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Relatively additive states on quantum logics

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Dedicated to Prof. Věra Trnková on the occasion of her birthday.

Abstract. In this paper we carry on the investigation of partially additive states on quantum logics (see [2], [5], [7], [8], [11], [12], [15], [18], etc.). We study a variant of weak states — the states which are additive with respect to a given Boolean subalgebra. In the first result we show that there are many quantum logics which do not possess any 2-additive central states (any logic possesses an abundance of 1-additive central state — see [12]). In the second result we construct a finite 3-homogeneous quantum logic which does not possess any two-valued 1-additive state with respect to a given Boolean subalgebra. This result strengthens Theorem 2 of [5] and presents a rather advanced example in the orthomodular combinatorics (see also [9], [13], [4], [6], [16], etc.). In the rest we show that Greechie logics allow for 2-additive three-valued states, and in case of Greechie lattices we show that one can even construct many 2-additive two-valued states. Some open questions are posed, too.

Keywords: (weak) state on quantum logic, Greechie paste job, Boolean algebra

Classification: 03G12, 46C05, 81P10

1. Introduction

The investigation of weak states on quantum logics began with the finding that there are (finite lattice) quantum logics which do not have any states ([4], see also [8], [18]). The intrinsic properties of quantum logics were then seen to be linked with the properties (and the size) of the set of weak states (see [12], [5], [15], etc.). In this note we take up a few questions which were left open in the previous study of weak states. (The Greechie past job is used in places and is assumed to be known by the reader — see [4], [13], etc. Our construction (Theorem 2.7) adds another dimension to the applicability of the Greechie pasting.)

2. Notions. Results

In the sequel a triple $(L, \leq, {}')$ is said to be a quantum logic (abbr., a logic) if L is an orthomodular poset. This means, a logic is a set L endowed with a partial

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ordering, $\leq$, and a binary operation, $'$, such that the following conditions are satisfied:

(i) $0, 1 \in L$ (i.e., L contains a least and a greatest element),
(ii) $a \leq b \quad \Rightarrow \quad b' \leq a'$ for any $a, b \in L$,
(iii) $(a')' = a$ for any $a \in L$,
(iv) $a \wedge a' = 0$, and $a \vee a' = 1$ for any $a \in L$,
(v) if $a, b \in L$ and if $a \leq b$, then $b = a \vee (b \wedge a')$.

The basic definition of this paper is as follows (recall that two elements $a, b \in L$ are said to be orthogonal if $a \leq b'$).

Definition 2.1. Let L be a quantum logic and let B be a Boolean subalgebra of L. Let n be a natural number. We say that a mapping $s: L \to \langle 0, 1 \rangle$ is an n-additive state on L relative to B (abbr., s is an n-B-state) if the following conditions are satisfied:

(i) $s(1) = 1$,
(ii) if $a \leq b$ $(a, b \in L)$, then $s(a) \leq s(b)$,
(iii) $s(a) = 1 - s(a')$ for any $a \in L$,
(iv) if $a_1, a_2, \dots, a_n \in L$ and $b \in B$, $b \neq 0$, and if all elements $a_1, a_2, \dots, a_n, b$ are mutually orthogonal, then $s(a_1 \vee a_2 \vee \dots \vee a_n \vee b) = s(a_1) + s(a_2) + \dots + s(a_n) + s(b)$.

We say that $s: L \to \langle 0, 1 \rangle$ is a B-state if s is an n-B-state for any $n \in \mathbb{N}$.

Of course, a mapping $s: L \to \langle 0, 1 \rangle$ is a (standard) state on L if s is a 1-B-state for any Boolean subalgebra of L. In the paper, we shall mainly deal with 2-B-states and 1-B-states since, as the following simple results says, these are essentially all possibilities.

Proposition 2.2. *Let B be a Boolean subalgebra of a logic L. Let $s: L \to \langle 0, 1 \rangle$ be a mapping which satisfies the conditions* (i), (ii), *and* (iii) *of Definition* 2.1. *Then the following conditions are equivalent.*

(i) *The mapping s is a B-state.*
(ii) *The mapping s is a 2-B-state.*
(iii) *The mapping s is additive on any set $a_1, a_2, \dots, a_n$ $(n \geq 2)$ of mutually orthogonal elements of L such that $\{a_1, a_2, \dots, a_n\} \cap (B \setminus \{0\}) \neq \emptyset$.*

Proof: The conditions (i) and (iii) are obviously equivalent and (i) implies (ii). To show (ii) implies (i), suppose that $a_1 \in B \setminus \{0\}$ and that the statement is true for any set $\{a_1, a_2, \dots, a_k\}$, where $k \leq n$. Then

$$\begin{aligned} s(a_1 \vee a_2 \vee \ldots \vee a_n \vee a_{n+1}) &= s(a_1) + s(a_2) + s(a_3 \vee \ldots \vee a_n \vee a_{n+1}) \\ &= s(a_2) + s(a_1 \vee a_3 \vee \ldots \vee a_n \vee a_{n+1}) \\ &= s(a_1) + s(a_2) + s(a_3) + s(a_4) + \ldots \\ &\quad + s(a_n) + s(a_{n+1}). \end{aligned}$$

□

Proposition 2.3. *Let L, B, and s be as in Proposition 2.2. Then s is a B-state if and only if for any Boolean subalgebra $\tilde{B}$ of L for which $\tilde{B} \cap B \neq \{0,1\}$ the mapping s restricted to $\tilde{B}$ is a state.*

Proof: It is a simple verification since any orthogonal family in L is contained in a Boolean subalgebra. $\square$

Let L be a logic and let B be a Boolean subalgebra of L. We see that we can think of three classes of "weak" states — the class $\mathcal{C}_1$ of 1-B-states, $\mathcal{C}_2$ of B-states (which coincide with 2-B-states), and $\mathcal{C}_3$ of B-states which are (standard) states when restricted to any $\tilde{B}$ with $\tilde{B} \cap B \neq \{0,1\}$. As checked above, $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3$. Let us note that all these inclusions are proper (this will also follow as a by-product of our investigation).

Before we launch on the investigation proper, let us exhibit some examples illustrating how the B-states may look like.

Examples:

1. If $B = \{0,1\}$, then L always contains an abundance of B-states. Indeed, in this case the condition (iv) does not apply and the conditions (i), (ii) and (iii) of Definition 2.1 are satisfied by an "order determining family" of such states (see e.g. [15]).
2. The logic depicted by the Greechie diagram in Figure 1 has the following property: If B is the Boolean subalgebra of L indicated in the diagram and if $s\colon L \to \langle 0,1\rangle$ is a two-valued 1-B-state then $s(a) = 1$ implies that $s(b) = 0$. (Indeed, if $s(a) = s(b) = 1$, then $s(c) = s(d) = 1$ which is absurd.)

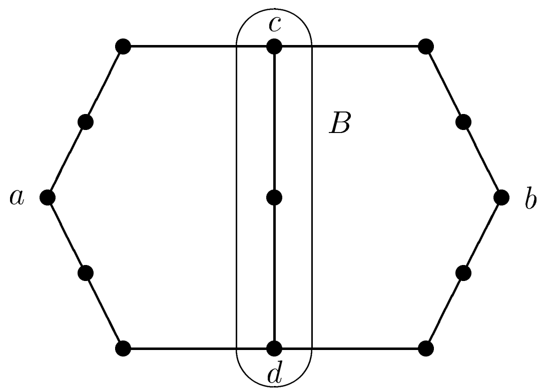

Figure 1: Greechie diagram

3. If $L = L(\mathbb{R}^3)$, where the latter symbol means the logic of subspaces in $\mathbb{R}^3$, and if a, b are two atoms of L, then we can easily show that there is a Boolean subalgebra B of L which contains b and a two-valued B-state s on L such

that $s(a) = 0$ and $s(b) = 1$. Indeed, we take $B = \{b, b', 0, 1\}$ and we set $s(b) = 1$, $s(b') = 0$; for $c \neq b$ we set $s(c) = 0$ if $\dim c \leq 1$ and $s(c) = 1$ if $\dim c \geq 2$ and c is different from b'. (As known, there are no (standard) two-valued states on $L(\mathbb{R}^3)$, see e.g. [13].)

Before we investigate on the existence and quantity of 2-B-states on a logic, let us remark that the abundance of these states, which is not automatically guaranteed as we will see later, allows us to generalize the following folklore Boolean results. Since the proofs follow the standard Boolean way, we omit them (see e.g. [15] for a detailed treatment of weak states in this connection).

Proposition 2.4. *Let L be a quantum logic and let B be a Boolean subalgebra of L. Let us suppose that for any pair $a \not\leq b$ there is a two-valued 2-B-state such that $s(a) = 1$ and $s(b) = 0$. Then the following statement holds true: There is a set, S, and a collection, Δ, of subsets of S such that we can find a one-to-one mapping $\varphi: L \to \Delta$ with the following properties:* (i) *$\varphi(1) = S$,* (ii) *$a \leq b$ in $L \Leftrightarrow \varphi(a) \subset \varphi(b)$,* (iii) *$\varphi(a') = S \setminus \varphi(a)$ for any $a \in L$, and* (iv) *if $a_1, a_2, \dots, a_n \in L$, $b \in B$, $b \neq 0$ and if the elements a_i, b $(i \leq n)$ are mutually orthogonal, then*

$$\varphi(a_1 \vee a_2 \vee \dots \vee a_n \vee b) = \varphi(a_1) \cup \varphi(a_2) \cup \dots \cup \varphi(a_n) \cup \varphi(b).$$

Proposition 2.5. *Let L be a logic and let B be a Boolean subalgebra of L. Let $s: B \to \langle 0, 1 \rangle$ be a (standard) state. Let us suppose that for any $a \in L, a \neq 0$ there is a 1-B-state on L (resp. 2-B-state on L) such that $s(a) = 1$. Then s can be extended over L as a 1-B-state (resp. 2-B-state).*

The question arises whether the assumptions on the existence (abundance) of states of the previous propositions are not redundant. In fact, in some cases they are — if e.g. B is the centre of L, then there is always an abundance of 1-B-states on L (see [5]). But if we wanted to generalize this result to 2-B-states, we are in for a mild surprise. (Recall that $C(L)$, for a logic L, denotes its centre. As known, $C(L)$ is the intersection of all maximal Boolean subalgebras (blocks) of L and, therefore, it is itself a Boolean subalgebra of L.)

Theorem 2.6. *Let B be a nontrivial Boolean algebra (i.e., $B \neq \{0, 1\}$). Then there exists a (lattice) logic, L, such that $B = C(L)$ and such that L does not possess any 2-$C(L)$-additive state.*

Proof: Let (S, Δ) be the Stone representation of B (thus, $\Delta = B$). Let G be the Greechie logic (i.e., let G be the (lattice) logic without any (standard) state — see [4]; it is easily seen that $C(G) = \{0, 1\}$). Let us take for L the (bounded) Boolean power of Δ and G (see also [3] for non-lattice versions of Boolean product). In other words, let L be the sublogic of $\Pi_{s \in S} G_s$, where $G_s = G$ for any $s \in S$, such that $k = (k_s \mid s \in S) \in L \Leftrightarrow$ there is a finite partition $\mathcal{P}$ of S,

$\mathcal{P} = \{S_1, S_2, \dots, S_n\}$, such that $S_i \in \Delta$ for any $i \leq n$ and such that $h_p = h_q$ whenever $p, q \in S_i$ $(i \leq n)$. Let us observe first that $C(L) = \Delta(= B)$. Indeed, if $h = (h_s \mid s \in S) \in C(L)$, then all coordinates h_s $(s \in S)$ of h must be either 0 or 1 (we make use of the fact that $C(G) = \{0, 1\}$). Let s be a 2-B-state on L. Take a partition of S consisting of two non-void sets S_1 and S_2. Thus, let us organize for $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ and both S_1, S_2 belong to Δ. Let k (resp. m) be the element of L such that $k_s = 1$ if $s \in S_1$, $k_s = 0$ otherwise (resp. $m_s = 1$ if $s \in S_2, m_s = 0$ otherwise). Then $k = m'$ in L, $k \neq 0$, $m \neq 0$, and both k, m belong to $C(L)$. Moreover, $k \vee m = 1$. Using 2-$C(L)$-additivity, it follows that $s(k) + s(m) = 1$. Suppose that $s(k) > 0$ (if not, we pass to m and have $s(m) > 0$). Since s is 2-$C(L)$-additive, we can easily construct a (standard) state, $\tilde{s}$, on the logic $K = \{p \in L \mid p \leq k\}$. Indeed, we let $\tilde{s}(p) = \dfrac{s(p \wedge k)}{s(k)}$ and realize that if $p_1 \leq p_2', p_1, p_2 \in K$, then $s(p_1 \vee p_2) = s(p_1 \vee p_2) + s(m) - s(m) = s(m \vee p_1 \vee p_2) - s(m) = s(m) + s(p_1) + s(p_2) - s(m) = s(p_1) + s(p_2)$. But this would imply the existence of a (standard) state on the logic $\Pi_{s \in S_1} G_s$. This is impossible — if we had a state on $\Pi_{s \in S_1} G_s$, we could easily construct a state on G_s. Indeed, such a state could be obtained as a composition with a natural embedding of G_s in $\Pi_{s \in S_1} G_s$. We have reached a contradiction. So there is no 2-$C(L)$-additive state on L and the proof is complete. □

If B is a general subalgebra of L, it seems conceivable that L has 1-B-states (since it has many (two-valued) 1-$C(L)$-states which at that are states when restricted to B — see [5]). We have not been able to answer this question in full generality. In view of the impact of Greechie example, which would then be strengthened if there is an L without any 1-B-state, it may present a fairly important question of orthomodular combinatorics. Also, the question of the existence of two-valued 1-B-states is of some importance in view of the set-representation of L. This question can be answered in the following stronger form, improving somewhat on a result of [5]. Let us recall that L is called 3-*homogeneous* if all maximal Boolean subalgebras of L have 3 atoms. These logics play a significant role in the combinatorial line of the theory of quantum logics (see [4], [13], [14], etc.).

Theorem 2.7. *There exists a finite* 3*-homogeneous quantum logic with a* 3*-atom Boolean subalgebra B such that L does not possess any two-valued* 1*-B-state.*

Proof: The construction we present is quite combinatorially involved. It will be obtained in several steps. We are to generalize the Greechie past job. Since this generalization seems to be useful in its own right, we shall explicitly formulate its basic ideas. First, recall briefly the technique of Greechie past job ([4], see also [9] and [10] for further analysis and more applications). □

The gist of Greechie technique is the following "loop lemma". Let Ω be a non-empty finite set. Let $\exp(\Omega)$ denote the set of all subsets of Ω and let $|A|$

denote the number of elements of A. Let $\mathbb{B} = \{\mathcal{L} \subset \exp(\Omega) \mid |B| \geq 3$ for any $B \in \mathcal{L}$ and $|A \cap B| \leq 1$ for any pair $A, B \in \mathcal{L}, A \neq B\}$. Let $\mathcal{L} \in \mathbb{B}$. By a *loop of order* 3 in $\mathcal{L}$ we mean a triple $A_1,\ A_2,\ A_3$ of elements of $\mathcal{L}$ such that, upon identifying $A_3 = A_0$, $A_{i-1} \cap A_i \neq \emptyset$ $(i = 1, 2, 3)$ and, moreover, $A_1 \cap A_2 \cap A_3 = \emptyset$. Suppose that $\mathcal{L}$ does not contain any loop of order 3 (resp. any loop of order 4). Then (the Greechie theorem, [4]) $\mathcal{L}$ can be viewed as a finite quantum logic (resp. lattice logic), $\mathcal{G}(\mathcal{L})$, such that $\bigcup_{A\in\mathcal{L}} A$ forms the set of all atoms of $\mathcal{G}(\mathcal{L})$ and the maximal Boolean subalgebras of $\mathcal{G}(\mathcal{L})$ are in a one-to-one correspondence with all $A \in \mathcal{L}$.

Let us denote by $\mathbb{L}$ the subset of $\mathbb{B}$ consisting of those $\mathcal{L}$ which do not contain loops of order 3. For our construction, we need to check that some elements of $\mathbb{L}$ allow for certain manipulations and give rise to new (more involved) logics. Let us formulate the results as lemmas. (We stick to the notation introduced above.)

Lemma 2.8. *Let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n \in \mathbb{B}$. Let $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}_i, \mathcal{D} = \bigcap_{i=1}^n \mathcal{L}_i$. Suppose that for all $i, j,\ i \neq j$ the following implication* (*denoted by* *) *holds true: If $A \in \mathcal{L}_i \setminus \mathcal{L}_j$ and $B \in \mathcal{L}_j \setminus \mathcal{L}_i$, then there is a $D \in \mathcal{D}$ such that $A \cap B \subset D$. Then $\mathcal{L} \in \mathbb{B}$.*

Proof: Obviously, $|A| \geq 3$ for any $A \in \mathcal{L}$. Let $A, B \in \mathcal{L}$. If both A, B belong to an $\mathcal{L}_i$ for some $i \leq n$, then obviously $|A \cap B| \leq 1$. Suppose that $A \in \mathcal{L}_i \setminus \mathcal{L}_j$ and $B \in \mathcal{L}_j \setminus \mathcal{L}_i$ $(i \neq j)$. Then $A \cap B \subset D$ for some $D \in \mathcal{D}$ and since A, D belong to $\mathcal{L}_i$, we infer that $|A \cap B| \leq |A \cap D| \leq 1$. □

Lemma 2.9. *Let the families $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n \in \mathbb{L}$ satisfy the assumptions of Lemma* 2.8*. Let $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}_i$ and let $\mathcal{D} = \bigcap_{i=1}^n \mathcal{L}_i = \{D_1, D_2\}$. Suppose further that $D_1 \cap D_2 \neq \emptyset$ and that $\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{D}$ provided $i \neq j$. Then $\mathcal{L} \in \mathbb{L}$ and therefore $\mathcal{L}$ induces a quantum logic, $\mathcal{G}(\mathcal{L})$.*

Proof: By Lemma 2.8, $\mathcal{L} \in \mathbb{B}$. We will show that $\mathcal{L}$ does not contain any loop of order 3. Suppose that $B_1,\ B_2,\ B_3$ is a loop in $\mathcal{L}$. Upon writing $B_0 = B_3$, there are elements $a_i \in \Omega$ $(i = 1, 2, 3)$ such that $B_{i-1} \cap B_i = \{a_i\}$. Since $B_1 \cap B_2 \cap B_3 = \emptyset$, it is obvious that the elements a_1, a_2, a_3 are distinct. We shall be discussing two cases using the implication that if $\{B_{i-1}, B_i\} \not\subset \mathcal{L}_k$ for any $k \leq n$, then $a_i \in D_1 \cup D_2$ as follows from the implication * of Lemma 2.8. Let us denote the previous implication by **.

In the first case, assume that there is an $i \in \{1, 2, 3\}$ such that $B_i \in \mathcal{D}$. Without any loss of generality, let us assume $B_1 = D_1$. Then for no k we have $\{B_2, B_3\} \subset \mathcal{L}_k$ (otherwise B_1, B_2, B_3 is a loop in $\mathcal{L}_k$). Therefore $B_2 \notin \mathcal{D}$ and $a_3 \in D$ for a $D \in \mathcal{D}$. Since $a_3 \notin B_1$, then $B_1 \neq D$. It follows that B_1, B_2, D is a loop of order 3 in $\mathcal{L}_k$ where $B_2 \in \mathcal{L}_k$ — a contradiction.

In the second case, assume that $\{B_1, B_2, B_3\} \cap \mathcal{D} = \emptyset$. Then there is an $i \in \{1, 2, 3\}$ such that $a_i \notin D_1 \cup D_2$. Otherwise $\{a_1, a_2, a_3\} \subset D_1 \cup D_2$ and therefore either D_1 or D_2 contains two elements of the set $\{a_1, a_2, a_3\}$. Suppose e.g. that $a_1, a_2 \in D_1$. Then $\{a_1, a_2\} \subset D_1 \cap B_1$ and therefore $|D \cap B_1| \geq 2$

which is a contradiction (if $B_1 \notin \mathcal{D}$, then $B_1 \neq D$). Suppose that $a_1 \notin D_1 \cup D_2$. Then there is an i such that $B_1, B_3 \in \mathcal{L}_i$ by **. Moreover, $B_1, B_2 \not\subset \mathcal{L}_k$ for any k. If not, we have $B_1 \in \mathcal{L}_i \cap \mathcal{L}_k$ for some k. Since $\{B_1, B_2, B_3\} \not\subset \mathcal{L}_i$ for any i (there are no loops of order 3 in $\mathcal{L}_i$'s), we see that $i \neq k$ and therefore $B_1 \in \mathcal{L}_i \cap \mathcal{L}_k = \mathcal{D}$ — a contradiction. As a result, $\{B_1, B_2\} \not\subset \mathcal{L}_k$ for any k. It follows that $a_2 \in D_1 \cup D_2$. By an analogous reasoning, $a_3 \in D_1 \cup D_2$. Since D_i cannot contain both a_2, a_3 as shown above, we see that $D_1 \cap B_2 \neq \emptyset$, $D_1 \neq D_2$, and $D_2 \cap B_2 \neq \emptyset$. Thus, B_2, D_1, D_2 is a loop of order 3 in $\mathcal{L}_k$ where $B_2 \in \mathcal{L}_k$ — a contradiction. □

In our construction that follows, we will apply Lemma 2.9 twice, first for $n = 2$ and $D_1 \neq D_2$, then for $n = 3$ and $D_1 = D_2$. The basic stone is the following Greechie diagram (let us call it “the first step construction”).

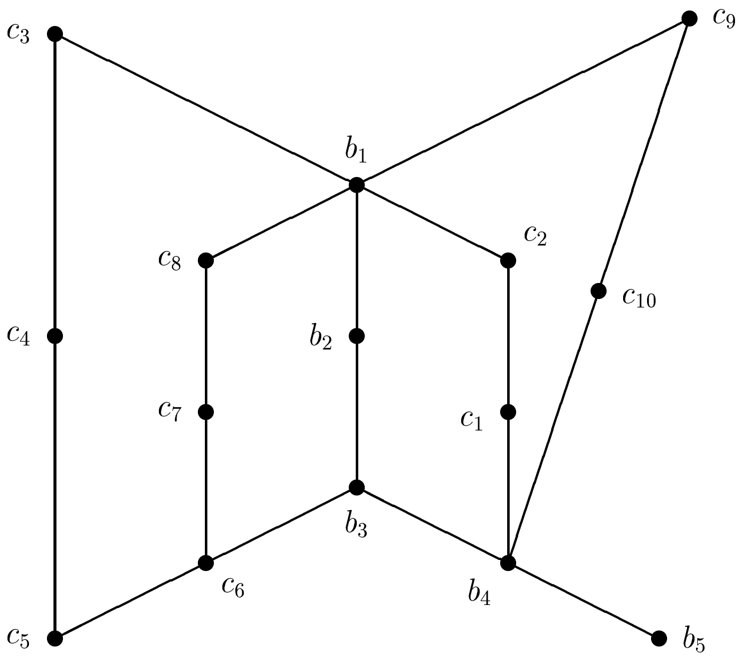

Figure 2: The first step construction

Lemma 2.10. *Let us consider the logic L given by the Greechie diagram of Figure 2 (the first step construction). Let B be the Boolean subalgebra of L the atoms of which are b_1, b_2, b_3. Then there is no two-valued 1-B-state on L such that $s(b_2) = 1$ and $s(b_4) = 1$.*

Proof: Suppose that s is a two-valued 1-B-state and $s(b_2) = s(b_4) = 1$. Then $s(b_1) = s(b_3) = s(c_1) = s(c_2) = 0$. It follows that $s(c_3) = 1$, $s(c_4) = s(c_5) = 0$, $s(c_6) = 1$, $s(c_7) = s(c_8) = 0$. Then $s(c_9) = 1$ which is a contradiction. This completes the proof. □

In what follows, we will extend, intuitively speaking, the example ("the first step construction") by pasting to it its copy in a suitable way, preserving B and interchanging only the atoms in the block $\tilde{B}$ generated by b_3, b_4, b_5. In the first approximation we produce "the second step construction" with the property that for any two-valued 1-B-state s we have $s(b_2) = 0$. In a formal way by making use of Lemma 2.9, let Ω be the set of atoms of "the first step construction" and $\mathcal{L}_0$ be the collection of its blocks. Let $B = \{b_1, b_2, b_3\}$, $\tilde{B} = \{b_3, b_4, b_5\}$. It is easy to show that if σ is an injective mapping defined on Ω such that $\sigma(\mathcal{L}_0) = \{\sigma(A) \mid A \in \mathcal{L}_0\}$ then any two-valued 1-$\sigma(B)$-state on $\mathcal{G}(\sigma(\mathcal{L}_0))$ has the property $s(\sigma(b_4)) = 1 \Rightarrow s(\sigma(b_2)) = 0$. Let us define two injective mappings σ_1, σ_2 on Ω as follows:

$$\begin{aligned}
\sigma_1(b_i) &= b_i && (i = 1, \dots, 5)\\
\sigma_1(b) &= (b, 1) && (b \in \Omega \setminus \{b_1, \dots, b_5\})\\
\sigma_2(b_i) &= b_i && (i = 1, 2, 3)\\
\sigma_2(b_4) &= b_5 &&\\
\sigma_2(b_5) &= b_4 &&\\
\sigma_2(b) &= (b, 2) && (b \in \Omega \setminus \{b_1, \dots, b_5\})
\end{aligned}$$

Then we have $\sigma_1(B) = \sigma_2(B) = B$ and $\sigma_1(\tilde{B}) = \sigma_2(\tilde{B}) = \tilde{B}$. Moreover, $\mathcal{L}_i = \sigma_i(\mathcal{L}_0)$ satisfy the assumption of Lemma 2.9. Therefore $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$ belongs to $\mathbb{L}$ (in particular, it does not contain a loop of order 3). Thus, $\mathcal{L}$ determines in the sense of the Greechie theorem a quantum logic, $\mathcal{G}(\mathcal{L})$, ("the second step construction"). Write $\mathcal{G}(\mathcal{L}) = R$. Let us show that $s(b_2) = 0$ for any two-valued 1-B-state on R.

Let s be a two-valued 1-B-state on R. Let R_1 be a sublogic of R determined by $\mathcal{L}_1$ (thus, $R_1 = \mathcal{G}(\mathcal{L}_1)$). Then $s_1 = s|R_1$ is a two-valued 1-B-state on R_1. Further, s_1 has the following property:

$$s(b_4) = s_1(\sigma_1(b_4)) = 1 \Rightarrow s_1(b_2) = 0.$$

Let R_2 be the sublogic of L determined by $\mathcal{L}_2$ (thus, $R_2 = \mathcal{G}(\mathcal{L}_2)$). Then setting $s_2 = s\,|R_2$ one obtains in the same way the validity of the implication

$$s(b_5) = s_2(\sigma_2(b_4)) = 1 \Rightarrow s_2(b_2) = 0.$$

Obviously, $s(b_3) = 1 \Rightarrow s(b_2) = 0$ since s is additive on the logic corresponding to B. But one of the values $s(b_3)$, $s(b_4)$, $s(b_5)$ must be 1. Therefore, in any case, $s(b_2) = 0$.

Finally, let us construct the desired example ("the final construction").

Let us change the notation. Let Ω be the set of all atoms of "the second step construction" and let $\mathcal{L}_0$ be the collection of its blocks. We have constructed the logic $R = \mathcal{G}(\mathcal{L}_0)$ with the property $s(b_2) = 0$ for any two-valued 1-B-state on R. In our notation adopted above we have

$$B = \{b_1, b_2, b_3\} \in \mathcal{L}_0.$$

Let us define injective mappings σ_1 , σ_2 , σ_3 on the underlying set Ω of atoms of R as follows

$$\begin{aligned} \sigma_i(b) &= (b, i) \quad \text{for} \quad && b \in \Omega \setminus \{b_1, b_2, b_3\}; \quad i = 1, 2, 3, \\ \sigma_i(b_j) &= b_{j+i} \pmod 3 && \text{in all other cases.} \end{aligned}$$

Then $\sigma_1(B) = \sigma_2(B) = \sigma_3(B) = B$. Let us apply Lemma 2.9 for $\mathcal{L}_i := \sigma_i(\mathcal{L}_0)$ $(i = 1, 2, 3)$. Let us show that the logic corresponding to $\mathcal{L} = \bigcup_{i=1}^{3} \mathcal{L}_i$ has no two-valued 1-B-state. Suppose that s is a two-valued 1-B-state on this logic. As before, each $s_i = s \,|\, \mathcal{G}(\mathcal{L}_i)$ defines a two-valued 1-B-state on the sublogics corresponding to $\mathcal{G}(\mathcal{L}_i)$ $(i = 1, 2, 3)$. Therefore $s_i(\sigma_i(b_2)) = 0$, i.e. $s(b_i) = 0$ $(i = 1, 2, 3)$. This is a contradiction since $\sum_{i=1}^{3} s(b_i) = 1$. The proof of Theorem 2.7 is complete.

A natural question arises whether we can construct a logic without any 1-B-state and whether there is a lattice logic without any 1-B-state. As mentioned before, if it is so, we would in a sense have a stronger result than that of the famous Greechie stateless one [4]. We are not able to answer these questions for the time being. However, we find out as a partial result that within the class of Greechie logics both questions answer in the negative — there is even an abundance of 2-B-states there. Let us recall that a logic L is said to be *Greechie* if it is finite and its blocks meet in at most one atom. Let us start with the question of the existence of a (general) 1-B-state on L.

Theorem 2.11. *Let L be a Greechie logic and let B be a block in L. Then there is a 2-B-state on L which ranges in the set $\{0, \frac{1}{2}, 1\}$.*

Before we take up the proof, let us formulate a simple lemma.

Lemma 2.12. *Let Ω be the set of atoms of L and let $s_0 \colon \Omega \to \langle 0, 1 \rangle$ be a mapping. For each block $\tilde{B}$ of L, let $s_{\tilde{B}} \colon \tilde{B} \to \langle 0, 1 \rangle$ be an order-preserving mapping such that $s_{\tilde{B}}(1) = 1$, $s_{\tilde{B}}(a) = 1 - s_{\tilde{B}}(a')$ for any $a \in \tilde{B}$ and $s_{\tilde{B}}$ agrees with s_0 on the atoms of $\tilde{B}$. Moreover, suppose that $s_{\tilde{B}}$ is a state if $\tilde{B} \cap B \neq \{0, 1\}$. Then if we set $s(x) = s_{\tilde{B}}(x)$ for each $x \in \tilde{B}$, then s is correctly defined and, moreover, it constitutes a 2-B-state on L.*

Proof: Let us verify that s is well-defined. Let $\tilde{B}_1$ and $\tilde{B}_2$ be blocks of L and let $x \in \tilde{B}_1 \cap \tilde{B}_2$. Then only four possibilities can occur: $x = 0$, $x = 1$, x is an atom, or x is a coatom. For $s_{\tilde{B}}$ we in order have four possibilities: $s_{\tilde{B}_1}(x) = 0 = s_{\tilde{B}_2}(x)$, $s_{\tilde{B}_1}(x) = 1 = s_{\tilde{B}_2}(x)$, $s_{\tilde{B}_1}(x) = s_0(x) = s_{\tilde{B}_2}(x)$, or $s_{\tilde{B}_1}(x) = 1 - s_0(x') = s_{\tilde{B}_2}(x)$. It is obvious that s has the properties (i), (ii), (iii) of Definition 2.1. Therefore s is a B-state by Proposition 2.3. □

Let us now return to the proof of Theorem 2.11. We are going to construct a suitable mapping $s_0 \colon \Omega \to \{0, \frac{1}{2}, 1\}$ on the set Ω of atoms of L. Choose an atom $b \in B$. Let $\mathcal{L}$ be the set of all blocks of L. Write $\mathcal{L}_1 = \{\tilde{B} \in \mathcal{L} : b \in \tilde{B}\}$. Let $\mathcal{L}_2$ be the set of all blocks of L containing an atom of B distinct from b. For every $\tilde{B} \in \mathcal{L}_2$, choose two distinct atoms $a_{\tilde{B}}, b_{\tilde{B}} \in \tilde{B} \setminus B$. Define $s_0(a_{\tilde{B}}) = s_0(b_{\tilde{B}}) = \frac{1}{2}$ for $\tilde{B} \in \mathcal{L}_2$ and $s_0(b) = 1$. Further, for all other atoms $a \in \Omega$ define $s_0(a) = 0$. Since $\mathcal{L}$ does not contain a loop of order 3, the sets $\Omega \cap (\tilde{B} \setminus B)$, $\tilde{B} \in \mathcal{L}_1 \cup \mathcal{L}_2$ are disjoint. It therefore follows from the definition of s_0 that $\sum_{a \in \tilde{B}} s_0(a) = 1$ for any $\tilde{B} \in \mathcal{L}_1 \cup \mathcal{L}_2$ and there is exactly one state $s_{\tilde{B}}$ on $\tilde{B}$ which agrees with s_0 on the atoms of $\tilde{B}$. If $\tilde{B} \in \mathcal{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$, we define $s_{\tilde{B}}$ as follows: $s_{\tilde{B}}(a) = s_0(a)$, $s_{\tilde{B}}(a') = 1 - s_0(a)$ for $a \in \Omega \cap \tilde{B}$ and $s_{\tilde{B}}(x) = \frac{1}{2}$ for all other elements $x \in \tilde{B}$. As a consequence of Lemma 2.12, there is a 2-B-state s on L which agrees on $\tilde{B}$ with $s_{\tilde{B}}$ for any $\tilde{B} \in \mathcal{L}$. The proof is complete.

Theorem 2.13. *Let L be a Greechie logic and let L be a lattice. Let B be a block in L and let a be an atom in L. Then there is a two-valued 2-B-state on L such that $s(a) = 1$.*

Before we present a proof, we will again formulate a lemma.

Lemma 2.14. *Let Ω be the set of all atoms of L and let $s_0 \colon \Omega \to \{0, 1\}$ be a (two-valued) mapping such that*

1. *for any block $\tilde{B}$ in L there is at most one atom $x \in \tilde{B}$ such that $s_0(x) = 1$,*
2. *for any block $\tilde{B}$ in L which contains an atom of B there is exactly one atom $x \in \tilde{B}$ such that $s_0(x) = 1$.*

Then there is a (two-valued) 2-B-state on L which agrees with s_0 on Ω.

Proof: Let $\tilde{B}$ be a block in L. If $s_0(x_0) = 1$ for exactly one atom $x_0 \in \tilde{B}$, then there is exactly one (two-valued) state $s_{\tilde{B}}$ on $\tilde{B}$ which agrees with s_0 on $\Omega \cap \tilde{B}$ (namely the Dirac measure δ_{x_0} concentrated in x_0). If $s_0(x) = 0$ for all $x \in \Omega \cap \tilde{B}$, choose $x_0 \in \tilde{B}$ and define $s_{\tilde{B}}$ as follows: $s_{\tilde{B}}(x_0) = 0$, $s_{\tilde{B}}(x_0') = 1$, and $s_{\tilde{B}}(x) = \delta_{x_0}(x)$ for all other elements $x \in \tilde{B}$. Then we can apply Lemma 2.12. □

Let us return to the proof of Theorem 2.13. Let us define a mapping, s_0, with the two properties stated in Lemma 2.14 and with an additional property of $s_0(a) = 1$. If $a \in B$, set $b = a$, otherwise choose an arbitrary atom $b \in B$. Define $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ as in the proof of Theorem 2.11. For $\tilde{B} \in \mathcal{L}_2$, choose $a_{\tilde{B}} \in \Omega \cap \tilde{B}$, where $a_{\tilde{B}} = a$ in case of $a \in \tilde{B}$, otherwise we can choose $a_{\tilde{B}} \in \tilde{B}$ in such a manner that no block contains both a and $a_{\tilde{B}}$ (the loop lemma). We put $s_0(a_{\tilde{B}}) = 1$ for $\tilde{B} \in \mathcal{L}_2$ and $s_0(a) = s_0(b) = 1$. For all other atoms $x \in \Omega$ we put $s_0(x) = 0$. Since, by the Greechie theorem, $\mathcal{L}$ does not contain any loop of order 4 or 3, the assumptions of Lemma 2.14 are satisfied. Therefore there is a (two-valued) 2-B-state s on L extending s_0. By the construction, $s(a) = 1$ and the proof is complete.

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