Oscillatory and nonoscillatory solutions for first order impulsive differential inclusions

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Abstract. In this paper we discuss the existence of oscillatory and nonoscillatory solutions of first order impulsive differential inclusions. We shall rely on a fixed point theorem of Bohnenblust-Karlin combined with lower and upper solutions method.

Keywords: impulsive differential inclusions, lower and upper solution, existence, nonoscillatory, oscillatory, fixed point

Classification: 34A37, 34A60, 34C10

1. Introduction

In this paper we prove the existence of nonoscillatory and oscillatory solutions for the following class of first order impulsive differential inclusions

$$y'(t) \in F(t, y(t)), \quad \text{a.e.} \quad t \in [t_0, \infty), \tag{1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \ k \in \mathbb{N}, \tag{2}$$

where $F : [t_0, \infty) \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact and convex values, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $t_0 < t_1 < \ldots < t_m < t_{m+1} \ldots$, $t_m \to \infty$ as $m \to \infty$, $y(t_k^-)$, $y(t_k^+)$ represent the left and right limit of $y(t)$ at $t = t_k$, respectively.

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotics, economics, etc., see the monographs of Bainov and Simeonov [4], Lakshmikantham *et al* [20], and Samoilenko and Perestyuk [21] and the references therein. Recently, by means of fixed point arguments, some extensions to impulsive differential inclusions have been obtained by Benchohra *et al* [6], [7], [9]. The questions of oscillation and nonoscillation for nonlinear differential equations have received much attention in the last three decades, we recommend, for instance, the monographs [1], [12], [16], [18] and the references cited therein. For oscillation and nonoscillation of impulsive differential equations see for instance the monograph of Bainov and Simeonov [5] and the papers of Graef *et al* [13], [14], [15] and Yong-shao and Weizhen [22]. However the theory of nonoscillatory solutions of differential inclusions has received much less attention. Very recently it was initiated by Agarwal, Grace and O'Regan in [2], [3]. The purpose of this paper is to give some sufficient conditions for the existence of oscillatory and nonoscillatory solutions to the class of impulsive differential inclusions (1)–(2). We shall rely on the Bohnenblust-Karlin [10] theorem and the concept of lower and upper solutions. Our results can be considered as a contribution to this field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. Let $[a,b]$ be an compact real interval and let $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions y from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in [a,b]\}.$$

Let $AC([a,b],\mathbb{R})$ be the space of absolutely continuous functions $y : [a,b] \to \mathbb{R}$.

The property

$$y \le \overline{y} \quad \text{if and only if} \quad y(t) \le \overline{y}(t) \quad \text{for all} \quad t \in [a,b]$$

defines a partial ordering in $C([a,b],\mathbb{R})$. If $\alpha,\beta \in C([a,b],\mathbb{R})$ and $\alpha \le \beta$, we let

$$[\alpha,\beta] = \{y \in C([a,b],\mathbb{R}) : \alpha \le y \le \beta\}.$$

Let $L^1([a,b],\mathbb{R})$ denote the Banach space of functions $y : [a,b] \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^b |y(t)|\,dt.$$

Let $(X,\|\cdot\|)$ be a Banach space. A multi-valued map $G : X \longrightarrow \mathcal{P}(X)$ has *convex* (*closed*) *values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded subset B of X (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open subset N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Finally, we say that G is *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$. In what follows, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cl,c}(X)$ and $\mathcal{P}_{cp,c}(X)$ denote the family of nonempty closed, nonempty closed convex and nonempty compact convex subsets of X, respectively. A multi-valued map $G : [t_0, \infty) \longrightarrow \mathcal{P}_{cl}(X)$ is said to be *measurable* if for each $x \in X$ the function $Y : [t_0, \infty) \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable, where d is the metric induced from the Banach space X. For more details on multi-valued maps see the book of Hu and Papageorgiou [17].

Definition 2.1. The multivalued map $F : [t_0, \infty) \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is L^1_{loc}-*Carathéodory* if

(i) $t \longmapsto F(t, y)$ is measurable for each $y \in \mathbb{R}$;
(ii) $y \longmapsto F(t, y)$ is upper semi-continuous for almost all $t \in [t_0, \infty)$;
(iii) for each $q > 0$, there exists $\phi_q \in L^1_{\text{loc}}([t_0, \infty), \mathbb{R}_+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \le \phi_q(t)$$

$$\text{for all } \ |y| \le q \ \text{ and for almost all } \ t \in [t_0, \infty).$$

For any $y \in C([a, b], \mathbb{R})$, we define the set

$$S^1_{F(y)} = \{v \in L^1([a, b], \mathbb{R}) : \ v(t) \in F(t, y(t)) \ \text{ for a.e. } \ t \in [a, b]\}.$$

This is known as the set of *selection functions*.

Lemma 2.1 ([19])**.** *Let J be an compact real interval and X be a Banach space. Let $F : J \times X \longrightarrow \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map with $S^1_{F(y)} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Gamma \circ S_F : C(J, X) \longrightarrow \mathcal{P}_{cp,c}(C(J, X)), \ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.2 (Bohnenblust-Karlin [10], see also [23, p. 452])**.** *Let X be a Banach space and $K \in \mathcal{P}_{cl,c}(X)$ and suppose that the operator $G : K \longrightarrow \mathcal{P}_{cl,c}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

3. Main results

Now, we are able to state and prove our main theorem for the impulsive differential inclusion (1)–(2). We give first the definition of a solution of the problem (1)–(2). Consider the following space

$$PC = \{y : [t_0, \infty) \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}), \ k \in \mathbb{N}^*, \text{ and there exist } y(t_k^-) \text{ and}$$

$$y(t_k^+), \ k \in \mathbb{N} \ \text{ with } \ y(t_k^-) = y(t_k)\}.$$

Definition 3.1. A function $y \in PC \cap AC(J', \mathbb{R})$ ($J' := [t_0, \infty) \backslash t_k,\ k \in \mathbb{N}$) is called *solution of the problem* (1)–(2) if there is $v \in L^1([t_0, \infty), \mathbb{R})$ with $v(t) \in F(t, y(t))$ a.e. $t \in [t_0, \infty)$ such that the differential equation $y'(t) = v(t)$, a.e. $t \in [t_0, \infty)$ is satisfied and $y(t_k^+) = I_k(y(t_k))$, $k \in \mathbb{N}$.

Now, we introduce the concept of lower and upper solutions for (1)–(2). It will be the basic tool in the approach that follows (see [7], [8]).

Definition 3.2. A function $\alpha \in PC \cap AC(J', \mathbb{R})$ is said to be a *lower solution* of (1)–(2) if there exists $v_1 \in L^1([t_0, \infty), \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on $[t_0, \infty)$, $\alpha'(t) \le v_1(t)$ a.e. on $[t_0, \infty)$ and $\alpha(t_k^+) \le I_k(\alpha(t_k))$, $k \in \mathbb{N}$. Similarly, a function $\beta \in PC \cap AC(J', \mathbb{R})$ is said to be an *upper solution* of (1)–(2) if there exists $v_2 \in L^1([t_0, \infty), \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on $[t_0, \infty)$, $\beta'(t) \ge v_2(t)$ a.e. on $[t_0, \infty)$ and $\beta(t_k^+) \ge I_k(\beta(t_k))$, $k \in \mathbb{N}$.

Definition 3.3. The solution y is said to be *regular* if it is defined on some halfline $[T_y, \infty)$ and $\sup\{|y(t)| : t \ge T\} > 0$ for all $T > T_y$. T_y depends on such solution y. This solution is said to be

(i) *eventually positive* if there exists $T \ge t_0$ such that y is defined for $t \ge T$ and $y(t) > 0$ for $t \ge T$;
(ii) *eventually negative* if there exists $T \ge t_0$ such that y is defined for $t \ge T$ and $y(t) < 0$ for $t \ge T$;
(iii) *nonoscillatory* if it is either eventually positive or eventually negative;
(iv) *oscillatory* if it is neither eventually positive nor eventually negative.

Theorem 3.1. *Assume that:*

(H1) $F : [t_0, \infty) \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ *is an L^1-Carathéodory multi-valued map;*
(H2) *there exist α and $\beta \in PC \cap AC(J', \mathbb{R})$ lower and upper solutions, respectively, for the problem* (1)-(2)*, such that $\alpha \le \beta$;*
(H3) $\alpha(t_k^+) \le \min_{[\alpha(t_k), \beta(t_k)]} I_k(y) \le \max_{[\alpha(t_k), \beta(t_k)]} I_k(y) \le \beta(t_k^+)$, $k \in \mathbb{N}$.

Then the problem (1)-(2) *has at least one solution y on $[t_0, \infty)$ such that $\alpha \le y \le \beta$.*

Proof: The proof will be given in several steps.

Step 1. Consider first the problem (1)-(2) on $J_0 = [t_0, t_1]$

$$y'(t) \in F(t, y(t)), \quad \text{a.e. } t \in [t_0, t_1). \tag{3}$$

Consider the modified problem

$$y'(t) \in F(t, (\tau y)(t)), \quad \text{a.e. } t \in [t_0, t_1), \tag{4}$$

where $\tau : C([t_0,t_1],\mathbb{R}) \to C([t_0,t_1],\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \alpha(t), & y(t) < \alpha(t), \\ y(t), & \alpha(t) \le y(t) \le \beta(t), \\ \beta(t), & y(t) > \beta(t). \end{cases}$$

Transform the problem into a fixed point problem. Consider the multivalued operator $N : C([t_0,t_1],\mathbb{R}) \to \mathcal{P}(C([t_0,t_1],\mathbb{R}))$ defined by:

$$N(y) = \left\{ h \in C([t_0,t_1],\mathbb{R}) : h(t) = \int_{t_0}^{t} g(s)\,ds,\ g \in \tilde{S}^1_{F,\tau y} \right\}$$

where

$$\begin{aligned} \tilde{S}^1_{F,\tau y} &= \{g \in S^1_{F,\tau y} : g(t) \ge v_1(t) \ \text{ a.e. on } \ A_1 \ \text{ and } \ g(t) \le v_2(t) \ \text{ a.e. on } \ A_2\}, \\ S^1_{F,\tau y} &= \{g \in L^1([t_0,t_1],\mathbb{R}) : g(t) \in F(t,(\tau y)(t)) \ \text{ for a.e. } \ t \in [t_0,t_1]\}, \\ A_1 &= \{t \in [t_0,t_1]) : y(t) < \alpha(t) \le \beta(t)\}, \\ A_2 &= \{t \in [t_0,t_1] : \alpha(t) \le \beta(t) < y(t)\}. \end{aligned}$$

Remark 3.1. (i) For each $y \in C([t_0,t_1],\mathbb{R})$, the set $\tilde{S}^1_{F,\tau y}$ is nonempty. In fact, (H1) implies there exists $g_3 \in S^1_{F,\tau y}$, so we set

$$g = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where

$$A_3 = \{t \in [t_0,t_1] : \alpha(t) \le y(t) \le \beta(t)\}.$$

Then, by the decomposability, $g \in \tilde{S}^1_{F,\tau y}$.

(ii) By the definition of τ it is clear that $F(\cdot,\tau y(\cdot))$ is an $L^1_{\mathrm{loc}}-$Carathéodory multi-valued map with compact convex values and there exists $\phi_1 \in L^1([t_0,t_1],\mathbb{R})$ such that

$$\|F(t,(\tau y)(t))\| \le \phi_1(t) \ \text{ for each } \ y \in \mathbb{R}.$$

We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in a series of Claims. Let

$$K_1 := \{y \in C([t_0,t_1],\mathbb{R}) : \|y\|_\infty \le \|\phi_1\|_{L^1}\}.$$

It is clear that K_1 is a closed bounded convex set.

Claim 1. *$N(y)$ is convex for each $y \in C([t_0,t_1],\mathbb{R})$.* This is a obvious since $\tilde{S}^1_{F,\tau y}$ is convex (because F has convex values).

Claim 2. $N(K_1) \subset K_1$.

Indeed, let $y \in K_1$ and fix $t \in [t_0,t_1)$. We must show that $N(y) \subset K_1$. If $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in [t_0,t_1]$ we have

$$h(t) = \int_{t_0}^{t} g(s)\,ds.$$

By the above remark we have for each $t \in [t_0,t_1]$

$$|h(t)| \leq \int_{t_0}^{t} |g(s)|\,ds \leq \|\phi_1\|_{L^1}.$$

Claim 3. *$N(K_1)$ is relatively compact.*

Since K_1 is bounded and $N(K_1) \subset K_1$, it is clear that $N(K_1)$ is bounded. $N(K_1)$ is equicontinuous. Let $u_1, u_2 \in [t_0,t_1]$ with $u_1 < u_2$. Let $y \in K_1$ and $h \in N(y)$. Then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in [t_0,t_1]$ we have

$$h(t) = \int_{t_0}^{t} g(s)\,ds.$$

Then

$$|h(u_2) - h(u_1)| = \left|\int_{t_0}^{u_2} g(s)\,ds - \int_{t_0}^{u_1} g(s)\,ds\right| \leq \int_{u_1}^{u_2} |g(s)|\,ds \leq \int_{u_1}^{u_2} \phi_1(s)\,ds.$$

The right-hand side tends to zero as $u_2 - u_1 \to 0$. Hence $N(K_1)$ is relatively compact in $C([t_0,t_1],\mathbb{R})$. Then $N(K_1)$ is relatively compact in $C([t_0,t_1],\mathbb{R})$.

Claim 4. *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h^*$. We shall prove that $h^* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in \tilde{S}^1_{F,\tau y_n}$ such that for each $t \in [t_0,t_1]$

$$h_n(t) = \int_{t_0}^{t} g_n(s)\,ds.$$

An application of Lemma 2.1 yields that there exists $g_* \in \tilde{S}^1_{F,\tau y_*}$ such that for each $t \in [t_0, t_1]$

$$h^*(t) = \int_{t_0}^{t} g_*(s)\, ds,$$

which means that N has a closed graph, and hence it is upper semicontinuous.

As a consequence of Lemma 2.2 we deduce that N has a fixed point which is a solution of (4).

Claim 5. *The solution* y *of the problem* (4) *satisfies*

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for all} \quad t \in [t_0, t_1].$$

Let y be solution to (4). We prove that

$$\alpha(t) \le y(t) \quad \text{for all} \quad t \in [t_0, t_1].$$

Suppose not. Then there exist $e_1, e_2 \in [t_0, t_1)$, $e_1 < e_2$ such that $\alpha(e_1) = y(e_1)$ and

$$y(t) < \alpha(t) \quad \text{for all} \quad t \in (e_1, e_2).$$

In view of the definition of τ one has

$$y(t) - y(e_1) \in \int_{e_1}^{t} F(s, \alpha(s))\, ds \quad \text{a.e.} \quad (e_1, e_2).$$

Thus there exists $g(t) \in F(t, \alpha(t))$ a.e. on (e_1, e_2) with $g(t) \ge v_1(t)$ a.e. on (e_1, e_2) and

$$y(t) = y(e_1) + \int_{e_1}^{t} g(s)\, ds.$$

Since α is a lower solution to (1)–(2) we have

$$\alpha(t) - \alpha(e_1) \le \int_{e_1}^{t} v_1(s)\, ds.$$

Since $y(e_1) = \alpha(e_1)$ and $g(t) \ge v_1(t)$, it follows that

$$\alpha(t) - \alpha(e_1) \le \int_{t}^{e_2} v_1(s)\, ds \le y(t) - \alpha(e_1) < \alpha(t) - \alpha(e_1),$$

which is a contradiction since $\alpha(t) > y(t)$ for all $t \in (e_1, e_2)$. Analogously, we can prove that

$$y(t) \le \beta(t) \quad \text{for all} \quad t \in [t_0, t_1).$$

This shows that the problem (3)–(2) has a solution in the interval $[\alpha, \beta]$ which is a solution of (3). Denote this solution by y_0.

Step 2: Consider now the following problem

$$y'(t) \in F(t, y(t)), \ a.e. \ t \in [t_1, t_2], \quad y(t_1^+) = I_1(y_0(t_1^-)). \tag{5}$$

Transform the problem (5) into a fixed point problem. Consider the following modified problem

$$y'(t) \in F(t, (\tau y)(t)), \quad \text{a.e. } t \in [t_1, t_2], \quad y(t_1^+) = I_1(y_0(t_1^-)). \tag{6}$$

A solution to (5) is a fixed point of the operator $N_1 : C([t_1, t_2], \mathbb{R}) \longrightarrow \mathcal{P}(C([t_1, t_2], \mathbb{R}))$ defined by:

$$N_1(y) = \left\{ h \in C([t_1, t_2], \mathbb{R}) : h(t) = I_1(y_0(t_1^-)) + \int_{t_1}^t g(s)\, ds, \quad g \in \tilde{S}^1_{F,\tau y} \right\}.$$

Since $y_0(t_1) \in [\alpha(t_1^-), \beta(t_1^-)]$, (H2) implies that

$$\alpha(t_1^+) \leq I_1(y_0(t_1^-)) \leq \beta(t_1^+),$$

that is

$$\alpha(t_1^+) \leq y(t_1^+) \leq \beta(t_1^+).$$

Using the same reasoning as that used for problem (4) we can conclude the existence of at least one solution y to (6). We now show that this solution satisfies

$$\alpha(t) \leq y(t) \leq \beta(t) \ \text{ on } \ J_1 = [t_1, t_2].$$

We first show that $\alpha(t) \leq y(t)$ on J_1. Assume this is false. Then since $y(t_1^+) \geq \alpha(t_1^+)$, there exist $e_3, e_4 \in J_1$ with $e_3 < e_4$ such that $y(e_3) = \alpha(e_3)$ and $y(t) < \alpha(t)$ on (e_3, e_4). Consequently,

$$y(t) - y(e_3) = \int_{e_3}^t g(s)\, ds, \quad t \in (e_3, e_4),$$

where $g(\cdot) \in F(\cdot, \alpha(\cdot))$ a.e. on J_1 with $g(t) \geq v_1(t)$ a.e. on (e_3, e_4) since α is a lower solution to (1). Thus

$$\alpha(t) - \alpha(e_3) \leq \int_{e_3}^t v_1(s)\, ds, \quad t \in (e_3, e_4).$$

It follows that
$$\alpha(t) \le y(t) \ \text{ on } \ (e_3, e_4),$$
which is a contradiction since $\alpha(t) > y(t)$ for all $t \in (e_3, e_4)$. Analogously, we can prove that
$$y(t) \le \beta(t) \ \text{ for all } \ t \in [t_1, t_2].$$
This shows that the problem (5) has a solution in the interval $[\alpha, \beta]$ which is a solution of (1)–(2) on J_1. Denote this solution by y_1.

Step 3: Take into account that $y_m := y\Big|_{[t_{m-1}, t_m]}$ is a solution to the problem

$$(7) \qquad y'(t) \in F(t, y(t)), \ a.e. \ t \in (t_{m-1}, t_m), \ y(t_m^+) = I_m(y_{m-1}(t_m^-)).$$

Consider the following modified problem

$$(8) \qquad y'(t) \in F(t, (\tau y)(t)), \ a.e. \ t \in [t_{m-1}, t_m], \ y(t_m^+) = I_m(y_{m-1}(t_{m-1}^-)).$$

Let the operator $N_m : C([t_{m-1}, t_m], \mathbb{R}) \longrightarrow \mathcal{P}(C([t_{m-1}, t_m], \mathbb{R}))$ be defined by:

$$N_m(y) = \left\{ h \in C([t_{m-1}, t_m], \mathbb{R}) : h(t) = I_m(y(t_{m-1}^-)) + \int_{t_m}^{t} g(s)\, ds, \ g \in \tilde{S}^1_{F,\tau y} \right\},$$

and set
$$K_m = \{ y \in C([t_{m-1}, t_m], \mathbb{R}) : \|y\|_\infty \le \|\phi_m\|_{L^1} \}.$$
Clearly, K_m is closed, bounded and convex. As in Step 1 we show that the operator $N_m : K_m \to \mathcal{P}(K_m)$ is completely continuous. As a consequence of Lemma 2.2 we deduce that N_m has a fixed point which is a solution of the problem (7). Denote this one by y_{m-1}. The solution y of the problem (1)–(2) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } \ t \in [t_0, t_1], \\ y_1(t), & \text{if } \ t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } \ t \in (t_m, t_{m+1}], \\ \dots \end{cases}$$

□

The following theorem gives sufficient conditions to ensure that the solutions of problem (1)–(2) are nonoscillatory.

Theorem 3.2. *Let α and β be lower and upper solutions respectively of* (1)–(2) *and assume that*

(H4) *α is eventually positive nondecreasing or β is eventually negative nonincreasing.*

Then every solution y of (1)–(2) *such that $y \in [\alpha, \beta]$ is nonoscillatory.*

PROOF: Assume that α is eventually positive. Thus there exist $T_\alpha > t_0$ such that

$$\alpha(t) > 0 \quad \text{for all} \quad t > T_\alpha.$$

Hence $y(t) > 0$ for all $t > T_\alpha$, and $t \neq t_k$, $k = 1, \dots$. For some $k \in \mathbb{N}$ and $t_k > T_\alpha$ we have $y(t_k^+) = I_k(y(t_k))$. From (H3) we get $y(t_k^+) \geq \alpha(t_k^+)$. Since for each $h > 0$, $\alpha(t_k + h) \geq \alpha(t_k) > 0$, we have $I_k(y(t_k)) > 0$ for all $t_k > T_\alpha$, $k = 1, \dots$ which means that y is nonoscillatory. Analogously, if β is eventually negative, then there exists $T_\beta > t_0$ such that

$$y(t) < 0 \quad \text{for all} \quad t > T_\beta,$$

which means that y is nonoscillatory. □

The following theorem discusses when solutions of (1)–(2) are nonoscillatory.

Theorem 3.3. *Let α and β be lower and upper solutions respectively of* (1)–(2) *and assume that the sequences $\alpha(t_k)$ and $\beta(t_k)$, $k = 1, \dots$ are oscillatory. Then every solution y of* (1)–(2) *such that $y \in [\alpha, \beta]$ is oscillatory.*

PROOF: Suppose on the contrary that y is a nonoscillatory solution of (1)–(2). Then there exists $T_y > 0$ such that $y(t) > 0$ for all $t > T_y$ or $y(t) < 0$ for all $t > T_y$. In the case that $y(t) > 0$ for all $t > T_y$ we have $\beta(t_k) > 0$ for all $t_k > T_y$, $k = 1, \dots$, which is a contradiction since $\beta(t_k)$ is an oscillatory upper solution. Analogously in the case $y(t) < 0$, for all $t > T_y$ we have $\alpha(t_k) < 0$ for all $t_k > T_y$, $k = 1, \dots$, which is also a contradiction since α is an oscillatory lower solution. □

4. An example

As an application of our results, we consider the following differential inclusion of the form

$$y' \in F(t, y), \quad a.e. \;\; t \in [t_0, \infty), \tag{9}$$

$$y(t_k^+) = I_k(y(t_k^-)), \;\; k \in \mathbb{N}, \tag{10}$$

where

$$F(t, y) = [f_1(t, y), f_2(t, y)] := \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

and $f_1, f_2 : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$. We assume that for each $t \in [t_0, \infty)$, $f_1(t, \cdot)$ is lower semicontinuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [t_0, \infty)$, $f_2(t, \cdot)$ is upper semicontinuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume, also, that there exist $g_1(\cdot),\ g_2(\cdot) \in L^1([t_0, \infty), \mathbb{R})$ such that

$$g_1(t) \leq f_1(t, y) \leq f_2(t, y) \leq g_2(t) \quad \text{for all} \quad t \in [t_0, \infty) \quad \text{and} \quad y \in \mathbb{R},$$

and for each $t \in [t_0, \infty)$

$$\begin{aligned} \int_{t_0}^{t} g_1(s)\, ds &\leq I_k \left(\int_{t_0}^{t} g_1(s)\, ds \right), \ k \in \mathbb{N}, \\ \int_{t_0}^{t} g_2(s)\, ds &\geq I_k \left(\int_{t_0}^{t} g_2(s)\, ds \right), \quad k \in \mathbb{N}. \end{aligned}$$

Consider the functions $\alpha(t) := \displaystyle\int_{t_0}^{t} g_1(s)\, ds$ and $\beta(t) := \displaystyle\int_{t_0}^{t} g_2(s)\, ds$. Clearly, α and β are lower and upper solutions of the problem (9)–(10), respectively, that is,

$$\alpha'(t) \leq f_1(t, y) \quad \text{for all} \quad t \in [t_0, \infty) \quad \text{and all} \quad y \in \mathbb{R},$$

and

$$\beta'(t) \geq f_2(t, y) \quad \text{for all} \quad t \in [t_0, \infty) \quad \text{and all} \quad y \in \mathbb{R}.$$

It is clear that F is compact, convex valued, and upper semicontinuous (see [11]). Since all the conditions of Theorem 3.1 are satisfied, the problem (9)–(10) has at least one solution y on $[t_0, \infty)$ with $\alpha \leq y \leq \beta$. If $g_1(t) > 0$ then α is positive and nondecreasing, thus $y(t)$ is nonoscillatory. If $g_2(t) < 0$ then β is negative and nonincreasing, thus $y(t)$ is nonoscillatory. If the sequences $\alpha(t_k)$ and $\beta(t_k)$ are both oscillatory, then $y(t)$ is oscillatory.

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