 4$ and the lattice $L_{\nu,\nu'}$;*
(5) *there is no lattice $(0,1)$-homomorphism from $I_1B(\delta)$ or $I_1B(\varepsilon)$ to $I_1B(\nu)$ or $I_1B(\nu')$;*

(6) *the only lattice* $(0,1)$*-endomorphism of* $L_{\nu,\nu'}$ *is the identity map.*

4. A full $(0,1)$-embedding

Let $\mathbb{U}$ denote the category whose objects are triples (D, p_0, p_1) in which D is a distributive $(0,1)$-lattice, p_0 and p_1 are $(0,1)$-homomorphisms from D onto the two-element lattice $\mathbf{2} = \{0 < 1\}$ such that $p_0^{-1}\{1\}$ and $p_1^{-1}\{1\}$ are incomparable prime filters of D, and $\mathbb{U}$-morphisms from (D, p_0, p_1) into (D', p_0', p_1') are all lattice $(0,1)$-homomorphisms $f : D \longrightarrow D'$ satisfying $p_i = p_i' \circ f$ for $i = 0, 1$.

Theorem 4.1 ([3])**.** *The category* $\mathbb{U}$ *is finite-to-finite universal.*

In this section, we construct a full embedding of $\mathbb{U}$ into the variety $\mathbb{K}$ of $(0,1)$-lattices generated by a simple non-distributive lattice K and all $(0,1)$-homomorphisms between members of $\mathbb{K}$. Throughout the section, the $(0,1)$-lattices L_i with $i \in 4$ are assumed to have the properties described by Statements 2.2 and/or 3.3. The full embedding $F : \mathbb{U} \longrightarrow \mathbb{K}$ we construct here is a modification of the full embedding presented in [3].

For the variety $\mathbb{D}$ of all distributive $(0,1)$-lattices and all their $(0,1)$-homomorphisms and for every $\mathbb{D}$-object D, let $\mathrm{Hom}(D, \mathbf{2})$ denote the set of all $\mathbb{D}$-morphisms from D to $\mathbf{2}$.

Let $H : \mathbb{D} \longrightarrow \mathbb{K}$ be the functor given by $H(D) = L_3^{\mathrm{Hom}(D,\mathbf{2})}$ for every distributive $(0,1)$-lattice $D \in \mathbb{D}$ and, for any $\mathbb{D}$-morphism $f : D \longrightarrow D'$, a mapping $\phi : \mathrm{Hom}(D, \mathbf{2}) \longrightarrow L_3$ and for any $\mu \in \mathrm{Hom}(D', \mathbf{2})$, by $\big(Hf(\phi)\big)(\mu) = \phi(\mu \circ f)$. Standard categorical calculus shows that $H : \mathbb{D} \longrightarrow \mathbb{K}$ is a well-defined functor. For any distributive $(0,1)$-lattice $D \in \mathbb{D}$ we identify any element $d \in D$ with the function $d : \mathrm{Hom}(D, \mathbf{2}) \longrightarrow L_3$ given by $d(p) = p(d)$ for every $p \in \mathrm{Hom}(D, \mathbf{2})$. The distributive $(0,1)$-lattice D then becomes a $(0,1)$-sublattice of $H(D)$ and for any $\mathbb{D}$-morphism $f : D \longrightarrow D'$, the $\mathbb{K}$-morphism Hf coincides with f on D. As is usual, for any $z \in L_3$, let z^* denote the constant mapping with the value z. Thus the mapping $z \mapsto z^*$ is the diagonal embedding of L_3 into $H(D)$ for any $D \in \mathbb{D}$, and $Hf(z^*) = z^*$ for every $z \in L_3$.

For any $\mathbb{U}$-object (D, p_0, p_1) and for $i, j \in \{0, 1\}$ define

$$D_{i,j} = \{d \in D \setminus \{1\} \mid p_0(d) = i,\ p_1(d) = j\}.$$

Obviously, $\{D_{0,0}, D_{0,1}, D_{1,0}, D_{1,1}\}$ is a decomposition of $D \setminus \{1\}$. Denote

$$\begin{aligned} &G(D, p_0, p_1) \\ &= \{d \vee z^* \mid (d, z) \in (D_{0,0} \times L_0) \cup (D_{0,1} \times L_1) \cup (D_{1,0} \times L_2) \cup (D_{1,1} \times L_3)\} \\ &\cup \{d \wedge z^* \mid (d, z) \in (D_{0,0} \times L_3) \cup (D_{0,1} \times L_2) \cup (D_{1,0} \times L_1) \cup (D_{1,1} \times L_0)\}. \end{aligned}$$

Since $d \vee z^*(p_i) = d(p_i) \vee z$, we conclude that for any $d \vee z^* \in G(D, p_0, p_1)$, if $d \in D_{0,0} \cup D_{0,1}$ then $d \vee z^*(p_0) = z$, if $d \in D_{1,0} \cup D_{1,1}$ then $d \vee z^*(p_0) = 1$, if

$d \in D_{0,0} \cup D_{1,0}$ then $d \vee z^*(p_1) = z$, and if $d \in D_{0,1} \cup D_{1,1}$ then $d \vee z^* = 1$. Analogously, for $d \wedge z^* \in G(D, p_0, p_1)$, if $d \in D_{0,0} \cup D_{0,1}$ then $d \wedge z^*(p_0) = 0$, if $d \in D_{1,0} \cup D_{1,1}$ then $d \wedge z^*(p_0) = z$, if $d \in D_{0,0} \cup D_{1,0}$ then $d \wedge z^*(p_1) = 0$, if $d \in D_{0,1} \cup D_{1,1}$ then $d \wedge z^*(p_1) = z$. Thus as a consequence we obtain

Corollary 4.2. *If $u \in G(D, p_0, p_1)$ for a $\mathbb{U}$-object (D, p_0, p_1), then $u(p_0) \in L_1$ and $u(p_1) \in L_2$.*

The lemma below follows immediately from the definition of $G(D, p_0, p_1)$.

Lemma 4.3. *Let (D, p_0, p_1) be a $\mathbb{U}$-object, and let M_1 and M_2 be $(0,1)$-sublattices of L_3. For each $d \in D \setminus \{1\}$, define a mapping $k_d : \mathrm{Dom}(k_d) \longrightarrow H(D)$ with $\mathrm{Dom}(k_d) = M_1/M_2$ by $k_d(z, 1) = d \vee z^*$ for $z \in M_1$ and $k_d(0, z) = d \wedge z^*$ for $z \in M_2$. Then $k_d(0, 1) = d$ for any d. Moreover, $\mathrm{Im}(k_d) \subseteq G(D, p_0, p_1)$ whenever $d \in D_{0,0} \setminus \{0\}$ and $\mathrm{Dom}(k_d) = L_0/L_3$, or $d \in D_{0,1}$ and $\mathrm{Dom}(k_d) = L_1/L_2$, or $d \in D_{1,0}$ and $\mathrm{Dom}(k_d) = L_2/L_1$, or $d \in D_{1,1}$ and $\mathrm{Dom}(k_d) = L_3/L_0$. And $\mathrm{Im}(k_0) \subseteq G(D, p_0, p_1)$ also for the mapping $k_0(z) = z^*$ with $\mathrm{Dom}(k_0) = L_0$. The mapping k_d is a $(0,1)$-homomorphism in all five cases.*

From the definition of $\mathbb{U}$ it follows that if $f : (D, p_0, p_1) \longrightarrow (D', p_0', p_1')$ is a $\mathbb{U}$-morphism, then $f(D_{i,j}) \subseteq D'_{i,j}$ for all $i, j \in \{0, 1\}$. Then $Hf(D_{i,j}) \subseteq D'_{i,j}$ for all $i, j \in \{0, 1\}$, and the lemma below follows since $Hf(z^*) = z^*$ for all $z \in L_3$ and because Hf is a $(0,1)$-homomorphism.

Lemma 4.4. *For any $\mathbb{U}$-morphism $f : (D, p_0, p_1) \longrightarrow (D', p_0', p_1')$,*

$$Hf(G(D, p_0, p_1)) \subseteq G(D', p_0', p_1').$$

From Lemma 4.4 it follows that we can define a functor $F : \mathbb{U} \longrightarrow \mathbb{K}$ as follows. For a $\mathbb{U}$-object (D, p_0, p_1), let $F(D, p_0, p_1)$ be a $(0,1)$-sublattice of $H(D)$ generated by the set $G(D, p_0, p_1)$. For a $\mathbb{U}$-morphism $f : (D, p_0, p_1) \longrightarrow (D', p_0', p_1')$ let Ff be the domain-range restriction of Hf to $F(D, p_0, p_1)$ and $F(D', p_0', p_1')$. Since $D \subseteq F(D, p_0, p_1)$, we obtain

Proposition 4.5. *The functor $F : \mathbb{U} \longrightarrow \mathbb{K}$ is correctly defined and faithful.*

Since the operations of the lattices $F(D, p_0, p_1)$ are defined componentwise, Corollary 4.2 yields

Corollary 4.6. *For every $\mathbb{U}$-object (D, p_0, p_1) and for every $u \in F(D, p_0, p_1)$ we have $u(p_0) \in L_1$ and $u(p_1) \in L_2$.*

We say that a function $f : \mathrm{Hom}(D, \mathbf{2}) \longrightarrow L_3$ is *skeletal* if $\mathrm{Im}(f) \subseteq \{0, 1\}$. Thus any element of $D \subseteq F(D, p_0, p_1)$ is skeletal, and the lemma below claims that $F(D, p_0, p_1)$ for a $\mathbb{U}$-object (D, p_0, p_1) has no other skeletal functions.

Lemma 4.7. *For any object (D, p_0, p_1), an element $\phi \in F(D, p_0, p_1)$ is a skeletal function if and only if $\phi \in D$.*

Proof: The proof follows the lines of [3]. For any $\phi \in F(D, p_0, p_1)$ and $z \in L_3$ define the *z-trim* ϕ_z of ϕ by

$$\phi_z(p) = \begin{cases} 1 & \text{if } \phi(p) \geq z, \\ 0 & \text{if } \phi(p) \not\geq z. \end{cases}$$

By a simple induction we prove that any z-trim ϕ_z with $\phi \in F(D, p_0, p_1)$ belongs to $D \subseteq F(D, p_0, p_1)$ and that any $\phi \in F(D, p_0, p_1)$ has only finitely many z-trims. Since $\phi = \phi_z$ for any skeletal function ϕ and for any $z > 0$, this will prove the lemma.

This claim is easily verified for all $\phi \in G(D, p_0, p_1)$ as follows. If $\phi = d \vee w^*$ for some $d \in D$ and $w \in L_3$ then $\phi_z = 1 \in D$ for all $z \leq w$ and $\phi_z = d \in D$ for all other $z \in L_3$. If $\phi = d \wedge w^*$ for some $d \in D$ and $w \in L_3$ then $\phi_0 = 1 \in D$, $\phi_z = d$ for $0 < z \leq w$, and $\phi_z = 0 \in D$ for all other $z \in L_3$.

Assume now that the claim is valid for some $\phi, \psi \in F(D, p_0, p_1)$.

First consider $\phi \wedge \psi$. For every $z \in L_3$ and every $p \in \mathrm{Hom}(D, \mathbf{2})$ we have $(\phi \wedge \psi)(p) \geq z$ if and only if $\phi(p), \psi(p) \geq z$; hence $(\phi \wedge \psi)_z = \phi_z \wedge \psi_z \in D$ for all $z \in L_3$ and $\phi \wedge \psi$ has only finitely many distinct z-trims.

Secondly consider $\mu = \phi \vee \psi$ and $z \in L_3$. We take $u, v \in L_3$ with $u \vee v \geq z$ and prove that $\mu_z \geq \phi_u \wedge \psi_v$. Since $\mathrm{Im}(\mu_z), \mathrm{Im}(\phi_u), \mathrm{Im}(\psi_v) \subseteq \{0, 1\}$, it suffices to show that $(\phi_u \wedge \psi_v)(p) = 1$ for some $p \in \mathrm{Hom}(D, \mathbf{2})$ implies $\mu_z(p) = 1$. Indeed, from $(\phi_u \wedge \psi_v)(p) = 1$ we obtain $\phi(p) \geq u$ and $\psi(p) \geq v$, and thus $\mu(p) = \phi(p) \vee \psi(p) \geq u \vee v \geq z$. Therefore $\mu_z(p) = 1$. By the induction hypothesis, ϕ and ψ have only finitely many distinct z-trims and all belong to D. Hence

$$\mu_z \geq \bigvee \{\phi_u \wedge \psi_v \mid u, v \in L_3,\ u \vee v \geq z\} = \sigma,$$

where $\sigma \in D$ is well-defined. If $\mu_z(p) = 1$, then for $u = \phi(p)$ and $v = \psi(p)$ we have $\mu(p) = u \vee v \geq z$ and $(\phi_u \wedge \psi_v)(p) = 1$, so that $\sigma = \mu_z$. Thus any z-trim of μ belongs to D. And μ has only finitely many distinct z-trims because, by the induction hypothesis, there are only finitely many distinct elements σ. □

Next we prove that F is full.

Lemma 4.8. *If $f : F(D, p_0, p_1) \to F(D', p'_0, p'_1)$ is a lattice $(0,1)$-homomorphism, then $f = Fg$ for some $\mathbb{U}$-morphism $g : (D, p_0, p_1) \to (D', p'_0, p'_1)$.*

Proof: First we prove that $f(D) \subseteq D'$. By Lemma 4.7, it suffices to show that $f(d)$ is skeletal for every $d \in D \setminus \{0, 1\}$. By Lemma 4.3, there exist $M_1, M_2 \in \{L_0, L_1, L_2, L_3\}$ and a one-to-one lattice $(0,1)$-homomorphism $g_d : M_1/M_2 \to F(D, p_0, p_1)$ with $g_d(0, 1) = d$, $g_d(0, z) = d \wedge z^*$ for all $z \in M_2$, $g_d(z, 1) = d \vee z^*$ for all $z \in M_1$. For $p \in \mathrm{Hom}(D', \mathbf{2})$, let π_p be the restriction of the p-th projection from $H(D')$ to its $(0,1)$-sublattice $F(D', p'_0, p'_1)$. Then $\pi_p : F(D', p'_0, p'_1) \longrightarrow L_3$ is a lattice $(0,1)$-homomorphism. By Lemma 4.3, the composite $\pi_p \circ f \circ g_d$ is thus a lattice $(0,1)$-homomorphism from M_1/M_2 to L_3 for all $p \in \mathrm{Hom}(D', \mathbf{2})$ and $d \in D$. By the hypothesis, $\pi_p \circ f \circ g_d(0,1) \in \{0,1\}$. Thus $f(d)(p) \in \{0,1\}$ for all $d \in D$ and $p \in \mathrm{Hom}(D', \mathbf{2})$, and hence $f(d)$ is a skeletal function for any $d \in D$. We conclude that $f(D) \subseteq D'$. Moreover, if $\pi_p \circ f \circ g_d(0,1) = 0$ then the restriction of $\pi_p \circ f \circ g_d$ to M_1 is the inclusion map, if $\pi_p \circ f \circ g_d(0,1) = 1$ then the restriction of $\pi_p \circ f \circ g_d$ on M_2 is the inclusion map. Hence we deduce that for $d \in D \setminus \{0,1\}$, if $d \vee z^* \in G(D, p_0, p_1)$ for some $z \in L_3$ then $f(d \vee z^*) = f(d) \vee z^*$ and if $d \wedge z^* \in G(D, p_0, p_1)$ for some $z \in L_3$ then $f(d \wedge z^*) = f(d) \wedge z^*$. Finally, by Lemma 4.3, there exists a lattice $(0,1)$-homomorphism $g_0 : L_0 \longrightarrow F(D, p_0, p_1)$ with $g_0(z) = z^*$ for all $z \in L_0$. By the hypothesis, $\pi_p \circ f \circ g_0$ is the inclusion map from L_0 into L_3 and whence $f(z^*) = z^*$ for all $z \in L_0$. Let g be the domain-range restriction of f to D and D'. Then $g : D \longrightarrow D'$ is a lattice $(0,1)$-homomorphism; and if g is a $\mathbb{U}$-morphism from (D, p_0, p_1) to (D', p'_0, p'_1), then $Fg = f$. Thus it remains to prove that $p_i = p'_i \circ g$ for $i = 0, 1$.

For $d \in D_{0,1}$ we have $d(p_0) = 0$ and $d(p_1) = 1$. By Lemma 4.3, there exists a lattice 0-homomorphism $h^0_d : L_2 \longrightarrow F(D, p_0, p_1)$ such that $h^0_d(z) = d \wedge z^*$ for all $z \in L_2$, in particular $h^0_d(1) = d$. If $f(d)(p'_0) = 1$, then, by Corollary 4.6, $\pi_{p'_0} \circ f \circ h^0_d$ is a lattice $(0,1)$-homomorphism from L_2 to L_1, contrary to the hypothesis. Therefore $f(d)(p'_0) = 0$. By Lemma 4.3, there exists a lattice 1-homomorphism $h^1_d : L_1 \longrightarrow F(D, p_0, p_1)$ such that $h^1_d(z) = d \vee z^*$ for all $z \in L_1$, in particular $h^1_d(0) = d$. If $f(d)(p'_1) = 0$, then, by Corollary 4.6, $\pi_{p'_1} \circ f \circ h^1_d$ is a lattice $(0,1)$-homomorphism from L_1 into L_2, and this is again a contradiction. Therefore $f(D_{0,1}) \subseteq D'_{0,1}$. By symmetry we obtain $f(D_{1,0}) \subseteq D'_{1,0}$. Using the fact that there are no $(0,1)$-homomorphisms from L_3 to L_0, we similarly find that $f(D_{0,0}) \subseteq D'_{0,0}$ and $f(D_{1,1}) \subseteq D'_{1,1}$. Hence $p_i = p'_i \circ f$ for $i = 0, 1$, and the proof is complete. □

Theorem 4.9. *The functor $F : \mathbb{U} \longrightarrow \mathbb{K}$ is a full embedding.*

Remark. In the subsequent section we shall use the following description of the functor F. For any $\mathbb{U}$-object (D, p_0, p_1), the elements of $F(D, p_0, p_1)$ are certain functions from $\mathrm{Hom}(D, \mathbf{2})$ into $L_3 = L_{\delta', \varepsilon'}$ on which the lattice operations are defined componentwise. We recall that elements of $L_{\delta', \varepsilon'}$ are pairs of functions from n to K where the lattice operations are also defined componentwise. Thus $\phi \in F(D, p_0, p_1)$ is a function from $\mathrm{Hom}(D, \mathbf{2})$ into $K^n \times K^n$ or a function ϕ' from $\mathrm{Hom}(D, \mathbf{2}) \times 2$ into K^n with $\phi(p) = (\phi'(p, 0), \phi'(p, 1))$ for all $p \in \mathrm{Hom}(D, \mathbf{2})$ or a function ϕ'' from $\mathrm{Hom}(D, \mathbf{2}) \times 2 \times n$ into K such that $\phi''(p, i, j) = \phi'(p, i)(j)$ for all $p \in \mathrm{Hom}(D, \mathbf{2})$, $i \in 2$ and $j \in n$. The function $(\beta, \beta)^*$ that is the constant function with the value $(\beta, \beta) \in L_{\delta', \varepsilon'}$ will play an important role in the final construction.

5. Almost universality

In this section we assume the conclusions of Statements 2.2 and 3.3, and that $\mathbb{K}$ is the variety generated by a $(0,1)$-lattice K whose every non-constant endomorphism preserves its bounds 0 and 1. To be able to deal with both short and tall lattices simultaneously, for any short lattice K we shall write β instead of b^*.

For a $\mathbb{U}$-object (D, p_0, p_1) let $M(D, p_0, p_1)$ be the sublattice of $F(D, p_0, p_1) \times L_{\nu,\nu'}$ consisting of all pairs $(\phi, \psi) \in F(D, p_0, p_1) \times L_{\nu,\nu'}$ satisfying

$$(\phi, \psi) \le ((\beta,\beta)^*, (\beta,\beta)) \text{ or } (\phi, \psi) \ge ((\beta,\beta)^*, (\beta,\beta)) \text{ or } \phi = (\beta,\beta)^* \text{ or } \psi = (\beta,\beta).$$

It is easy to see that $M(D, p_0, p_1)$ is a sublattice of $F(D, p_0, p_1) \times L_{\nu,\nu'}$. From Proposition 2.1 and Proposition 3.2 it follows

Corollary 5.1. *Let $f : K \longrightarrow M(D, p_0, p_1)$ be a one-to-one lattice homomorphism for some $\mathbb{U}$-object (D, p_0, p_1). Then one of the following possibilities occurs:*

(1) $f(0) = ((\beta,\beta)^*, (\chi_i, \beta))$ *and* $f(1) = ((\beta,\beta)^*, (\chi_{j-1}, \beta))$ *for some* $i, j \in \{1, 2, \dots, n\}$*;*
(2) $f(0) = ((\beta,\beta)^*, (\beta, \chi_i))$ *and* $f(1) = ((\beta,\beta)^*, (\beta, \chi_{j-1}))$ *for some* $i, j \in \{1, 2, \dots, n\}$*;*
(3) $f(0) = (\phi, (\beta,\beta))$, $f(1) = (\psi, (\beta,\beta))$ *and there exists* $p \in \mathrm{Hom}(D, \mathbf{2})$ *with* $\phi(p) = (\chi_i, \beta)$ *and* $\psi(p) = (\chi_{j-1}, \beta)$ *for some* $i, j \in \{1, 2, \dots, n\}$*;*
(4) $f(0) = (\phi, (\beta,\beta))$, $f(1) = (\psi, (\beta,\beta))$ *and there exists* $p \in \mathrm{Hom}(D, \mathbf{2})$ *with* $\phi(p) = (\beta, \chi_i)$ *and* $\psi(p) = (\beta, \chi_{j-1})$ *for some* $i, j \in \{1, 2, \dots, n\}$.

In what follows, we assume that for some $\mathbb{U}$-objects (D, p_0, p_1) and (D', p'_0, p'_1), a non-constant lattice 0-homomorphism $f : M(D, p_0, p_1) \to M(D', p'_0, p'_1)$ is given. Of particular interest will be the quadruple

$$Q = \{((\beta,\beta)^*, (\beta, \chi_n)), ((\beta,\beta)^*, (\chi_n, \beta)), ((\beta, \chi_n)^*, (\beta,\beta)), ((\chi_n, \beta)^*, (\beta,\beta))\}$$

of distinct elements of $M(D, p_0, p_1)$ and its image $f(Q)$ in $M(D', p'_0, p'_1)$. We note that

$$\bigwedge Q = ((\chi_n, \chi_n)^*, (\chi_n, \chi_n)) = 0 \in M(D, p_0, p_1), \quad \text{and hence} \quad \bigwedge f(Q) = 0.$$

Below are some useful relations of the elements of Q to other members of $M(D, p_0, p_1)$.

$$\begin{aligned}
((\beta,\beta)^*, (\beta, \chi_0)) &\ge ((\beta,\beta)^*, (\chi_n, \beta)),\\
((\beta,\beta)^*, (\chi_0, \beta)) &\ge ((\beta,\beta)^*, (\beta, \chi_n)),\\
((\chi_0, \beta)^*, (\beta,\beta)) &\ge ((\beta, \chi_n)^*, (\beta,\beta)),\\
((\beta, \chi_0)^*, (\beta,\beta)) &\ge ((\chi_n, \beta), (\beta,\beta)),\\
((\chi_0, \chi_0)^*, (\beta,\beta)) &\ge ((\beta,\beta)^*, (\chi_n, \beta)) \vee ((\beta,\beta)^*, (\beta, \chi_n)),\\
((\beta,\beta)^*, (\chi_0, \chi_0)) &\ge ((\chi_n, \beta)^*, (\beta,\beta)) \vee ((\beta, \chi_n)^*, (\beta,\beta)).
\end{aligned}$$

Notation. Denote A the subset of $M(D, p_0, p_1)$ consisting of all elements $((\beta,\beta)^*, (\beta,\chi_i))$ and $((\beta,\beta)^*, (\chi_i,\beta))$ with $i = 1, 2, \ldots, n$ and of all elements $(\phi, (\beta,\beta))$ for which there exists $p \in \mathrm{Hom}(D', \mathbf{2})$ with $\phi(p) \in \{(\beta,\chi_i), (\chi_i,\beta) \mid i = 1, 2, \ldots, n\}$.

We recall that the members of A are exactly the zeros of isomorphic copies of the lattice K in $M(D, p_0, p_1)$, see Propositions 2.1 and 3.2.

Lemma 5.2. *If Z is a finite subset of A and $\bigwedge Z = 0 \in M(D, p_0, p_1)$, then*

(1) $((\beta,\beta)^*, (\beta,\chi_n)), ((\beta,\beta)^*, (\chi_n,\beta)) \in Z$;
(2) $|Z \cap \big(F(D, p_0, p_1) \times \{(\beta,\beta)\}\big)| \geq 2$;
(3) $\bigwedge Z \cap \big(F(D, p_0, p_1) \times \{(\beta,\beta)\}\big) = ((\chi_n,\chi_n)^*, (\beta,\beta))$.

In particular, $|Z| \geq 4$.

Proof: Assume that $Z \subseteq A$ is finite and $\bigwedge Z = 0 = ((\chi_n,\chi_n)^*, (\chi_n,\chi_n))$. Let $Z_1 = Z \cap \big(F(D, p_0, p_1) \times \{(\beta,\beta)\}\big)$ and $Z_2 = Z \cap \big(\{(\beta,\beta)^*\} \times L_{\nu,\nu'}\big)$. Then $\{Z_1, Z_2\}$ is a decomposition of Z and $\bigwedge Z_1 = (\phi_1, (\beta,\beta))$ and $\bigwedge Z_2 = ((\beta,\beta)^*, \phi_2)$ for some ϕ_1 and ϕ_2. Since $\bigwedge Z = (\bigwedge Z_1) \wedge (\bigwedge Z_2)$, we conclude that $\phi_1 = (\chi_n,\chi_n)^*$ and $\phi_2 = (\chi_n, \chi_n)$. Clearly $\phi_1, \phi_2 \notin A$, and thus $|Z_1|, |Z_2| \geq 2$. By Proposition 2.1(4) or 3.2(4), $((\beta,\beta)^*, (\beta,\chi_n)), ((\beta,\beta)^*, (\chi_n,\beta)) \in Z_2 \subseteq Z$. The remainder is obvious. $\square$

Lemma 5.3. *The set $f(Q)$ is a four-element subset of $A \subseteq M(D', p'_0, p'_1)$. It always contains $((\beta,\beta)^*, (\beta,\chi_n))$ and $((\beta,\beta)^*, (\chi_n,\beta))$ and satisfies*

(1) $|f(Q) \cap (F(D', p'_0.p'_1) \times \{(\beta,\beta)\})| = 2$;
(2) $\bigwedge f(Q) \cap \big(F(D', p'_0.p'_1) \times \{(\beta,\beta)\}\big) = ((\chi_n,\chi_n)^*, (\beta,\beta))$.

Proof: First we show that $f((\beta,\beta)^*, (\beta,\chi_n)) \notin A$ implies that

$$f((\beta,\beta)^*, (\beta,\chi_0)) = f((\beta,\beta)^*, (\beta,\chi_n)).$$

Indeed, for every $i = 1, 2, \ldots, n$, $g_i(z) = ((\beta,\beta)^*, (\beta, z^*_{i-1,1}))$ is a one-to-one lattice homomorphism from K to $M(D, p_0, p_1)$ with

$$g_i(0) = ((\beta,\beta)^*, (\beta,\chi_i)) \quad \text{and} \quad g_i(1) = ((\beta,\beta)^*, (\beta,\chi_{i-1})),$$

and the claim follows by Corollary 5.1. In the same manner, if $f((\beta,\beta)^*, (\chi_n,\beta)) \notin A$ then $f((\beta,\beta)^*, (\chi_0,\beta)) = f((\beta,\beta)^*, (\chi_n,\beta))$, if $f((\chi_n,\beta)^*, (\beta,\beta)) \notin A$ then $f((\chi_0,\beta)^*, (\beta,\beta)) = f((\chi_n,\beta)^*, (\beta,\beta))$, if $f((\beta,\chi_n)^*, (\beta,\beta)) \notin A$ then $f((\beta,\chi_0)^*, (\beta,\beta)) = f((\beta,\chi_n)^*, (\beta,\beta))$.

Suppose that $f((\beta,\beta)^*, (\beta,\chi_n)), f((\beta,\beta)^*, (\chi_n,\beta)) \notin A$. Then

$$\begin{aligned} f((\beta,\beta)^*, (\chi_n,\beta)) = f((\beta,\beta)^*, (\chi_0,\beta)) \geq f((\beta,\beta)^*, (\beta,\chi_n)) = f((\beta,\beta)^*, (\beta,\chi_0)) \\ \geq f((\beta,\beta)^*, (\chi_n,\beta)), \end{aligned}$$

that is, all these elements coincide. Since $(\chi_n,\beta)\wedge(\beta,\chi_n)=(\chi_n,\chi_n)$ and $(\chi_0,\beta)\vee(\beta,\chi_0)=(\chi_0,\chi_0)$, it follows that $f((\beta,\beta)^*,(\chi_0,\chi_0))=f((\beta,\beta)^*,(\chi_n,\chi_n))$. But from $(\chi_n,\chi_n)<(\beta,\beta)<(\chi_0,\chi_0)$ it then follows that $f((\chi_n,\chi_n)^*,(\beta,\beta))\le f((\beta,\beta)^*,(\chi_n,\chi_n))$. Using $(\chi_n,\beta)\wedge(\beta,\chi_n)=(\chi_n,\chi_n)$ we then conclude that

$$\text{(a)} \qquad f((\chi_n,\chi_n)^*,(\beta,\beta))=((\chi_n,\chi_n)^*,(\chi_n,\chi_n)).$$

Therefore $f((\chi_n,\beta)^*,(\beta,\beta))\wedge f((\beta,\chi_n)^*,(\beta,\beta))=((\chi_n,\chi_n)^*,(\chi_n,\chi_n))=0\in M(D',p_0',p_1')$ and, by Lemma 5.2 applied to these two elements of $f(Q)$ it follows that $f((\chi_n,\beta)^*,(\beta,\beta))\notin A$ or $f((\beta,\chi_n)^*,(\beta,\beta))\notin A$. In the first case

$$f((\chi_n,\beta)^*,(\beta,\beta))=f((\chi_0,\beta)^*,(\beta,\beta))\ge f((\beta,\chi_n)^*,(\beta,\beta)),$$

and thus, meeting with $f((\beta,\chi_n)^*,(\beta,\beta))$ and using (a), we obtain

$$f((\beta,\chi_n)^*,(\beta,\beta))=((\chi_n,\chi_n)^*,(\chi_n,\chi_n))=0\notin A,$$

and hence also

$$f((\beta,\chi_0)^*,(\beta,\beta))=f((\beta,\chi_n)^*,(\beta,\beta))=0.$$

From $(\chi_n,\beta)<(\beta,\chi_0)$ we then get $f((\chi_n,\beta)^*,(\beta,\beta))=0\notin A$, and therefore

$$f((\chi_0,\beta)^*,(\beta,\beta))=f((\chi_n,\beta)^*,(\beta,\beta))=0.$$

From this we immediately obtain that

$$0=f((\chi_0,\chi_0)^*,(\beta,\beta))\ge f((\beta,\beta)^*,(\chi_n,\beta))=f((\beta,\beta)^*,(\chi_0,\chi_0))$$

and hence $0=f((\chi_0,\chi_0)^*,(\chi_0,\chi_0))$. Thus f is the constant mapping with the value 0, and this contradicts the hypothesis. The second case has the same proof, and analogously we obtain a contradiction if $f((\chi_n,\beta)^*,(\beta,\beta)), f((\beta,\chi_n)^*,(\beta,\beta))\notin A$. Therefore

$$\{f((\chi_n,\beta)^*,(\beta,\beta)), f((\beta,\chi_n)^*,(\beta,\beta))\}\cap A\neq\emptyset \quad\text{and}$$
$$\{f((\beta,\beta)^*,(\chi_n,\beta)), f((\beta,\beta)^*,(\beta,\chi_n))\}\cap A\neq\emptyset.$$

Using the latter property, we note that if $f((\beta,\beta)^*,(\chi_n,\beta))\notin A$ then

$$A\ni f((\beta,\beta)^*,(\beta,\chi_n))\le f((\beta,\beta)^*,(\chi_0,\beta))=f((\beta,\beta)^*,(\chi_n,\beta)),$$

and analogously, if $f((\beta,\beta)^*,(\beta,\chi_n))\notin A$ then $A\ni f((\beta,\beta)^*,(\chi_n,\beta))\le f((\beta,\beta)^*,(\beta,\chi_n))$, if $f((\chi_n,\beta)^*,(\beta,\beta))\notin A$ then $A\ni f((\beta,\chi_n)^*,(\beta,\beta))\le f((\chi_n,\beta)^*,((\beta,\beta))$ and, finally, if $f((\beta,\chi_n)^*,(\beta,\beta))\notin A$ then $A\ni f((\chi_n,\beta)^*,(\beta,\beta))\le f((\beta,\chi_n)^*,(\beta,\beta))$. Therefore

$$0=\bigwedge f(Q)=\bigwedge(f(Q)\cap A),$$

and Lemma 5.2 completes the proof. $\square$

Lemma 5.4. *If* $(\phi,\psi)\in L_{\nu,\nu'}$ *then* $f(((\beta,\beta)^*,(\phi,\psi))=((\beta,\beta)^*,(\phi,\psi))$.

PROOF: By Lemma 5.3, $f(((\beta,\beta)^*,(\beta,\chi_n))$ and $f(((\beta,\beta)^*,(\chi_n,\beta))$ are distinct elements of A. First assume that $f(((\beta,\beta)^*,(\beta,\chi_n)),f(((\beta,\beta)^*,(\chi_n,\beta))\in A\cap \big(F(D',p_0',p_1')\times\{(\beta,\beta)\}\big)$. Then, by Lemma 5.3,

$$f(((\beta,\beta)^*,(\chi_n,\chi_n))=f(((\beta,\beta)^*,(\beta,\chi_n))\wedge f(((\beta,\beta)^*,(\chi_n,\beta))=((\chi_n,\chi_n)^*,(\beta,\beta))$$

and, by Corollary 5.1,

$$f(((\beta,\beta)^*,(\beta,\chi_{n-1})),f(((\beta,\beta)^*,(\chi_{n-1},\beta))\in F(D',p_0',p_1')\times\{(\beta,\beta)\}.$$

By an easy induction using Corollary 5.1, we find that

$$f(((\beta,\beta)^*,(\beta,\chi_0)),f(((\beta,\beta)^*,(\chi_0,\beta))\in F(D',p_0',p_1')\times\{(\beta,\beta)\}.$$

Hence f maps the intervals

$$[((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\beta,\chi_0))] \text{ and } [((\beta,\beta)^*,(\chi_n,\beta)),((\beta,\beta)^*,(\chi_0,\beta))]$$

into $F(D',p_0',p_1')\times\{(\beta,\beta)\}$. Since the union of these two intervals generates the sublattice $\{(\beta,\beta)^*\}\times L_{\nu,\nu'}$, the restriction h_p of $\pi_p\circ\rho_1\circ f$ to $(\{\beta,\beta)^*\}\times L_{\nu,\nu'}$ is a lattice 0-homomorphism from $L_{\nu,\nu'}$ to $L_3=L_{\delta',\varepsilon'}$ (here ρ_1 is the restriction of the first projection from $F(D',p_0',p_1)\times L_{\nu,\nu'}$ to $M(D',p_0',p_1')$ and π_p is the restriction of the p-th projection to $F(D',p_0',p_1')$ for $p\in\mathrm{Hom}(D',\mathbf{2})$). By (4) from Statement 2.2 or 3.3, any 0-homomorphism h_p is a constant mapping, and this contradicts Lemma 5.3. Thus, by Lemma 5.3, we can assume that

$$\{f(((\beta,\beta)^*,(\beta,\chi_n)),f(((\beta,\beta)^*,(\chi_n,\beta))\}\cap\{((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\chi_n,\beta))\}\neq\emptyset.$$

If $\{f(((\beta,\beta)^*,(\beta,\chi_n)),f(((\beta,\beta)^*,(\chi_n,\beta))\}\neq\{((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\chi_n,\beta))\}$ then, by Lemma 5.3 either the element $f(((\beta,\chi_n)^*,(\beta,\beta))$ or the element $f(((\chi_n,\beta)^*,(\beta,\beta))$ belongs to the set $\{((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\chi_n,\beta))\}$. If $f(((\beta,\chi_n)^*,(\beta,\beta))=((\beta,\beta)^*,(\beta,\chi_n))$ then, by Corollary 5.1, $f(((\beta,\chi_0)^*,(\beta,\beta))=((\beta,\beta)^*,(\beta,\chi_i))$ for some $i\le n$. Since

$$((\beta,\chi_0)^*,(\beta,\beta))\ge((\beta,\beta)^*,(\beta,\beta))=((\beta,\beta)^*,(\chi_n,\beta))\vee((\beta,\beta)^*,(\beta,\chi_n)),$$

we conclude that $i=0$; this is because for $i>0$ we have $((\beta,\beta)^*,(\beta,\chi_i))\not\ge((\beta,\beta)^*,(\chi_n,\beta))$. Thus the domain-range restriction of f to the intervals $[((\beta,\chi_n)^*,(\beta,\beta)),((\beta,\chi_0)^*,(\beta,\beta))]$ and $[((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\beta,\chi_0))]$ is a lattice $(0,1)$-homomorphism from $I_2A(\delta)$ into $I_2A(\nu')$ or $I_1B(\delta)$ into $I_1B(\nu')$ — and this contradicts the item (5) of either Statement 2.2 or 3.3. Similarly, if $f(((\beta,\chi_n)^*,(\beta,\beta))=((\beta,\beta)^*,(\chi_n,\beta))$ then we obtain a lattice $(0,1)$-homomorphism from $I_2A(\delta)$ into $I_2A(\nu)$ or $I_1B(\delta)$ into $I_1B(\nu)$, if $f((\chi_n,\beta,)^*,(\beta,\beta)) = ((\beta,\beta)^*,(\beta,\chi_n))$ then we obtain a lattice $(0,1)$-homomorphism from $I_2A(\varepsilon)$ into $I_2A(\nu')$ or $I_1B(\varepsilon)$ into $I_1B(\nu')$, if $f((\chi_n,\beta)^*,(\beta,\beta)) = ((\beta,\beta)^*,(\chi_n,\beta))$ then we obtain a lattice $(0,1)$-homomorphism from $I_2A(\varepsilon)$ into $I_2A(\nu)$ or $I_1B(\varepsilon)$ into $I_1B(\nu)$.

Thus

$$\{f((\beta,\beta)^*,(\beta,\chi_n)), f((\beta,\beta)^*,(\chi_n,\beta))\} = \{((\beta,\beta)^*,(\beta,\chi_n)),((\beta,\beta)^*,(\chi_n,\beta))\}.$$

By Corollary 5.1, there exist i and j with $0 \le i,j \le n$ and

$$\{f((\beta,\beta)^*,(\beta,\chi_0)), f((\beta,\beta)^*,(\chi_0,\beta))\} = \{((\beta,\beta)^*,(\beta,\chi_i)),((\beta,\beta)^*,(\chi_j,\beta))\}.$$

From $((\beta,\beta)^*,(\beta,\chi_0)) \ge ((\beta,\beta)^*,(\chi_n,\beta))$ and $((\beta,\beta)^*,(\chi_0,\beta)) \ge ((\beta,\beta)^*,(\beta,\chi_n))$ it follows that $i=j=0$ because for $i>0$ neither $((\beta,\beta)^*,(\chi_i,\beta)) \ge ((\beta,\beta)^*,(\beta,\chi_n))$ nor $((\beta,\beta)^*,(\beta,\chi_i)) \ge ((\beta,\beta)^*,(\chi_n,\beta))$. Therefore $f((\beta,\beta)^*,(\chi_n,\chi_n)) = ((\beta,\beta)^*,(\chi_n,\chi_n))$ and $f((\beta,\beta)^*,(\chi_0,\chi_0)) = ((\beta,\beta)^*,(\chi_0,\chi_0))$. Thus $f(\{(\beta,\beta)^*\}\times L_{\nu,\nu'}) \subseteq \{(\beta,\beta)^*\}\times L_{\nu,\nu'}$ and the domain-range restriction of f to $\{(\beta,\beta)^*\}\times L_{\nu,\nu'}$ is a lattice $(0,1)$-endomorphism of $L_{\nu,\nu'}$. Using (6) of Statement 2.2 or 3.3, we then obtain $f((\beta,\beta)^*,(\phi,\psi)) = ((\beta,\beta)^*,(\phi,\psi))$ for all $(\phi,\psi)\in L_{\nu,\nu'}$. □

Lemma 5.5. *There exists a lattice $(0,1)$-homomorphism $g: F(D,p_0,p_1) \to F(D',p'_0,p'_1)$ such that f is the domain-range restriction of $g\times 1$ to $M(D,p_0,p_1)$ and $M(D',p'_0,p'_1)$ where $1 = 1_{L_{\nu,\nu'}}$ is the identity endomorphism of $L_{\nu,\nu'}$.*

Proof: By Lemma 5.3 and 5.4, $f((\chi_n,\beta)^*,(\beta,\beta)) = (\phi,(\beta,\beta))$ and $f((\beta,\chi_n)^*,(\beta,\beta)) = (\psi,(\beta,\beta))$ with $\phi,\psi \in F(D',p'_0,p'_1)$ satisfying $\phi\wedge\psi = (\chi_n,\chi_n)^*$. From $f(Q)\subseteq A$ and the definition of A it follows that there are $r,q\in \mathrm{Hom}(D',\mathbf{2})$ with $\phi(r),\psi(q)\in\{(\chi_n,\beta),(\beta,\chi_n)\}$.

By Corollary 5.1, $f((\chi_{n-1},\beta)^*,(\beta,\beta)) = \phi'',(\beta,\beta))$ with $\phi''\in F(D',p'_0,p'_1)$ such that for every $p\in\mathrm{Hom}(D',\mathbf{2})$ either $\phi(p)=\phi''(p)$ or else there exists $j_p\in\{1,2,\dots,n\}$ such that either $\phi(p) = (\chi_{j_p},\beta)$ and $\phi''(p) = (\chi_{j_p-1},\beta)$ or else $\phi(p) = (\beta,\chi_{j_p})$ and $\phi''(p) = (\beta,\chi_{j_p-1})$. For $i\in n$, define $g_i : K \to M(D,p_0,p_1)$ by $g_i(x) = ((x^*_{i,1},\beta)^*,(\beta,\beta))$; then g_i is a one-to-one lattice homomorphism from K to $M(D,p_0,p_1)$ with $g_i(0) = ((\chi_{i+1},\beta)^*,(\beta,\beta))$ and $g_i(1) = ((\chi_i,\beta)^*,(\beta,\beta))$ for all $i\in n$. Continuing an easy induction using these maps shows that $f((\chi_0,\beta)^*,(\beta,\beta)) = (\phi',(\beta,\beta))$, where for each $p\in\mathrm{Hom}(D',\mathbf{2})$, either $\phi(p)=\phi'(p)$ or else there exist i_p and j_p such that $0\le i_p<j_p\le n$ and either $\phi(p)=(\chi_{j_p},\beta)$ and $\phi'(p)=(\chi_{i_p},\beta)$, or $\phi(p)=(\beta,\chi_{j_p})$ and $\phi'(p)=(\beta,\chi_{i_p})$. In the same manner, we find that $f((\beta,\chi_0)^*,(\beta,\beta)) = (\psi',(\beta,\beta))$, where for each $p\in\mathrm{Hom}(D',\mathbf{2})$, either $\psi(p)=\psi'(p)$ or else there exist k_p and l_p such that $0\le k_p<l_p\le n$ and either $\psi(p)=(\chi_{l_p},\beta)$ and $\psi'(p)=(\chi_{k_p},\beta)$, or $\psi(p)=(\beta,\chi_{l_p})$ and $\psi'(p)=(\beta,\chi_{l_p})$.

Suppose that $j_p < n$ for some $p \in \mathrm{Hom}(D', \mathbf{2})$. Then $\phi(p) \in \{(\beta, \chi_{j_p}), (\chi_{j_p}, \beta)\}$ and, since $\phi(p) \wedge \psi(p) = (\chi_n, \chi_n)$, claim (4) of Propositions 2.1 or 3.2 implies that $\psi(p) = (\chi_n, \chi_n)$ and thus $\psi'(p) = \psi(p)$ by Corollary 5.1. From $((\beta, \chi_0)^*, (\beta, \beta)) \geq ((\chi_n, \beta)^*, (\beta, \beta))$ it follows that

$$\text{(b)} \qquad (\psi, (\beta, \beta)) = f((\beta, \chi_0)^*, (\beta, \beta)) \geq f((\chi_n, \beta)^*, (\beta, \beta)) = (\phi, (\beta, \beta)),$$

and hence $(\chi_n, \chi_n) = \psi'(p) \geq \phi(p)$, a contradiction. Therefore $j_p = n$ for every $p \in \mathrm{Hom}(D', \mathbf{2})$ with $\phi(p) \neq \phi'(p)$. The same argument shows that $l_p = n$ for every $p \in \mathrm{Hom}(D', \mathbf{2})$ with $\psi(p) \neq \psi'(p)$.

Next assume that $p \in \mathrm{Hom}(D', \mathbf{2})$ is such that $\phi(p) \neq \phi'(p)$ and $\psi(p) = \psi'(p)$. Then (b) implies that $\psi(p) = \psi'(p) \geq \phi(p)$, and this is a contradiction because $(\chi_n, \chi_n) \neq \phi(p) = \phi(p) \wedge \psi(p) = (\chi_n, \chi_n)$. Thus we conclude that $\phi(p) \neq \phi'(p)$ if and only if $\psi(p) \neq \psi(p')$ for each $p \in \mathrm{Hom}(D, \mathbf{2})$.

Now consider $p \in \mathrm{Hom}(D', \mathbf{2})$ for which $\phi(p) = \phi'(p)$. Then $\psi(p) = \psi'(p)$ by the previous paragraph, and (b) and $f((\chi_0, \beta)^*, (\beta, \beta)) \geq f((\beta, \chi_n)^*, (\beta, \beta))$ imply that $\phi(p) = \phi'(p) \geq \psi(p) \geq \phi(p)$. Using $\phi(p) \wedge \psi(p) = (\chi_n, \chi_n)$, we find that $\phi(p) = \psi(p) = (\chi_n, \chi_n)$. By Lemma 5.4, $f((\beta, \chi_0)^*, (\beta, \beta)) \geq f((\beta, \beta)^*, (\beta, \chi_n)) = ((\beta, \beta)^*, (\beta, \chi_n))$ and hence $\psi(p) \geq (\beta, \beta)$ — and this is a contradiction. Whence $\phi(p) \neq \phi'(p)$ and $\psi(p) \neq \psi'(p)$ for every $p \in \mathrm{Hom}(D', \mathbf{2})$.

Thus $\{\phi(p), \psi(p)\} \subseteq \{(\beta, \chi_n), (\chi_n, \beta)\}$, and $\phi(p) \wedge \psi(p) = (\chi_n, \chi_n)$ implies that $\phi(p) \neq \psi(p)$, and hence $\{\phi(p), \psi(p)\} = \{(\beta, \chi_n), (\chi_n, \beta)\}$. From (b) and $f((\chi_0, \beta)^*, (\beta, \beta)) \geq f((\beta, \chi_n)^*, (\beta, \beta))$ it follows that $\phi'(p) \geq \psi(p)$ and $\psi'(p) \geq \phi(p)$. Hence $k_p = i_p = 0$ because for no $i > 0$ we have $(\chi_i, \beta) \geq (\beta, \chi_n)$ or $(\beta, \chi_i) \geq (\chi_n, \beta)$. Altogether, $f(F(D, p_0, p_1) \times \{(\beta, \beta)\}) \subseteq F(D', p'_0, p'_1) \times \{(\beta, \beta)\}$, $f((\chi_n, \chi_n)^*, (\beta, \beta)) = ((\chi_n, \chi_n)^*, (\beta, \beta))$ and $f((\chi_0, \chi_0)^*, (\beta, \beta)) = ((\chi_0, \chi_0)^*, (\beta, \beta))$. Let $g : F(D, p_0, p_1) \to F(D', p'_0, p'_1)$ be the mapping defined by $f(\phi, (\beta, \beta)) = (g(\phi), (\beta, \beta))$ for all $\phi \in F(D, p_0, p_1)$. Then g is a lattice $(0,1)$-homomorphism from $F(D, p_0, p_1)$ to $F(D', p'_0, p'_1)$. By Lemma 5.4 and by the definition of F, we conclude that $g((\beta, \beta)^*) = (\beta, \beta)^*$. Thus f and $g \times 1_{L_{\nu,\nu'}}$ coincide on the intervals $F(D, p_0, p_1) \times \{(\beta, \beta)\}$ and $\{(\beta, \beta)^*\} \times L_{\nu,\nu'}$. Since the union of these intervals generates $M(D, p_0, p_1)$, it follows that f is the domain-range restriction of $g \times 1_{L_{\nu,\nu'}}$ to $M(D, p_0, p_1)$ and $M(D', p'_0, p'_1)$, and the proof is complete. $\square$

We extend M to a functor from $\mathbb{U}$ into $\mathbb{K}$. Since $M(D, p_0, p_1)$ is a $(0,1)$-sublattice of $F(D, p_0, p_1) \times L_{\nu,\nu'} \in \mathbb{K}$ we conclude that $M(D, p_0, p_1)$ is a 0-lattice from $\mathbb{K}$ for any $\mathbb{U}$-object (D, p_0, p_1). For a $\mathbb{U}$-morphism $f : (D, p_0, p_1) \to (D, p_0, p_1)$, the mapping Ff is a lattice $(0,1)$-homomorphism from $F(D, p_0, p_1)$ to $F(D', p'_0, p'_1)$ with $Ff((\beta, \beta)^*) = (\beta, \beta)^*$. Thus $Ff \times 1_{L_{\nu,\nu'}}(F(D, p_0, p_1) \times \{(\beta, \beta)\}) \subseteq F(D', p'_0, p'_1) \times \{(\beta, \beta)\}$ and $Ff \times 1_{L_{\nu,\nu'}}(\{(\beta, \beta)^*\} \times L_{\nu,\nu'}) \subseteq \{(\beta, \beta)^*\} \times L_{\nu,\nu'}$. Therefore $Ff \times 1_{L_{\nu,\nu'}}(M(D, p_0, p_1)) \subseteq M(D', p'_0, p'_1)$, and we define Mf

as the domain-range restriction of $Ff \times 1_{L_{\nu,\nu'}}$ to $M(D,p_0,p_1)$ and $M(D',p_0',p_1')$. Then Mf is a $(0,1)$-homomorphism from $M(D,p_0,p_1)$ to $M(D',p_0',p_1')$. Whence M is correctly defined faithful functor from $\mathbb{U}$ into $\mathbb{K}$. By Lemma 5.5, M is an almost full embedding. Now we are ready to complete

Proof of Theorem 0: Clearly (2) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (4). The implication (4) $\Longrightarrow$ (3) was proved in [3]. Since $\mathbb{V}$ is finitely generated any simple lattice in $\mathbb{V}$ must be finite. The functor M defined on the basis of Lemma 5.5 establishes the implication (3) $\Longrightarrow$ (2). □

The lattice K generating the variety $\mathbb{K}$ need not be finite — our arguments also give

Corollary 5.6. *Any variety $\mathbb{K}$ of lattices containing a finitely generated non-distributive simple lattice K such that every non-constant endomorphism of K preserves* 0 *and* 1 *is almost universal.*

Remark. The observations made in the introductory section and the properties of the functor M imply that any variety $\mathbb{K}$ satisfying the hypothesis of Corollary 5.6 or that of Theorem 0 is $\mathbb{V}$-relatively almost universal for any proper subvariety $\mathbb{V}$ of $\mathbb{K}$, and also weakly almost var-universal.

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