Observations on spaces with zeroset or regular G_δ-diagonals

Raushan Z. Buzyakova

Abstract. We show that if X^2 has countable extent and X has a zeroset diagonal then X is submetrizable. We also make a couple of observations regarding spaces with a regular G_δ-diagonal.

Keywords: zeroset diagonal, regular G_δ-diagonal, submetrizable, countable extent

Classification: 54E99, 54A25

1. Introduction

In [MAR], H. Martin proved that a separable space with a zeroset diagonal is submetrizable. In general, having a zeroset diagonal does not guarantee submetrizability as announced in [FRW]. In [MAR], H. Martin asks for what classes of spaces the presence of a zeroset diagonal implies submetrizability. And in this paper we show that if X^2 has countable extent and X has a zeroset diagonal then X is submetrizable. Martin's theorem and our main result motivate the following questions.

Question 1.1. *Let X have countable extent and a zeroset diagonal. Is X submetrizable? What if X is additionally locally compact or Čech-complete?*

Question 1.2 (A.V. Arhangelskii)**.** *Let X have countable Souslin number and a zeroset diagonal. Is X submetrizable?*

A property that lies between separability and countable Souslin number is $\aleph_1$-calibre. A space X has *$\aleph_1$-calibre* if every uncountable collection $\mathcal{U}$ of open sets in X contains an uncountable subcollection $\mathcal{U}'$ such that $\bigcap \mathcal{U}' \neq \emptyset$.

Question 1.3. *Let X have $\aleph_1$-calibre and a zeroset diagonal. Is X submetrizable?*

In notation and terminology we will follow [ENG]. For a space X, by Δ_X we denote the set $\{(x,x) : x \in X\}$. A space X has *countable extent* if every uncountable subset of X has a limit point in X. A space X has a *zeroset diagonal* if Δ_X is a zeroset in $X \times X$, that is, there exists a continuous function $f : X \times X \to [0,1]$ such that $\Delta_X = f^{-1}(0)$. A space X has a *regular G_δ-diagonal* if there exists a collection $\{W_n\}_n$ of open sets in $X \times X$ such that $\Delta_X = \bigcap_n W_n = \bigcap_n \overline{W}_n$. A space X *condenses* into a space Y if there exists a continuous injection of X into Y.

2. Results

Given $f : X \times X \to [0,1]$ and $a \in X$, by f_a we denote the mapping defined by $f_a(x) = f(x,a)$ (as H. Martin does in [MAR]). We use the following idea of H. Martin [MAR]: *if $x \in \overline{A}$ then for any $y \in X \setminus \{x\}$ there exists $a \in A$ such that $f_a(x) \neq f_a(y)$.*

Although in our main results, we do not assume any separation axioms, we will make use of a known folklore-type fact that any space with a G_δ-diagonal is a T_1-space.

Theorem 2.1. *Let $X \times X$ have countable extent and let X have a zeroset diagonal. Then X is submetrizable.*

PROOF: Fix a continuous $f : X^2 \to [0,1]$ such that $f^{-1}(0) = \Delta_X$. Assume that for each $\beta < \alpha$, (x_β, y_β) and a family $\mathcal{U}_\beta$ of open boxes in X^2 are defined.
Definition of (x_α, y_α) and $\mathcal{U}_\alpha$: Let $O_\alpha = \bigcup\{U \times V : U \times V \in \mathcal{U}_\beta, \beta < \alpha\}$. If $(X^2 \setminus \Delta_X) \subset O_\alpha$ then stop construction. Otherwise, take any $(x_\alpha, y_\alpha) \in X^2 \setminus (\Delta_X \cup O_\alpha)$. Let $\mathcal{U}_\alpha$ consist of all elements in the following form:

$$f_{y_\alpha}^{-1}((1/n, 1]) \times f_{y_\alpha}^{-1}([0, 1/n)),$$

where $n \in \omega \setminus \{0\}$.

Let us show that for some $\alpha < \omega_1$, the co-diagonal part is in O_α. Assume the contrary. Since $f^{-1}(0) = \Delta_X$, there exists $n_0 \in \omega$ such that $A = \{(x_\alpha, y_\alpha) : f(x_\alpha, y_\alpha) > 1/n_0, \alpha < \omega_1\}$ is uncountable. Since X^2 has countable extent, there exists a limit point (a,b) for A. By continuity, there exists n_1 such that $f(a,b) > 1/n_1$, whence $a \neq b$. Let $\alpha \leq \omega_1$ be the smallest ordinal such that (a,b) is a limit point for $\{(x_\beta, y_\beta) : \beta < \alpha\}$. Clearly α is limit (follows from T_1-axiom).

On one hand, $(a,b) \notin O_\alpha = \bigcup_{\beta<\alpha} O_\beta$. Indeed, since each (x_β, y_β) is selected outside of O_β, no element of O_β can be limit for $\{(x_\gamma, y_\gamma) : \beta \leq \gamma < \alpha\}$. Therefore, if (a,b) were in O_β for some $\beta < \alpha$, then (a,b) would have been a limit point for $\{(x_\gamma, y_\gamma) : \gamma < \beta\}$ contradicting the choice of α.

On the other hand, there exist open $U_a \ni a$ and $V_b \ni b$ such that $f(V_b \times V_b) \subset [0, 1/n_1)$ while $f(U_a \times V_b) \subset (1/n_1, 1]$. Since b is a limit point for $\{y_\beta : \beta < \alpha\}$ there exists $y_\beta \in V_b$ for some $\beta < \alpha$. Then $V = f_{y_\beta}^{-1}([0, 1/n_1))$ contains b while $U = f_{y_\beta}^{-1}((1/n_1, 1])$ contains a. Then $U \times V \in \mathcal{U}_\beta$ contains (a,b). Hence $(a,b) \in O_\alpha$, a contradiction. (This "on the other hand" part is based on ideas of H. Martin mentioned above.)

Thus, $X^2 \setminus \Delta_X = O_\alpha$ for some $\alpha < \omega_1$. Clearly, $\Delta_{\beta \leq \alpha} f_{y_\beta}$ is a continuous injection of X to $[0,1]^\omega$. $\square$

Karpov proved in [KAR] that if X is Čech-complete and ω_1-Lindelöf then X^2 is ω_1-Lindelöf as well. Recall that X is *ω_1-Lindelöf* if every ω_1-sized open cover of X contains a countable subcover. Since every ω_1-Lindelöf space has countable extent we arrive at the following.

Corollary 2.2. *Let X be a Čech-complete ω_1-Lindelöf space. If X has a zeroset diagonal, then X is submetrizable.*

In [ARH], A.V. Arhangelskii proved that a submetrizable Čech-complete Lindelöf space is metrizable. This motivates the following question.

Question 2.3. *Let X be a submetrizable Čech-complete ω_1-Lindelöf space. Is X metrizable?*

This is related to a question of A.V. Arhangelskii whether ω_1-Lindelöf Tychonoff (regular) spaces with G_δ-diagonal are submetrizable.

In Theorem 2.1, we do not really know if "zeroset diagonal" can be replaced by "regular G_δ-diagonal". The author does not even know if this can be done in Martin's theorem. However, we believe that it is not possible. And therefore, it is interesting to know if any trace of submetrizability is left if we replace "zeroset diagonal" with "regular G_δ-diagonal" in Martin's theorem and in Theorem 2.1. In the next two results a family $\mathcal{U}$ of open sets in X is called *Hausdorff separating* if for any distinct $x, y \in X$ there exist disjoint $U_x, U_y \in \mathcal{U}$ containing x and y, respectively. Clearly, the presence of a countable Hausdorff separating family in X guarantees that X condenses onto a second-countable Hausdorff space. The following theorem is an analogue of Martin's theorem.

Theorem 2.4. *Let X be separable with a regular G_δ-diagonal. Then X condenses onto a second-countable Hausdorff space.*

Proof: Let D be a countable dense subspace of X. Let $\{W_n\}_n$ be a collection of open sets in X^2 such that $\bigcap_n W_n = \bigcap_n \overline{W}_n = \Delta_X$. Let $\mathcal{B}$ be the collection of all open sets in X that are of one of the following types:

1. $\{x : (x,d) \in W_n\}$ for some $d \in D$ and $n \in \omega$;
2. $\{x : (x,d) \in X \setminus \overline{W}_n\}$ for some $d \in D$ and $n \in \omega$.

The family $\mathcal{B}$ is clearly countable. We only need to show that $\mathcal{B}$ is Hausdorff separating. First, observe that every element of $\mathcal{B}$ is open in X.

Fix any distinct $a, b \in X$. There exists $n \in \omega$ such that $(a,b) \notin \overline{W}_n$. Let $U_a \ni a$ and $V_b \ni b$ be open neighborhoods such that $V_b \times V_b \subset W_n$ and $U_a \times V_b \subset X^2 \setminus \overline{W}_n$. Due to the density property, there exists $d \in D$ such that $d \in V_b$. Then $B_b = \{x : (x,d) \in W_n\}$ contains b while $B_a = \{x : (x,d) \in X \setminus \overline{W}_n\}$ contains a. Clearly, B_a and B_b are disjoint elements of $\mathcal{B}$. □

The proof of the next theorem is almost identical to the proof of Theorem 2.1, however we have decided to handle it separately for better readability.

Theorem 2.5. *Let $X \times X$ have countable extent and let X have a regular G_δ-diagonal. Then X condenses onto a second-countable Hausdorff space.*

Proof: Fix $\{W_n\}_n$ a family of open sets in X^2 such that $\bigcap_n W_n = \bigcap_n \overline{W}_n = \Delta_X$. Assume that for each $\beta < \alpha$, (x_β, y_β) and a family $\mathcal{U}_\beta$ of open boxes in X^2 are defined.
Definition of (x_α, y_α) *and* $\mathcal{U}_\alpha$*:* Let $O_\alpha = \bigcup\{U \times V : U \times V \in \mathcal{U}_\beta, \beta < \alpha\}$. If $(X^2 \setminus \Delta_X) \subset O_\alpha$ then stop construction. Otherwise, take any $(x_\alpha, y_\alpha) \in X^2 \setminus (\Delta_X \cup O_\alpha)$. Let $\mathcal{U}_\alpha$ consist of all elements in the following form:

$$\{x : (x, y_\alpha) \in X \setminus \overline{W}_n\} \times \{x : (x, y_\alpha) \in W_n\},$$

where $n \in \omega$.

Let us show that for some $\alpha < \omega_1$, the co-diagonal part is in O_α. Assume the contrary. Since $\bigcap_n \overline{W}_n = \Delta_X$, there exists $n_0 \in \omega$ such that W_{n_0} misses uncountably many (x_α, y_α)'s. That is, $A = [X^2 \setminus W_{n_0}] \cap \{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$ is uncountable. Since X^2 has countable extent, there exists (a, b) a limit point for A. Since A is outside of W_{n_0}, while $\Delta_X \subset W_{n_0}$, we have $a \neq b$. Hence there exists $n_1 \in \omega$ such that $(a, b) \in X^2 \setminus \overline{W}_{n_1}$. Let $\alpha \leq \omega_1$ be the smallest ordinal such that (a, b) is a limit point for $\{(x_\beta, y_\beta) : \beta < \alpha\}$. On one hand, $(a, b) \notin O_\alpha$ (see the proof of Theorem 2.1). On the other hand, there exist open $U_a \ni a$ and $V_b \ni b$ such that $V_b \times V_b \subset W_{n_1}$ while $U_a \times V_b \subset X^2 \setminus \overline{W}_{n_1}$. Since b is a limit point for $\{y_\beta : \beta < \alpha\}$ there exists $y_\beta \in V_b$ for some $\beta < \alpha$. Then $V = \{(x, y_\beta) : x \in W_{n_1}\}$ contains b while $U = \{(x, y_\beta) : x \in X \setminus \overline{W}_{n_1}\}$ contains a. Then $U \times V \in \mathcal{U}_\beta$ contains (a, b). Therefore $(a, b) \in O_\alpha$, a contradiction.

Thus, $X^2 \setminus \Delta_X = O_\alpha$ for some $\alpha < \omega_1$. Clearly, $\mathcal{B} = \{U, V : U \times V \in \mathcal{U}_\beta, \beta \leq \alpha\}$ is a countable Hausdorff separating family. □

The last two theorems suggest the following question.

Question 2.6. *Let a Tychonoff (regular) X have a G_δ-diagonal. Suppose that X is separable or X^2 has countable extent. Is it true that X condenses onto a second-countable T_1-space?*

The assumption of regularity in the above question is important. Indeed, let kN be the Katětov extension of the natural numbers. The space kN is Hausdorff and is a countable union of closed discrete subspaces. Hence, kN has a G_δ-diagonal. However, kN does not condense onto any second countable T_1-space since $|kN| = 2^{2^\omega}$, while any second-countable T_1-space has cardinality at most 2^ω.

References

[ARH] Arhangelskii A.V., *On a class of spaces containing all metric spaces and all locally compact spaces*, Mat. Sb **67** (1965); English Translation: Amer. Math. Soc. Transl. **92** (1970), 1–39.

[ENG] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed., 1989.
[FRW] Fleissner W.G., Reed G.M., Wage M.L., *Normality versus countable paracompactness in perfect spaces*, Bull. Amer. Math. Soc. **82** (1976), no. 4, 635–639.
[KAR] Karpov A.N., *Countable products of Čech-complete linearly Lindelöf and initially-compact spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (2000), no. 5, 7–9, 67.
[MAR] Martin H.W., *Contractibility of topological spaces onto metric spaces*, Pacific J. Math. **61** (1975), no. 1, 209–217.

Mathematics Department, Brooklyn College, Brooklyn, NY 11210, USA
E-mail: RaushanB@brooklyn.cuny.edu

(*Received* November 12, 2004, *revised* January 10, 2005)