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The almost lattice isometric copies of c_0 in Banach lattices

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Abstract. In this paper it is shown that if a Banach lattice E contains a copy of c_0, then it contains an almost lattice isometric copy of c_0. The above result is a lattice version of the well-known result of James concerning the almost isometric copies of c_0 in Banach spaces.

Keywords: Banach lattice, lattice isometry, almost lattice isometry, copy of c_0

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1. Introduction

The classical result due to Lozanovskii and Meyer-Nieberg ([1, Theorem 14.12]) asserts that *a Banach lattice E is a KB-space iff E contains no lattice copy of c_0 iff E contains no copy of c_0*. Here and in what follows the term ‘copy’ means ‘topological copy’, and ‘lattice copy’ means ‘both lattice and topological copy’, and ‘lattice isometric copy’ means ‘both lattice and isometric copy’. Note that the above result implies that c_0 is quite often embeddable in Banach lattices. It is known that Banach lattices which contain a (lattice) copy of c_0 need not contain a lattice isometric copy of c_0, not even an isomorphically isometric copy of c_0 (see [4, pp. 521–522]).

In this paper we give a criterion in order that a Banach lattice contains a lattice isometric copy of c_0.

Recall that two Banach spaces X, Y are said to be $(1+\varepsilon)$-*isometric* provided that there exists a linear isomorphism $T : X \to Y$ with $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$, equivalently, that there exists a linear isomorphism $T : X \to Y$ such that

$$\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$$

for all $x \in X$. We say that a Banach space X contains an *almost isometric copy* of Y if for any $\varepsilon > 0$ there exists a subspace Z of X such that Z, Y are $(1+\varepsilon)$-isometric.

Let E, F be two Banach lattices. It is interesting to know whether or not for any $\varepsilon > 0$ there exists a lattice isomorphism T from E onto F such that $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$. Namely, it is expected that not only E, F are $(1+\varepsilon)$-isometric, but their respective lattice structures are preserved. For our purpose we introduce the following definition.

Definition. E, F are called to be $(1+\varepsilon)$-*lattice isometric* if there exists a lattice isomorphism $T : E \to F$ such that $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$. E is said to contain an *almost lattice isometric copy* of Y if for any $\varepsilon > 0$ there exists a Banach sublattice L of E such that L, F are $(1+\varepsilon)$-lattice isometric.

Note that Hudzik and Mastylo [3] proved that if a σ-Dedekind complete Banach lattice with an order semicontinuous norm contains a copy of l_∞, it contains an almost isometric copy of l_∞. Moreover, a routine verification implies that it contains an almost lattice isometric copy of l_∞.

Let us recall that the well-known result of James [5] (see also [2, pp. 241–242]) shows that a Banach space X contains an almost isometric copy of c_0 whenever it contains a copy of c_0. One can set a natural question: does the Banach lattice E contain an almost lattice isometric copy of c_0 whenever it contains a copy of c_0? In this paper we give a positive answer. In a sense our result is a lattice version of the preceding result due to James.

Our notions in this paper are standard. For the undefined notions and basic facts concerning Banach lattices we refer the reader to the monographs [1], [7].

2. Results

We start with a result which characterizes the lattice isometric embeddings of c_0 in Banach lattices.

Theorem 1. *A Banach lattice E contains a lattice isometric copy of c_0 if and only if there exists a disjoint sequence $\{x_n\}$ of E^+ such that $\|x_n\| = 1$ for all n, and $\|\sum_{i=1}^n x_i\| = 1$ for all n.*

Proof: We shall use the following result that is included in the proof of Theorem 14.3 of [1]:

- $(*)$ A Banach lattice E contains a lattice copy of c_0 if and only if there exist a disjoint sequence $\{u_n\}$ of E^+ and two positive constants K, M such that
  - (1) $\|u_n\| \ge K$ for all n; and
  - (2) $\|\sum_{i=1}^n u_i\| \le M$ for all n.

  Moreover, if the pairwise disjoint sequence $\{u_n\}$ satisfies (1) and (2), then $T : c_0 \to E$, defined by $T(\alpha_1, \alpha_2, \dots) = \sum_{n=1}^\infty \alpha_n u_n$ for every $(\alpha_1, \alpha_2, \dots) \in c_0$, is a lattice embedding (= lattice and topological isomorphism into) satisfying

$$K\|(\alpha_1, \alpha_2, \dots)\|_\infty \le \|T(\alpha_1, \alpha_2, \dots)\| \le M\|(\alpha_1, \alpha_2, \dots)\|_\infty.$$

Now let $\{x_n\}$ be a sequence of pairwise disjoint unit vectors in E^+ such that $\|\sum_{i=1}^n x_i\| = 1$ for all n, and apply $(*)$. Clearly, $T : c_0 \to E$, defined by $T(\alpha_1, \alpha_2, \dots) = \sum_{n=1}^\infty \alpha_n x_n$ for every $(\alpha_1, \alpha_2, \dots) \in c_0$, is a lattice isometry (into), as desired.

For the converse, take the sequence $e_n = (0, \cdots, 0, 1, 0, \cdots)$, where 1 stands on the n-th place, and let $T : c_0 \to E$ be a lattice isometry. Then the vectors $x_n = T(e_n)$ satisfy the desired properties. $\square$

Let us recall that a Banach lattice E is said to be an AM-space whenever $x \wedge y = 0$ in E implies $(\|x \vee y\| =)\|x + y\| = \max\{\|x\|, \|y\|\}$. It is known ([6, p. 152]) that any infinite dimensional Banach lattice has an infinite system of pairwise disjoint nonzero elements. Then an immediate consequence of the preceding theorem is the following.

Corollary 1. *Every infinite dimensional AM-space contains a lattice isometric copy of* c_0.

Now we present our main theorem in this paper, which is a lattice version of the result due to James concerning the almost isometric copies of c_0 in Banach spaces ([5]). Our proof uses the idea of James.

Theorem 2. *Let E be a Banach lattice. If E contains a copy of c_0, then it contains an almost lattice isometric copy of c_0.*

Proof: Since E contains a copy of c_0, it contains a lattice copy of c_0. Therefore, in virtue of $(*)$ (in the proof of Theorem 1), it follows that there exist a pairwise disjoint sequence $\{u_n\}$ if E^+ and two positive constants m, M such that $\inf_n\{\|u_n\|\} \geq m$ and $\|\sum_{i=1}^n u_i\| \leq M$ for all n.

For each $n \in \mathbb{N}$, let us define

$$K_n = \sup\Big\{\Big\|\sum_{i=n}^{m} u_i\Big\| : m \geq n\Big\}.$$

Note that $\{K_n\}$ is a decreasing sequence all values of which lie between m and M. So it is convergent and $m \leq K = \lim_n K_n \leq M$. Let $0 < \varepsilon < 1$ be fixed, and take $0 < \theta < 1 < \lambda$ such that $\theta/\lambda \geq 1 - \varepsilon$. We pick $p_1 \in \mathbb{N}$ so that $K_{p_1} < \lambda K$. Take advantage of the definition of K_n to select a certain $p_2 > p_1$ such that

$$\Big\|\sum_{i=p_1}^{p_2-1} u_i\Big\| > \theta K_{p_1} \geq \theta K.$$

By the definition we can construct an increasing sequence $\{p_n\}$ in $\mathbb{N}$ satisfying

$$K_{p_n} \leq K_{p_1} < \lambda K, \qquad \Big\|\sum_{i=p_n}^{p_{n+1}-1} u_i\Big\| \geq \theta K.$$

Let us put

$$x_n = \sum_{i=p_n}^{p_{n+1}-1} (\lambda K)^{-1} u_i.$$

Observe that $\{x_n\}$ is a mutually disjoint sequence of E^+ such that $\inf_n\{\|x_n\|\} \geq \theta/\lambda$ and $\|\sum_{i=1}^n x_i\| \leq (\lambda K)^{-1} M$ for all n, which allows us to apply $(*)$ again. Now if for each $\alpha = (\alpha_1, \alpha_2, \dots) \in c_0$ we define $T(\alpha) = \sum_{n=1}^\infty \alpha_n x_n$, then T is a lattice embedding from c_0 into E.

On the one hand, for each $\alpha = (\alpha_1, \alpha_2, \dots) \in c_0$ we have

$$\begin{aligned} \|T(\alpha)\| &= \|T(|\alpha|)\| = \Big\| \sum_{n=1}^{\infty} |\alpha_n| \, x_n \Big\| \\ &= \lim_n \Big\| \sum_{i=1}^{n} |\alpha_i| \, x_i \Big\| \leq \|\alpha\|_\infty \sup_n \Big\| \sum_{i=1}^{n} x_i \Big\| \\ &\leq (\lambda K)^{-1} K_{p_1} \|\alpha\|_\infty \leq \|\alpha\|_\infty . \end{aligned}$$

On the other hand, $\|T(\alpha)\| \geq |\alpha_n| \, \|x_n\| \geq (\theta/\lambda) \, |\alpha_n| \geq (1-\varepsilon) \, |\alpha_n|$ for each n. Hence,

$$(1-\varepsilon) \, \|\alpha\|_\infty \leq \|T(\alpha)\| \leq \|\alpha\|_\infty$$

for all $\alpha \in c_0$. □

With the preceding theorem the following result should be clear.

Corollary 2. *For a Banach lattice E the following statements are equivalent:*

1. *E is not a KB-space;*
2. *E contains a copy of c_0;*
3. *E contains a lattice copy of c_0;*
4. *E contains an almost isometric copy of c_0;*
5. *E contains an almost lattice isometric copy of c_0.*

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