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A note on the structure of WUR Banach spaces

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Abstract. We present an example of a Banach space E admitting an equivalent weakly uniformly rotund norm and such that there is no $\Phi : E \to c_0(\Gamma)$, for any set Γ, linear, one-to-one and bounded. This answers a problem posed by Fabian, Godefroy, Hájek and Zizler. The space E is actually the dual space Y^* of a space Y which is a subspace of a WCG space.

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Introduction

Our motivation for the present work was two questions posed to us, in Paseky's conference (2004), by G. Godefroy and V. Zizler. Shortly after, we suspected that one of the examples of our recent paper [A-Me] is a possible candidate for answering both questions. Furthermore, discussing with S. Troyanski during his visit in Athens, we realized that Zizler's question is closely related to a problem posed by M. Fabian, G. Godefroy, P. Hájek and V. Zizler [F-G-H-Z]. Thus our goal is to show that the second example of [A-Me] answers negatively the following two questions.

Q1. Let X be a Banach space with a WUR norm. Does there exist a bounded, linear, one-to-one operator $\Phi : X \to c_0(\Gamma)$, for some set Γ?

Q2. Let X be a Banach space such that X is a subspace of a WCG and also there exists a norm-one projection $P : X^{**} \to X$. Is then X a WCG space?

The example from [A-Me] answering the aforementioned questions is a subspace Y of a Banach space X with the following properties.

(i) The space X is WCG and it does not contain ℓ_1.

(ii) Both spaces X and Y are duals, $X^{**} = X \oplus \ell_2(\Gamma)$ and $Y^{**} = Y \oplus \ell_2(\Gamma)$. In particular X^{**} is WCG.

(iii) The space Y is not WCG and X/Y is reflexive.

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The space X is of the form $\big(\sum_{n=1}^{\infty} \oplus J(T_n)\big)_2$, where $(T_n)_n$ is the remarkable Rezničenko sequence of trees. This is a sequence of trees with each T_n of height ω and which satisfy a strong Baire property. The original construction of $(T_n)_n$ was based on a transfinite (for $\xi < \omega_1$) recursive argument. In the present paper we provide a new construction with the use of a coding function σ. Each T_n consists of all σ-admissible sequences with first element the natural number n, ordered by the initial segment inclusion. It is worth pointing out that the space X is also a James tree space with $T = \bigcup_{n=1}^{\infty} T_n$, which shows that the WCG $J(T)$ spaces are not hereditarily WCG. The following is the key property for our results.

Proposition. *Let Y be the subspace of X mentioned before. Then there is no $\Phi : Y^* \to c_0(\Gamma)$ linear, one-to-one and bounded.*

This proposition in conjunction with the property that X^{**} is Hilbert-generated yields a negative answer to Q1. Let us recall that if E admits an equivalent WUR norm, then E^* is a subspace of a WCG ([F-H-Z]). In particular, if E is isomorphic to Y^* for some Banach space Y, then Y could not contain ℓ_1. This actually shows that any example Y^* answering in negative Q1, should satisfy the following properties. First ℓ_1 does not embed in Y and second, Y^{**} is a non-WCG subspace of a WCG Banach space. Namely the space Y must satisfy the basic properties of the example presented here.

Rezničenko sequences of trees

We start with the construction of the sequence $(T_n)_n$ mentioned above. First we fix a well ordering $\prec$ of the set $\mathbb{R}$ of real numbers.

Let $\{I_\alpha : \alpha < c\}$, with $|I_\alpha| = c$ for $\alpha < c$, be a disjoint family of subsets of the set $\mathbb{R} \setminus \mathbb{N}$, where c denotes the cardinality of the continuum. We denote by $\mathcal{L}$ the set of all sequences $\vec{s} = (s_1, s_2, \dots)$ with the following properties:

(1) for every $k \in \mathbb{N}$, $s_k = (t_0, t_1, \dots, t_{d_k})$, where $t_0 \in \mathbb{N}$, $d_k \geq 0$, $t_i \in \mathbb{R} \setminus \mathbb{N}$ for $1 \leq i \leq d_k$, $t_i \neq t_j$ for $1 \leq i < j \leq d_k$ and

(2) $s_k \cap s_m = \emptyset$ for $k < m$.

Fix a one-to-one mapping $\sigma : \mathcal{L} \to [0, c)$, where $[0, c)$ is the interval of all ordinals smaller than c.

Definition 1. A finite sequence $(t_0, t_1, \dots, t_d)$, where $t_0 \in \mathbb{N}$, $d \geq 1$, $t_i \in \mathbb{R} \setminus \mathbb{N}$ for $1 \leq i \leq d$, is said to be *σ-admissible* if $t_0 \prec t_1 \prec \cdots \prec t_d$ and for all $i = 1, 2, \dots, d$, there exists $\vec{s}_i \in \mathcal{L}$ such that $(t_0, t_1, \dots, t_{i-1}) \in \vec{s}_i$ and $t_i \in I_{\sigma(\vec{s}_i)}$.

Define for every $k \in \mathbb{N}$ a partial order $<_k$ in $\mathbb{R}$ as follows:

If $t, s \in \mathbb{R}$, then $t <_k s$ iff there exist a $\sigma-$admissible sequence $(t_0, t_1, \dots, t_d)$ with $t_0 = k$ and $0 \leq i < j \leq d$ such that $t = t_i$ and $s = t_j$.

Set $T_k = \{t \in \mathbb{R} \setminus \mathbb{N} : k <_k t\} \cup \{k\}$ for $k \in \mathbb{N}$. Then the sequence of partially ordered sets $(T_k, <_k)$, $k \in \mathbb{N}$, has the properties of a sequence of Rezničenko trees (see also Definition 3.1 and Proposition 3.2 in [A-Me]). In fact we have the following

Theorem 2. (i) *For every $k \in \mathbb{N}$, the partially ordered set $(T_k, <_k)$ is a tree of height ω with root k.*

(ii) *If $k_1 \neq k_2$ and I_i is a segment of T_{k_i}, $i = 1, 2$, then $|I_{k_1} \cap I_{k_2}| \leq 1$.*

(iii) *For every non empty subset M of $\mathbb{N}$ and I_n initial segment of T_n, $n \in M$, such that $I_n \cap I_m = \emptyset$ for $n \neq m$, there exist uncountable many $t \in \mathbb{R} \setminus \mathbb{N}$ such that $t \in S^n_{\max I_n}$, for all $n \in M$, (where for $t \in T_k$ we denote by S^k_t the set of all immediate successors of t in the tree T_k).*

PROOF: (i) Let us observe that the definition of the σ-admissible sequences yields that for any $k \in \mathbb{N}$ and every pair $(k = t_0, t_1, \ldots, t_{d_1})$, $(k = s_0, s_1, \ldots, s_{d_2})$ of σ-admissible sequences, there exists $0 \leq i_0 \leq \min\{d_1, d_2\}$ such that for all $i \leq i_0$ we have $t_i = s_i$ and the sets $\{t_{i_0+1}, \ldots, t_{d_2}\}$, $\{s_{i_0+1}, \ldots, s_{d_2}\}$ are disjoint. This shows that $(T_k, <_k)$ is a tree of height ω.
(ii) By (i), it is enough to show the property only for initial segments. Let $k_1 \neq k_2$ and $(k_1, t_1, \ldots, t_{d_1})$, $(k_2, s_1, \ldots, s_{d_2})$ be σ-admissible sequences. Assume that $|\{k_1, t_1, \ldots, t_{d_1}\} \cap \{k_2, s_1, \ldots, s_{d_2}\}| \geq 2$. Namely, there exist $1 \leq i_1 < i_2 \leq d_1$ and $1 \leq j_1 < j_2 \leq d_2$ such that $\{t_{i_1}, t_{i_2}\} = \{s_{j_1}, s_{j_2}\}$. Since $t_{i_1} \prec t_{i_2}$ and $s_{j_1} \prec s_{j_2}$ for the fixed well ordering $\prec$ of $\mathbb{R}$, we conclude that $t_{i_1} = s_{j_1}$ and $t_{i_2} = s_{j_2}$. This yields a contradiction since the σ-admissible sequences $(k_1, t_1, \ldots, t_{i_2-1})$, $(k_2, s_1, \ldots, s_{j_2-1})$ have common σ-extension although they are not disjoint.
(iii) It follows immediately from the definitions of the function σ and the σ-admissible sequences. □

Any sequence of trees T_k, $k \in \mathbb{N}$, satisfying the assertions (i) to (iii) of the above theorem is called a sequence of Rezničenko trees. As it is shown in [A-Me] (Proposition 3.3), any sequence of Rezničenko trees satisfies a sort of Baire category property. To this end we need the following definition.

Definition 3. Let T be a tree. A subset D of T is said to be *successively dense in* T if there exists $t_0 \in T$ such that for every $t \in T$ with $t_0 \leq t$ we have $D \cap S_t \neq \emptyset$.

Let us point out that if T has the additional property that for each $t \in T$ $S_t \neq \emptyset$, then every successively dense subset D of T must contain an infinite segment. Under the above terminology we have the following fundamental property of Rezničenko sequences of trees.

Theorem 4. *Let T_n, $n \geq 1$ be any sequence of Rezničenko trees, so that each T_n has as a root the number $n \in \mathbb{N}$ and $T = \bigcup_{n=1}^{\infty} T_n$. If D_n, $n \geq 1$ is any sequence of subsets of T with $T = \bigcup_{n=1}^{\infty} D_n$, then there exists $k_0 \in \mathbb{N}$ such that the set D_{k_0} is successively dense in T_{k_0}. In particular, there exists $t_0 \in S^{k_0}_{k_0}$ such that for every $t \in T_{k_0}$ with $t_0 \leq_{k_0} t$ we have $|S^{k_0}_t \cap D_{k_0}| \geq \omega_1$.*

The proof follows the arguments of [A-Me, Proposition 3.3].

Remarks. (1) It follows in particular from Theorem 4 that the set $D_{k_0} \cap T_{k_0}$ contains an infinite segment.

(2) An interesting modification, in a countable setting, of the concept of the sequences of Rezničenko trees, is given in [A-M]. This modification is used there for other purposes.

We briefly describe in the sequel the properties of the example from [A-Me, Theorem 4.3], which we are interested in. Theorem 4.3 states that

Theorem 5. *There exists a WCG Banach space X such that X^{**} is also WCG not containing ℓ_1. Moreover there exists a closed subspace Y of X such that:*

(a) *the spaces Y and Y^{**} are not WCG;*
(b) *the quotient X/Y is a reflexive space.*

The space X

We first recall the definition of a James space $J(T)$, for a given tree $(T, \leq)$. So $J(T)$ is the completion of the linear space $c_{00}(T)$ of finitely supported real functions on T under the norm

$$\|x\|_{J(T)} = \sup\Big\{\sum_{i=1}^{n}\Big(\sum_{t\in S_i} x(t)^2\Big)^{1/2} : S_1, \dots, S_n \text{ are disjoint segments of } (T, \leq)\Big\}.$$

The space X in the above theorem is of the form $(\sum_{m=1}^{\infty} \oplus X_m)_2$, where X_m is the James space $J(T_m \times \{m\})$ and T_m, $m \geq 1$, is a sequence of Rezničenko trees. Since each tree T_m is of height ω, each X_m has the following properties:

(i) X_m is a WCG, $X_m \cong Z_m^*$ and $X_m^*/Z_m \cong \ell^2(B_m)$, where Z_m is the closed linear span of the set $\{e^*_{(t,m)} : t \in T_m\}$ in X_m^* and $\mathcal{B}_m$ the set of branches of the tree T_m (clearly Z_m is a WCG, since the set $\{e^*_{(t,m)} : t \in T_m\} \cup \{0\}$ is weakly compact in X_m^*).

Using properties of Dixmier's projection $P_m : Z_m^{***} \to Z_m^*$ we find that,

(ii) $X_m^{**} \cong X_m \oplus \ell_2(\mathcal{B}_m)$ (cf. [F-Z, Example 5.7, pp. 148 and Examples 6.49–6.54, pp. 199–201]).

Set $Z = (\sum_{m=1}^{\infty} \oplus Z_m)_2$. Then using properties (i) and (ii) (and Dixmier 's projection $P : Z^{***} \to Z^*$) we get that,

(iii) $X \cong Z^*$, $X^*/Z \cong \ell_2(\mathcal{B})$ and $X^{**} \cong X \oplus \ell_2(\mathcal{B})$, where $B = \bigcup_{m=1}^{\infty} \mathcal{B}_m$. It follows in particular that X is complemented in X^{**} by Dixmier' s projection $P : X^{**} \to X$.

We notice that, it follows for the definition of X and properties (i) and (iii) that both of the spaces X and X^{**} are WCG. These spaces have the additional property to be Hilbert-generated. We recall that a Banach space Z is Hilbert-generated if there exists a bounded linear operator from a Hilbert space onto a dense subspace of Z (see [F-G-H-Z]).

Lemma 6. *The spaces X and X^{**} are Hilbert generated.*

Proof: It clearly follows from the definition of X and property (iii) that it is enough to show that each James space $Z = J(T)$, where T is any tree of height ω, is Hilbert-generated. Indeed, let $T(n)$ be the n-th level of T, $n \geq 0$. Then $T = \bigcup_{n=0}^{\infty} T(n)$ and each of the subspaces $Z_n = \overline{\text{span}}\{e_t : t \in T(n)\}$ of Z is isometric to the Hilbert space $\ell_2\big(T(n)\big)$. Since the union of $\bigcup_{n=0}^{\infty} Z_n$ generates Z, it is easily verified that the operator $F : \ell_2(T) \to Z$ defined by $F(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^n}$, where $x_n = x|_{T(n)}$ for $x \in \ell_2(T)$ and $n \geq 0$, makes Z a Hilbert-generated space. $\square$

The space Y

The space Y is defined as follows: for every $t \in T = \bigcup_{m=1}^{\infty} T_m$, set

$$D_t = \{m \in \mathbb{N} : t \in T_m\} \quad \text{and} \quad x_t = \sum_{m \in D_t} \frac{1}{2^{m/2}} e_{(t,m)}.$$

Finally set, $Y = \overline{\text{span}}\{x_t : t \in T\} \subset X$. Then the following facts can be proved (see [A-Me, Theorem 4.3]).

(1) There exists a family $\{f_t : t \in T\} \subset Y^*$ so that the family $\{(x_t, f_t) : t \in T\}$ is an M-basis for Y, where for $t \in T$ and $m \in D_t$, $f_t = 2^{m/2} I^*(e^*_{(t,m)})$ and $I : Y \to X$ is the natural embedding of Y into X.

(2) Let $m \in \mathbb{N}$ and $b = \{t_1 < \ldots < t_n < \ldots\}$ be any branch of the tree T_m. Then the series $\sum_{k=1}^{\infty} f_{t_k}$ is weak* convergent in Y^*.

Facts (1) and (2) together imply that Y is not WCG.

(3) $Y^{**} \cong Y \oplus \ell_2(\mathcal{B})$.

Since Y is not a WCG, it clearly follows from fact (3) that neither Y^{**} is WCG.

The following lemma is the analogue for trees T of height ω of a known property of the James tree space [J].

Lemma 7. *The space Y is complemented in Y^{**} by a norm-one projection and hence it is a dual space of a WCG space Y_0 (having a shrinking M-basis).*

Proof: Let $P : X^{**} \cong X \oplus \ell_2(\mathcal{B}) \to X$ be Dixmier's projection and $y^{**} \in Y^{**} \subset X^{**}$. Then $y^{**} = y + w$, where $y \in X$ and $w \in \ell_2(\mathcal{B})$. Since from fact (3), $Y^{**} \cong Y \oplus \ell_2(\mathcal{B})$ we find that $X \cap Y^{**} = Y$, so $y = y^{**} - w \in X \cap Y^{**} = Y$. Therefore the restriction of P on the subspace Y^{**} of X^{**} is a norm-one projection of Y^{**} onto Y.

We define Y_0 to be the closed linear span of the set $\{f_t : t \in T\}$ in the space Y^*. We shall show that $Y_0^{**} \cong Y$. So we define the operator $F : Y \to Y_0^*$ by $F(y) = y_{|Y_0}$. It is clear that F is a well defined linear bounded ($\|F(y)\| \leq \|y\|$) operator and since the family $\{f_t : t \in T\}$ separates the points of Y it is also one-to-one.

Let $g \in Y_0^*$. Then by Hahn-Banach theorem there exist $\hat{g} \in Y^{**} : \hat{g}_{|Y_0} = g$ and $\|g\| = \|\hat{g}\|$. Set $P(\hat{g}) = y$; then clearly $y \in Y$ and

$$w = \hat{g} - y \in \ell_2(\mathcal{B}).$$

So we have that for all $t \in T$,

$$g(f_t) = \hat{g}(f_t) = (y + w)(f_t) = y(f_t) + w(f_t) = y(f_t),$$

because $w(f_t) = 0$, for every $t \in T$ (recall that, $f_t = 2^{m/2} I^*(e^*_{(t,m)})$, for $m \in D_t$). It follows that $g(y^*) = y(y^*)$ for all $y^* \in Y_0$, which implies that $F(y) = g$. Therefore the operator F is surjective and thus an isomorphism between the spaces Y and Y_0^*.

It is obvious from the above that the family $\{(f_t, x_t) : t \in T\}$ is a shrinking M-basis for Y_0. □

Now we are able to prove the main result of this note.

Proposition 8. *There is no bounded linear one-to-one operator $F : Y^* \to c_0(\Gamma)$ for any set Γ.*

Proof: Assume, for the purpose of contradiction, that there exists a bounded linear one-to-one operator $F : Y^* \to c_0(\Gamma)$ for some set Γ. Let $F^* : \ell_1(\Gamma) \to Y^{**}$ be the dual operator of F. Then we may assume without loss of generality that $F^*(e^*_\gamma) \neq 0$ for all $\gamma \in \Gamma$ and note that the set $\{F^*(e^*_\gamma) : \gamma \in \Gamma\} \cup \{0\}$ is a *weak** compact (and *weak** total) in Y^{**}, so that for every sequence $(\gamma_n)_n$ of distinct points of Γ we have that $w^* - \lim_{n\to\infty} F^*(e^*_{\gamma_n}) = 0$. By Lemma 7, the M-basis $\{(f_t, x_t) : t \in T\}$ of the predual Y_0 of Y is shrinking, therefore the set

$\Omega = \left\{\frac{f_t}{\|f_t\|} : t \in T\right\}$ is weakly discrete and the set $\Omega \cup \{0\}$ is weakly compact in Y_0.

We consider the map

$$\Phi : T \times \Gamma \to \mathbb{R} : \Phi(t,\gamma) = F^*(e^*_\gamma)(f_t) \ \text{ for } \ (t,\gamma) \in T \times \Gamma.$$

It follows that there exist partitions $\{T_\delta : \delta \in \Delta\}$ and $\{\Gamma_\delta : \delta \in \Delta\}$ of T and Γ into countable sets, such that for every $\delta_1, \delta_2 \in \Delta$ with $\delta_1 \neq \delta_2$ and for every $t \in T_{\delta_1}$, $\gamma \in \Gamma_{\delta_2}$, we have that $\Phi(t,\gamma) = 0$ (see [F, Lemma 1.6.2] and [A-Me, Proposition 2.1]).

We enumerate each Γ_δ and T_δ as $\{\gamma^\delta_n : n \geq 1\}$, $\{t^\delta_n : n \geq 1\}$ and for $n, m \in \mathbb{N}$ we put

$$D_{n,m} = \{t \in T : t = t^\delta_n \ \text{ for some } \delta \in \Delta \text{ and there exists } \gamma \in \Gamma_\delta : |\Phi(t,\gamma)| \geq \frac{1}{m}\}$$

and

$$\Gamma_{n,m} = \Big\{ \gamma \in \Gamma : \gamma = \gamma_n^\delta \text{ for some } \delta \in \Delta \text{ and there exists } t \in T_\delta : |\Phi(t,\gamma)| \geq \frac{1}{m} \Big\}.$$

Set $D_m = \bigcup_{n=1}^\infty D_{n,m}$ and $\Gamma_m = \bigcup_{n=1}^\infty \Gamma_{n,m}$ for $m \in \mathbb{N}$. Then we have

(a) $T = \bigcup_{m=1}^\infty D_m$;

(b) if $(t,\gamma) \in T \times \Gamma$ and $\Phi(t,\gamma) \neq 0$ then there exists $m \in \mathbb{N}$ such that $(t,\gamma) \in D_m \times \Gamma_m$ and

(c) for every $m \in \mathbb{N}$ and $x \in D_m \cup \Gamma_m$ there exists $y \in D_m \cup \Gamma_m$ such that,

$$\begin{aligned} &\text{either } \ x \in D_m,\ y \in \Gamma_m \ \text{ and } \ |\Phi(x,y)| \geq \frac{1}{m} \\ &\text{or } \ x \in \Gamma_m,\ y \in D_m \ \text{ and } \ |\Phi(y,x)| \geq \frac{1}{m}\,. \end{aligned}$$

We get from fact (3) that for every $\gamma \in \Gamma$ there exists a unique pair $y_\gamma \in Y$ and $w_\gamma \in \ell_2(\mathcal{B})$ such that $F^*(e_\gamma^*) = y_\gamma + w_\gamma$.

Let $m_0 \in \mathbb{N}$ be such that D_{m_0} is successively dense in the tree T_{m_0} (see Theorem 4 and also (a)). Using this fact and also properties (a)–(c) above, we can choose by induction sequences $(\gamma_n)_n \subset \Gamma_{m_0}$ and $(t_n)_n \subset T_{m_0}$ such that:

(d) $\{t_1 < \ldots < t_n < \ldots\}$ is an infinite segment of the tree T_{m_0};

(e) for every $n \geq 1$, $|\Phi(t_{n+1},\gamma_{n+1})| \geq \frac{1}{m_0}$ and $t_{n+1} \notin b$ for all branches $b \in \mathcal{B}$ with $w_{\gamma_n}(b) \neq 0$. Note that $w_\gamma \in \ell_2(\mathcal{B})$ thus the set $\{b \in \mathcal{B} : w_\gamma(b) \neq 0\}$ is at most countable.

Fact (2) and (d) above imply that the series $\sum_{k=1}^\infty f_{t_k}$ is weak*-convergent in Y^*, say $x^* = w^* - \sum_{k=1}^\infty f_{t_k}$. It also follows from (e) that $w_{\gamma_n}(x^*) = 0$ for all $n \geq 1$. We shall show that the sequence $(F^*(e_{\gamma_n}^*))_n$ is not weakly* null. Indeed

$$\begin{aligned} F^*(e_{\gamma_n}^*)(x^*) &= (w_{\gamma_n} + y_{\gamma_n})(x^*) = y_{\gamma_n}(x^*) = \lim_{\ell\to\infty} \sum_{k=1}^{\ell} f_{t_k}(y_{\gamma_n}) \\ &= f_{t_n}(y_{\gamma_n}) = (w_{\gamma_n} + y_{\gamma_n})(f_{t_n}) = F^*(e_{\gamma_n}^*)(f_{t_n}) = \Phi(t_n,\gamma_n). \end{aligned}$$

Therefore

$$|F^*(e_{\gamma_n}^*)(x^*)| = |\Phi(t_n,\gamma_n)| \geq \frac{1}{m_0} \ \text{ for all } \ n \geq 1,$$

which proves the claim and so the proof of the theorem is complete. □

Remarks. (1) It is clear that the space Y obtained above provides a counterexample to question Q2 stated in the introduction (cf. Lemma 7).

(2) It is well known that the property of a Banach space E to admit a bounded linear one-to-one operator into some $c_0(\Gamma)$ is not a three space property (see [D-G-Z, Chapter VI, Theorem 8.8.3 and Chapter VII, Example 4.9]). The space Y^*

from Theorem 8 also proves the same result. Indeed, according to assertion (ii) of this theorem, Y^* admits no bounded linear one-to-one operator into some $c_0(\Gamma)$. On the other hand it is easy to prove that $Y^*/Y_0 \cong \ell_2(\mathcal{B})$ and of course Y_0 and $\ell_2(\mathcal{B})$ both admit such operators as WCG spaces.

(3) We note that the space Y^* as a dual of a weakly $\mathcal{K}$-analytic space Y with $\dim Y \leq c$, admits a bounded linear one-to-one operator into the space $c_1(\Sigma)$, Σ =the Baire space of irrationals, which is also weak* to pointwise continuous. Thus Y^* admits a dual rotund norm (see [M] and also [D-G-Z, Chapter VI., Theorem 6.7, Chapter VII, Theorem 1.16]). We also note that since Y^{**} is weakly $\mathcal{K}$-analytic, the space Y^* admits an equivalent locally uniformly rotund (LUR) norm $|||\cdot|||$, the dual norm of which is also LUR. In particular $|||\cdot|||$ is a Fréchet differentiable norm and Y^* is an Asplund space (see [D-G-Z, Chapter VII, Theorem 2.7]).

Applications

We first recall that a norm $\|\cdot\|$ of a Banach space X is said to be weakly uniformly rotund (WUR for sort) if $w-\lim(x_n - y_n) = 0$ whenever $\|x_n\| = \|y_n\| = 1$ for all n and $\lim \|x_n + y_n\| = 2$. Fabian, Hájek, and Zizler have proved that if X is a WUR Banach space, then its dual X^* is a subspace of a WCG. More exactly, they proved that the space X admits an equivalent WUR norm if and only if the bidual unit ball $B_{X^{**}}$ of X^{**} in its weak* topology is a uniform Eberlein compact space ([F-H-Z]). The following result is an easy consequence of the theorem of Fabian, Hájek and Zizler.

Corollary 9. *Let E be a Banach space such that E^* is a subspace of a Hilbert generated F. Then E admits a WUR renorming.*

Proof: We simply observe that $(B_{E^{**}}, w^*)$ is a continuous image of a uniform Eberlein compact space (i.e., of the ball of (B_{F^*}, w^*) of F^*), hence a well-known result of Benyamini, Rudin and Wage yields that the space $(B_{E^{**}}, w^*)$ is a uniform Eberlein compact ([B-R-W]). Now by the above mentioned result of Fabian, Hájek and Zizler we get the conclusion. □

Summing up all the previous results, we get a negative answer to the problem of Fabian, Godefroy, Hájek and Zizler mentioned in the introduction as question Q1.

Theorem 10. *There exists a WUR renormable Banach space E that does not admit any bounded, linear, one-to-one operator into some $c_0(\Gamma)$.*

Proof: Set $E = Y^*$, where Y is the space of Proposition 8, so there is no bounded, linear, one-to-one operator from E to $c_0(\Gamma)$. On the other hand, $E^* = Y^{**}$ is a subspace of the Hilbert generated space X^{**} (see Lemma 6) and hence, by the above corollary, E admits a WUR renorming. The proof of the theorem is completed. □

The following describes a peculiar property of James tree spaces.

Proposition 11. *Let T be a tree. Then the following are equivalent.*

(i) *$J(T)$ is weakly countably determined.*

(ii) *There exists a sequence $(A_n)_{n\in\mathbb{N}}$ such that each A_n is an antichain of T and $T = \bigcup_{n=1}^{\infty} A_n$.*

(iii) *$J(T)$ is Hilbert generated (hence it is WCG).*

Proof: (i)⇒(ii) Let us observe that every branch b of T is at most countable (otherwise the ordinal ω_1 will be subset of $B_{J(T)^*}$ yielding a contradiction) and moreover the set

$$\mathcal{D} = \{S^* : S \text{ is a segment of } T\}$$

is a w*-compact subset of $B_{J(T)^*}$. Hence $\mathcal{D}$ is a Gulko compact subset of $\Sigma\{0,1\}^T$. Clearly the adequate closure of $\mathcal{D}$,

$$\hat{\mathcal{D}} = \{A \subseteq T : \exists S \in \mathcal{D} \text{ with } A \subseteq S\}$$

remains Gulko compact. This follows from Theorem 3.6 [M]. Theorem 4.2 of [L-S] yields that $T = \bigcup_{n=1}^{\infty} A_n$ with each A_n an antichain of T.

(ii)⇒(iii) As we have mentioned in Lemma 6, for A antichain of T, the space $\overline{\text{span}}\{e_t : t \in A\}$ is isometric to $\ell_2(A)$. The result follows from arguments similar to the proof of Lemma 6.

(iii)⇒(i) Obvious. □

Remarks. It follows from the above proposition that the notions Hilbert-generated, WCG, weakly $\mathcal{K}$-analytic and WCD coincide within the class of James tree spaces. However WCG $J(T)$ are not necessarily hereditarily WCG. Indeed, as we have mentioned in the introduction, the space X in this paper is a $J(T)$ space and its subspace Y is not WCG.

We conclude with the following open questions for a Banach space E.

(1) Assume that E is a subspace of a WCG Banach space F and that E^* admits a bounded, linear, one-to-one operator into some $c_0(\Gamma)$. Is then E a WCG? The following special cases are important:

(a) $F = E^{**}$. This is a well-known open problem posed by Johnson and Lindenstrauss [J-L] (see also [D-G-Z, Problem VI.4] and [Z, Problem 9]).

(b) F has an unconditional basis. Note that if E itself has an unconditional basis, then by a result of Johnson the answer is positive (see [A-Me]).

(c) $F = L^1(\mu)$ for some finite measure μ.

(d) $E = X^*$ for some WCG Banach space X.

(2) Assume that E is a WCG, so that for every closed linear subspace X of E, the space X^* admits a bounded, linear, one-to-one operator into some $c_0(\Gamma)$. Is then E hereditarily WCG? This question is still of interest with the further assumption that ℓ_1 does not embed into E.

It is clear that a positive answer to the first question will provide a positive answer to the second question.

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