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Remarks on the cardinality of a power homogeneous space

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Abstract. We provide a further estimate on the cardinality of a power homogeneous space. In particular we show the consistency of the formula $|X| \leq 2^{\pi\chi(X)}$ for any regular power homogeneous ccc space.

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Our notations follow [En] and [Ho]. All spaces are assumed to be regular.

As usual, $c(X)$, $\pi(X)$, $\chi(X)$ and $\pi\chi(X)$ denote the cellularity, the π-weight, the character and the π-character of the space X. In addition, we put $c^*(X) = \sup\{c(X^n) : n < \omega\}$.

A topological space X is homogeneous if for any $p, q \in X$ there exists a homeomorphism $f : X \to X$ such that $f(p) = q$. A space X is power homogeneous if X^λ is homogeneous for some cardinal λ.

The ordered spaces $\omega + 1$ and ω_1 and the real interval $[0, 1]$ are examples of non-homogeneous power homogeneous spaces. On the other hand, $\omega_1 + 1$, $\omega \cup \{p\}$, where $p \in \beta\omega \setminus \omega$, and $[0, 1] \cup \{2\}$ are not power homogeneous.

A substantial contribution in the study of power homogeneous spaces was given by van Douwen in [vD]. In that paper he proved the following quite unexpected result:

Proposition 1. *If X is a power homogeneous space, then $|X| \leq 2^{\pi(X)}$.*

Later on Ismail [Is, Theorem 1.6] and Arhangel'skii [A1, Theorem 1.5] obtained:

Proposition 2. *If X is a homogeneous space, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

These two results naturally suggest the following question, which is essentially Remark 2.7 in [vM]:

Question 1. *Is it true that the cardinality of a power homogeneous space X does not exceed $2^{c(X)\pi\chi(X)}$?*

Recently, van Mill [vM] obtained a good partial solution by proving:

Proposition 3. *If X is a compact power homogeneous space, then the inequality $|X| \leq 2^{c(X)\pi\chi(X)}$ holds.*

The main purpose of this note is to work on Question 1. We will present a partial result, which is however an improvement of van Douwen's theorem. To obtain it, we follow the approach of van Mill, which in turn was inspired by the ideas developed by van Douwen in [vD].

In the sequel, $\mathrm{RO}(X)$ denotes the collection of all regular open subsets of the space X. Given a cardinal κ and an element $\phi \in \mathrm{RO}(X)^\kappa$, according to van Douwen [vD], we define the *way ϕ clusters at x* as the set $w(\phi, x) = \{A : A \subseteq \kappa$ and $x \in \overline{\bigcup\{\phi(\alpha) : \alpha \in A\}}\}$. Then we put $W(x, \kappa, X) = \{w(\phi, x) : \phi \in \mathrm{RO}(X)^\kappa\}$. The set $W(x, \kappa, X)$ describes the ways all possible κ-sequences of elements of $\mathrm{RO}(X)$ cluster at x.

The family $\mathrm{RO}(X)$ is invariant under homeomorphisms. This immediately gives:

Lemma 1. *If X is a homogeneous space then $W(x, \kappa, X) = W(y, \kappa, X)$ for any $x, y \in X$.*

An easy, but relevant, fact in dealing with homogeneous spaces is the following observation of van Douwen ([vD, 2.2]):

Lemma 2. *If X is a homogeneous space, $p \in X$ and $\phi \in RO(X)^\kappa$, then $\{w(\phi, x) : x \in X\} \subseteq W(p, \kappa, X)$.*

Proof: If for some $q \in X$ we had $w(\phi, q) \notin W(p, \kappa, X)$, then $W(q, \kappa, X) \neq W(p, \kappa, X)$, contradicting Lemma 1. $\square$

A key role played by the way a certain κ-sequence clusters at different points of the space is in the possibility to measure the size of certain sets.

Lemma 3 ([vD, 3.1])**.** *Let X be a space and $\kappa = \pi\chi(X)$. If Y is a subset of X of cardinality not exceeding κ, then there exists an element $\phi \in RO(X)^\kappa$ such that the map $x \mapsto w(\phi, x)$ is injective on the set $\overline{Y}$.*

Proof: Observe first that the regularity of the space implies that $\mathrm{RO}(X)$ is a π-base. For any $x \in Y$ fix a local π-base $\mathcal{U}_x$ at x consisting of elements of $\mathrm{RO}(X)$ and fix an onto map $\phi : \kappa \to \bigcup\{\mathcal{U}_x : x \in Y\}$. We claim that the κ-sequence ϕ serves to our purpose. Let p, q be distinct points of $\overline{Y}$ and choose an open set $V \subseteq X$ such that $p \in V$ and $q \notin \overline{V}$. Now, let A be the set of those $\alpha \in \kappa$ such that $\phi(\alpha) \subseteq V$. Since the set $\phi(A)$ still contains a local π-base at p, we have $p \in \overline{\bigcup \phi(A)}$ and so $A \in w(\phi, p)$. On the other hand, it is clear that $A \notin w(\phi, q)$ and we are done. $\square$

Later, we will need the following two well-known facts.

Fact 1 ([Ho, Theorem 11.6])**.** *If X is a space and λ an infinite cardinal, then $c(X^\lambda) = c^*(X)$.*

Fact 2 ([Ho, Theorem 6.2]). *If X is a space then $|RO(X)| \leq \pi\chi(X)^{c(X)}$.*

The next important lemma is a slight modification of Lemma 3.7 in [vD].

Lemma 4. *Let X be a space and $\kappa = c^*(X)$. If $\lambda \geq \kappa$, then for any $p \in X^\lambda$ the formula $W(p, \kappa, X^\lambda) = \bigcup\{W(p \restriction A, \kappa, X^A) : A \subseteq \lambda$ and $|A| = \kappa\}$ holds.*

Proof: For any $A \subseteq \lambda$ let $\pi_A : X^\lambda \to X^A$ be the projection. If $\phi \in \mathrm{RO}(X^A)^\kappa$ and ψ is the κ-sequence defined by letting $\psi(\alpha) = \pi_A^{-1}(\phi(\alpha))$, then it is clear that $\psi \in \mathrm{RO}(X^\lambda)^\kappa$ and $w(\psi, p) = w(\phi, p \restriction A)$. This immediately gives $\bigcup\{W(p \restriction A, \kappa, X^A) : A \subseteq \lambda$ and $|A| = \kappa\} \subseteq W(p, \kappa, X^\lambda)$. For the converse, fix an element $\psi \in \mathrm{RO}(X^\lambda)^\kappa$. By Fact 1 the cellularity of X^λ is κ and consequently any regular open subset of X^λ depends of at most κ-many coordinates ([En, 2.7.12a]). Taking into account that the range of ψ consists of at most κ regular open sets, we can find a set $A \subseteq \lambda$ such that $|A| = \kappa$ and the formula $\psi(\alpha) = \pi_A^{-1}(\pi_A(\psi(\alpha)))$ holds for any $\alpha \in \kappa$. This implies that $w(\psi, p) = w(\pi_A \circ \psi, p \restriction A) \in W(p \restriction A, \kappa, X^A)$ and the proof is complete. □

Now, we are in the position to establish the main lemma.

Lemma 5. *Let X be a power homogeneous space and $\kappa = c^*(X)\pi\chi(X)$. If Y is a subset of X and $|Y| \leq \kappa$, then $|\overline{Y}| \leq 2^\kappa$.*

Proof: Let λ be a cardinal such that X^λ is homogeneous. Without any loss of generality, we may assume $\lambda \geq \kappa$. Let $\pi : X^\lambda \to X$ be any projection and fix a set $Z \subseteq X^\lambda$ such that $\pi : Z \to \overline{Y}$ is bijective. By Lemma 3 there exists an element $\phi \in \mathrm{RO}(X)^\kappa$ such that the map $x \mapsto w(\phi, x)$ is injective on $\overline{Y}$. Define $\psi \in \mathrm{RO}(X^\lambda)^\kappa$ by letting $\psi(\alpha) = \pi^{-1}(\phi(\alpha))$. As π is an open map, we have $w(\psi, x) = w(\phi, \pi(x))$ for any $x \in X^\lambda$. Now, let $p \in X^\lambda$ be a point which is a constant function. This implies that $W(p \restriction A, \kappa, X^A) = W(p \restriction \kappa, \kappa, X^\kappa)$ for any $A \subseteq \lambda$ satisfying $|A| = \kappa$ and therefore, by Lemma 4, we obtain $W(p, \kappa, X^\lambda) = W(p \restriction \kappa, \kappa, X^\kappa)$. Since by Lemma 3 and the openness of π, the map $x \mapsto w(\psi, x)$ is injective for any $x \in Z$, we have $|\overline{Y}| = |Z| \leq |\{w(\psi, x) : x \in Z\}| \leq |W(p, \kappa, X^\lambda)|$ (by Lemma 2) $= |W(p \restriction \kappa, \kappa, X^\kappa)| \leq |\mathrm{RO}(X^\kappa)^\kappa| \leq$ (Fact 2) $(\pi\chi(X^\kappa)^{c(X^\kappa)})^\kappa = ((\kappa \cdot \pi\chi(X))^{c^*(X)})^\kappa = 2^\kappa$. □

Theorem 1. *If X is a power homogeneous space then $|X| \leq 2^{c^*(X)\pi\chi(X)}$.*

Proof: Let $\kappa = c^*(X)\pi\chi(X)$. For any $x \in X$ fix a local π-base $\mathcal{U}_x$ at x of size not exceeding κ. By Fact 2, we may select a dense set $D \subseteq X$ of size not exceeding 2^κ. For any $U \in \mathcal{U}_x$ pick a point in $U \cap D$ and let D_x be the set so obtained. It is clear that $x \in \overline{D_x}$ and $|D_x| \leq \kappa$. By applying Lemma 5, an easy counting argument implies $|X| \leq |D|^\kappa 2^\kappa = 2^\kappa$. □

Although the previous result is not the full answer to Question 1, it is however an improvement of van Douwen's Theorem (Proposition 1), as the inequality

$c^*(X)\pi\chi(X) \leq \pi(X)$ holds for any space. It is a real improvement, since even for compact homogeneous spaces X we may have $c^*(X)\pi\chi(X) < \pi(X)$: the compact right topological group X, constructed under [CH] by Kunen in [K2], satisfies $c^*(X)\chi(X) = \omega$ and $\pi(X) = \omega_1$.

Recall that a cardinal κ is a precaliber of a space X if any family of non-empty open sets of cardinality κ has a centered subfamily of the same cardinality. It is a known fact that if $c(X)^+$ is a precaliber of the space X then $c^*(X) = c(X)$ (see the proof of Theorem II.2.24 in [K1]). Thus, we immediately get:

Corollary 1. *If X is a power homogeneous space and $c(X)^+$ is a precaliber of X, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

MA(ω_1) implies that $\aleph_1$ is a precaliber of any ccc space (see [K1, Lemma 2.23]). This yields a consistent positive answer to Question 1 for spaces having countable cellularity.

Corollary 2 [MA(ω_1)]**.** *If X is a power homogeneous ccc space, then $|X| \leq 2^{\pi\chi(X)}$.*

Van Mill in [vM, Theorem 3.2] proved that if X is a hereditarily normal compact homogeneous space then $|X| \leq 2^{c(X)}$, raising the question whether such a result may hold more generally for power homogeneous spaces. We present here a partial solution.

Theorem 2. *If X is a compact hereditarily normal space and X^λ is homogeneous for some $\lambda \leq c(X)$, then $|X| \leq 2^{c(X)}$.*

Proof: Since $I^{\aleph_1}$ is not hereditarily normal (it contains a copy of $(\omega+1)\times(\omega_1+1)$), it follows that X cannot be mapped onto $I^{c(X)^+}$. By applying a result of Šapirovskiĭ [Sa, Theorem 5], namely if a product of fewer than κ compact spaces maps onto I^κ (for a regular cardinal κ) then one factor must map onto I^κ, we deduce that the space X^λ cannot be mapped onto $I^{c(X)^+}$. By the fundamental result of Šapirovskiĭ (see again [Sa]), there exists a point $p \in X^\lambda$ such that $\pi\chi(p, X^\lambda) \leq c(X)$. But X^λ is homogeneous and consequently $\pi\chi(X^\lambda) \leq c(X)$. This in turn implies $\pi\chi(X) \leq c(X)$ and from Proposition 3 we get $|X| \leq 2^{c(X)}$. □

Corollary 3 [GCH]**.** *If X is a compact hereditarily normal space and X^λ is homogeneous for some $\lambda \leq c(X)$, then $\chi(X) \leq c(X)$.*

Proof: Since X^λ is compact and homogeneous, we have $|X^\lambda| = 2^{\chi(X^\lambda)}$ ([Is, Theorem 1.13]). By Theorem 2, we obtain $|X^\lambda| \leq 2^{c(X)\lambda}$ and therefore $2^{\chi(X^\lambda)} \leq 2^{c(X)\lambda}$. Since $\chi(X^\lambda) = \chi(X)\lambda$ and $\lambda \leq c(X)$, from [GCH] we get $\chi(X) \leq c(X)$. □

Of course, the inequality $\chi(X) \leq c(X)$ does not hold for every compact power homogeneous space. It can fail even for compact groups and actually the gap can be arbitrarily large, for this it suffices to consider the group 2^λ. However, it was pointed out in [vM] that a role is played by the absence of subspaces homeomorphic to $\beta\lambda$ for $\lambda \leq c(X)$. More generally, we can prove the following extension of Corollary 3.5 in [vM]:

Corollary 4 [GCH]**.** *If X is compact, X^λ is homogeneous for $\lambda = c(X)$ and X contains no copy of $\beta\lambda$, then $\chi(X) \leq c(X)$.*

Proof: It is enough to observe that [GCH] implies $\beta\lambda \subseteq I^{\lambda^+}$ and consequently X cannot be mapped onto $I^{c(X)^+}$ (this is because a closed irreducible map onto an extremally disconnected space is a homeomorphism [En, 6.3.19c]). Then argue as in Theorem 2 and Corollary 3. $\square$

In connection with Theorem 2 and Corollaries 3 and 4, notice that several power homogeneous spaces are actually ω-power homogeneous, this is for instance the case of all examples presented at the beginning of the paper.

At the end of his paper [vM], van Mill asked whether it is true that any hereditarily normal homogeneous compactum has cardinality at most $\mathfrak{c}$. Of course, the same could be asked even for power homogeneous spaces. We do not know the full answer to this question, but we would like to finish with a partial result in that direction.

Recall that the Novak number of a space is the smallest size of a cover consisting of nowhere dense sets. We use in the sequel the Novak number $n(\mathbb{R})$ of the real line.

Theorem 3. *Let X be a power homogeneous compactum satisfying $w(X) < n(\mathbb{R})$. If X is either hereditarily normal or countably tight, then X is first countable and so $|X| \leq \mathfrak{c}$.*

Proof: Juhász has shown in [Ju, Corollary 6] that a compact space X satisfying $w(X) < n(\mathbb{R})$, which is either hereditarily normal or countably tight, has a dense set of points of countable character. On the other hand, Arhangel'skii has recently obtained in [A2, Theorem 7] that if a power homogeneous Hausdorff space has a point of countable character, then the set of all G_δ-points is closed. By combining the previous two results, we quickly derive our conclusion. $\square$

As MA(countable), i.e. Martin's Axiom restricted to countable partial orders, implies $n(\mathbb{R}) = \mathfrak{c}$, we have:

Corollary 5 [MA(countable)]**.** *If X is a compact power homogeneous hereditarily normal space satisfying $w(X) < \mathfrak{c}$, then X is first countable and $|X| \leq \mathfrak{c}$.*

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