Bi-ideal-simple semirings

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Abstract. Commutative congruence-simple semirings were studied in [2] and [7] (but see also [1], [3]–[6]). The non-commutative case almost (see [8]) escaped notice so far. Whatever, every congruence-simple semiring is bi-ideal-simple and the aim of this very short note is to collect several pieces of information on these semirings.

Keywords: bi-ideal-simple, semiring, zeropotent

Classification: 16Y60

1. Introduction

A semiring is a non-empty set equipped with two binary operations, denoted as addition and multiplication, such that the addition makes a commutative semigroup, the multiplication is associative and distributes over the addition from both sides. The additive (multiplicative, resp.) semigroup of the semiring may, but need not, contain a neutral and/or an absorbing element. An element will be called *bi-absorbing* if it is absorbing for both the operations. If such an element exists, it will be denoted by the symbol o $(= o_S)$. We thus have $o+x = ox = xo = o$ for every $x \in S$.

Let S be a semiring. We put $A + B = \{a + b;\ a \in A, b \in B\}$, $AB = \{ab;\ a \in A, b \in B\}$ and $2A = \{a + a;\ a \in A\}$ for any two subsets A and B of S.

A semiring S is called *congruence-simple* if it has just two congruence relations.

2. Bi-ideals

Let S be a semiring. A non-empty subset I of S is called *a bi-ideal* of S if $(S+I) \cup SI \cup IS \subseteq I$ (i.e., I is an ideal both of the additive and the multiplicative semigroup of the semiring S).

The following seven lemmas are easy.

2.1 Lemma. *A one-element subset $\{w\}$ of S is a bi-ideal if and only if $w = o_S$ is a bi-absorbing element of S.*

This research has been supported by the Grant Agency of the Charles University, Grant #444/2004 and by the institutional grant MSM 0021620839.

2.2 Lemma. *The subsets $S, S+S, SS+S, SSS+S, \dots$ are bi-ideals of S.*

2.3 Lemma. *$SaS+S$ is a bi-ideal of S for every $a \in S$.*

In the remaining lemmas, assume that $o \in S$.

2.4 Lemma. *The set $\{x \in S;\, xSx = o\}$ is a bi-ideal.*

2.5 Lemma. *The set $\{x \in S;\, x + xSx = o\}$ is a bi-ideal.*

2.6 Lemma. *The set $\{x \in S;\, 2x = o\}$ is a bi-ideal.*

2.7 Lemma. *The sets $(o:S)_l = \{x \in S;\, xS = o\}$, $(o:S)_r = \{x \in S;\, Sx = o\}$ and $(o:S)_m = \{x \in S;\, SxS = o\}$ are bi-ideals.*

3. Bi-ideal-free semirings

3.1 Proposition. *A semiring S is bi-ideal-free (i.e., it has no proper bi-ideal) if and only if $S = SaS + S$ for every $a \in S$.*

Proof: The assertion follows easily from 2.3. □

4. Bi-ideal-simple semirings — introduction

A semiring S will be called *bi-ideal-simple* if $|S| \geq 2$ and $I = S$ whenever I is a bi-ideal of S with $|I| \geq 2$.

4.1 Proposition. *A semiring S is bi-ideal-simple if and only if at least one (and then just one) of the following five cases takes place:*

(1) $|S| = 2$;
(2) $|S| \geq 3$ *and* $S = SaS + S$ *for every* $a \in S$;
(3) $|S| \geq 3$, $o \in S$ *and* $S = SaS + S$ *for every* $a \in S$, $a \neq o$;
(4) $|S| \geq 3$, $o \in S$, $S + S = o$ *and* $SaS = S$ *for every* $a \in S$, $a \neq o$;
(5) $|S| \geq 3$, $o \in S$, $SS = o$ *and* $S + a = S$ *for every* $a \in S$, $a \neq o$.

Proof: We prove only the direct implication, the converse one being trivial. So, let S be a bi-ideal-simple semiring. We will assume that $|S| \geq 3$ and, moreover, in view of 3.1, that S is not bi-ideal-free. Then $o \in S$ by 2.1 and we have to distinguish the following three cases.

(i) Let $S + S = o$. If $SS = o$, then every subset of S containing o is a bi-ideal and then $|S| = 2$, a contradiction. Thus $SS \neq o$, and hence $(o:S)_l = o$ by 2.7. Similarly $(o:S)_r = o$ and, by combination, we get $(o:S)_m = o$ (use 2.7 again). Now, if $a \in S$, $a \neq o$, then $SaS \neq o$ and, since $S + S = o$, the set SaS is a bi-ideal. Thus $SaS = S$ and (4) is true.
(ii) Let $SS = o$. Then every ideal of $S(+)$ is a bi-ideal, and so the semigroup $S(+)$ is ideal-simple. Now, (5) is clear.
(iii) Let $S + S \neq o \neq SS$. We have $(o:S)_l = o = (o:S)_r$ by 2.7, and hence $(o:S)_m = o$, too. Further, $S + S = S$ by 2.2. Now, if $a \in S$,

$a \neq o$, then $bac \neq o$ for some $b, c \in S$, and $b = d + e$, $d, e \in S$. Then $o \neq bac = dac + eac \in SaS + S$ and consequently, $SaS + S = S$ by 2.3. It means that (3) is true. □

In the rest of this section, we assume that S is a bi-ideal-simple semiring with $o \in S$.

4.2 Proposition. *If S is a nil-semiring (i.e., for every $x \in S$ there exists $n \geq 1$ with $x^n = o$), then $SS = o$.*

Proof: The result is clear for $|S| = 2$, and so let $|S| \geq 3$. If $a \in S$, $a \neq o$, is such that either $S = SaS + S$ or $S = SaS$, then $a = bac + w$, $b, c \in S$, $w \in S \cup \{0\}$, and we have $a = b^2ac^2 + bwc + w = b^3ac^3 + b^2wc^2 + bwc + w = \dots$, a contradiction with $b^n = o$. Thus $SaS + S \neq S \neq SaS$ and the rest is clear from 4.1. □

4.3 Lemma. *If $|S| \geq 3$ and $SS \neq o$, then for every $a \in S$, $a \neq o$, there exist elements $b, c, d \in S$ such that $bac \neq o \neq ada$.*

Proof: Combine 2.4, 2.7 and 4.2. □

4.4 Lemma. *Just one of the following two cases takes place:*

(1) *$x + xSx = o$ for every $x \in S$;*
(2) *for every $a \in S$, $a \neq o$, there exists at least one $b \in S$ with $a + aba \neq o$.*

Proof: Use 2.5. □

4.5 Lemma. *Just one of the following two cases takes place:*

(1) *$2x = o$ for every $x \in S$;*
(2) *$2a \neq o$ for every $a \in S, a \neq o$.*

Proof: Use 2.6. □

5. Congruence-simple semirings

5.1 Proposition. *Every congruence-simple semiring is bi-ideal-simple.*

Proof: If I is a bi-ideal, then the relation $(I \times I) \cup \mathrm{id}_S$ is a congruence of S. □

5.2 Proposition. *Let S be a bi-ideal-simple semiring with $o \in S$ and let $a \in S$, $a \neq o$. If r is a congruence of S maximal with respect to $(a, o) \notin r$, then S/r is a congruence-simple semiring.*

Proof: S/r is non-trivial. If s is a congruence of S such that $r \subseteq s$ and $r \neq s$, then $(a, o) \in s$ and $I = \{x \in S;\ (x, o) \in s\}$ is a bi-ideal of S. Since $|I| \geq 2$, we have $I = S$ and $s = S \times S$. □

5.3 Corollary. *Let S be a bi-ideal-simple semiring with $o \in S$. Then S can be imbedded into the product of congruence-simple factors of S.*

6. Bi-ideal-simple semirings of type 4.1(4)

6.1. If S is a bi-ideal-simple semiring of type 4.1(4), then the multiplicative semigroup of S is ideal-simple.

6.2. Let S be a multiplicative ideal-simple semigroup with $|S| \geq 3$ and $o \in S$. Setting $S + S = o$ we get a bi-ideal-simple semiring of type 4.1(4).

7. Bi-ideal-simple semirings of type 4.1(5)

7.1. If S is a bi-ideal-simple semiring of type 4.1(5), then $T(+)$ is an (abelian) subgroup of $S(+)$, where $T = S \setminus \{o\}$.

7.2. Let $T(+)$ be an abelian group, $|T| \geq 2$, $o \notin T$ and $S = T \cup \{o\}$. Setting $SS = S + o = o + S = o$ we get a bi-ideal-simple semiring of type 4.1(5).

8. Additively zeropotent semirings

In this section, let S be an additively zeropotent semiring (a *zp-semiring* for short). That is, $o \in S$ and $2S = o$. We define a relation $\leq$ on S by $a \leq b$ iff $b \in (S + a) \cup \{a\}$. It is easy to check that $\leq$ is a relation of order which is compatible with the two operations defined on S. That is, $\leq$ is an ordering of the semiring S. Clearly, o is the greatest element of S.

8.1 Lemma. *If $|S| \geq 2$, then an element $a \in S$, $a \neq o$, is maximal in $S \setminus \{o\}$ if and only if $S + a = o$.*

PROOF: Obvious. □

In the rest of this section, we will assume that $S = S + S$.

8.2 Lemma. *If $|S| \geq 2$, then S has no minimal elements.*

PROOF: If $a \in S$, $a \neq o$, then $a = b + c$, $b \leq a$. If $b = a$, then $a = a + c = a + 2c = a + o = o$, a contradiction. □

8.3 Corollary. *Either $|S| = 1$ or S is infinite.*

8.4 Lemma. *The only idempotent element of S is the bi-absorbing element o.*

PROOF: Let $b^2 = b$ for some $b \in S$. Then $b = c + d$ and $b = b^3 = b(c + d)b = bcb + bdb$. Of course, $c \leq b$, $d \leq b$, and hence $cd \leq bd$, $cd \leq cb$, $o = 2cd \leq bd + cb$ and $bd + cb = o$. Finally, $o = bob = bdb + bcb = b$. □

8.5 Corollary. *If S contains a left (or right) unit, then $|S| = 1$.*

8.6 Lemma. *If $a^k = a^l$ for some $a \in S$ and $1 \leq k < l$, then $a^k = o$.*

PROOF: There are positive integers m, n such that $m(l - k) = k + n$. Now, if $b = a^{k+n}$, then $b = a^k a^n = a^l a^n = a^k a^{l-k} a^n = a^l a^{l-k} a^n = a^k a^{l-k} a^{l-k} a^n = \dots = a^k a^{m(l-k)} a^n = a^{2k+2n} = b^2$. By 8.4, $b = o$, and hence $a^k = a^l = a^k a^{l-k} = a^k a^{l-k} a^{l-k} = \dots = a^k a^{m(l-k)} = a^k a^{k+n} = a^k b = o$. □

8.7 Lemma. *Let $a, b \in S$ and $k, l \geq 1$ be such that $a^k = a^l + b$. Then $a^{2k} = o$. Moreover, if $2k \leq l$, then $a^k = o$.*

PROOF: We have $a^{2l} + a^l b = a^l(a^l + b) = a^{k+l} = (a^l + b)a^l = a^{2l} + ba^l$. Consequently, $a^{2k} = (a^l + b)^2 = a^{2l} + a^l b + ba^l + b^2 = a^{2l} + ba^l + ba^l + b^2 = o$. If $2k \leq l$, then $a^{2k} = o$ implies $a^l = o$ and hence $a^k = a^l + b = o$. $\square$

8.8 Lemma. *If $a \in S$ is a non-nilpotent element, then the powers $a^1, a^2, a^3, \dots$ are pair-wise incomparable.*

PROOF: Combine 8.6 and 8.7. $\square$

8.9 Lemma. *If $a, b \in S$ are such that $aba \leq a$, then $a = o$.*

PROOF: If $aba = a$, then $(ab)^2 = ab$, $ab = o$ by 8.4 and $a = aba = oa = o$. If $aba + c = a$, then $ab = abab + cb = (ab)^2 + cb$, $ab = o$ by 8.7 and $a = o$, too. $\square$

8.10 Lemma. *If $a, b \in S$ are such that $ab = a \neq o$ ($ab = b \neq o$, resp.), then $a \nleq b$ ($b \nleq a$, resp.).*

PROOF: Firstly, $a \neq b$ by 8.4. Now, if $a \leq b$, then $b = a + c$, $ac \leq bc$, $c \leq b$, $ac \leq ab = a$, $o = 2ac \leq a + bc$ and $a + bc = o$. Thus $o = a(a + bc) = a^2 + abc = a^2 + ac = a(a + c) = ab = a$, a contradiction. Similarly the second case. $\square$

8.11 Proposition. *Let S be a zp-semiring with $S + S = S$ and $|S| \geq 2$. Then:*

(i) *S is infinite;*
(ii) *the ordered set $(S, \leq)$ has no minimal elements;*
(iii) *the bi-absorbing element o is the only idempotent element of S;*
(iv) *S contains neither a left nor a right unit;*
(v) *if $a \in S$ is not nilpotent, then the elements a^i, $i \geq 1$, are pair-wise incomparable in $(S, \leq)$;*
(vi) *if $a \neq o$, then $aba \nleq a$ for every $b \in S$;*
(vii) *if $o \neq a \leq b$, then $ab \neq a$;*
(viii) *if $o \neq b \leq a$, then $ab \neq b$.*

PROOF: See 8.2, 8.3, 8.4, 8.5, 8.8, 8.9 and 8.10. $\square$

9. Bi-ideal-simple zp-semirings

9.1 Proposition. *Let S be a zp-semiring with $|S| \geq 3$. Then S is bi-ideal-simple if and only if at least one (and then just one) of the following two cases takes place:*

(1) *$S + S = o$ and $SaS = S$ for every $a \in S$, $a \neq o$;*
(2) *$S = S + SaS$ for every $a \in S$, $a \neq o$.*

PROOF: The result follows easily from 4.1. $\square$

In the rest of this section, let S be a bi-ideal-simple semiring such that $S + S \neq o$. Then $S + S = S$ and S is infinite (see 8.11). Moreover, by 4.2, S is not nil.

9.2 Lemma. *Let V be a finite subset of $S \setminus \{o\}$. Then there exists at least one element $a \in S$ such that $a \neq o$ and $a \not\leq v$ for every $v \in V$.*

Proof: Firstly, by 9.1, we have $S = S + SbS$ for every $b \in S$, $b \neq o$. In particular, $SbS \neq o$, $SS \neq o$ and, by 4.3, for every $w \in V$ there is at least one $a_w \in S$ with $wa_ww \neq o$. Then $a_w \neq o$ and $wa_ww \not\leq w$ by 8.9. Now, a sequence $v_1, \dots, v_k$, $k \geq 2$, of elements from V will be called *admissible* in the sequel if these elements are pair-wise distinct and $v_ia_iv_i \leq v_{i+1}$ for some $a_i \in S$, $1 \leq i \leq k-1$.

If there is no admissible sequence, then $wa_ww \neq o$ and $wa_ww \not\leq v$ for all $w, v \in V$. The result is proved in this case, and hence we can assume that $v_1, \dots, v_k$, $k \geq 2$, is an admissible sequence with maximal length k.

Let m be maximal with respect to $1 \leq m \leq k$ and $v_kbv_k \leq v_m$ for at least one $b \in S$. Then $m < k$ by 8.9 and $v_ma_mv_m \leq v_{m+1}$ implies $v_kbv_ka_mv_kbv_k \leq v_{m+1}$, a contradiction with the maximality of m. We have thus shown that $v_kcv_k \not\leq v_i$ for all $1 \leq i \leq k$ and $c \in S$. In particular, $o \neq v_ka_kv_k \not\leq v_i$ for some $a_k \in S$ and all i, $1 \leq i \leq k$. Finally, it follows from the maximality of k that $v_ka_kv_k \not\leq v$ for every $v \in V$. □

9.3 Corollary. *Denote by A the set of maximal elements of $(S \setminus \{o\}, \leq)$ (see 8.1) and assume that every element from $S \setminus \{o\}$ is smaller or equal to an element from A. Then the set A is infinite.*

9.4 Proposition. *Just one of the following two cases takes place:*

(1) *$x + xSx = o$ and $x^m + x^n = o$ for every $x \in S$ and all positive integers m, n;*

(2) *for every $a \in S$, $a \neq o$, there exists at least one $b \in S$ such that $a + aba \neq o$.*

Proof: Taking into account 4.4, we may assume that $x + xSx = o$ for every $x \in S$. Then $x + x^3 = o$ and we put $I = \{a \in S;\, a + a^2 = o\}$. Clearly, I is an ideal of $S(+)$. Moreover, if $a \in I$ and $b \in S$, then $ab + abab = (a + aba)b = o$. Thus $ab \in I$, similarly $ba \in I$ and we see that I is a bi-ideal of S.

If $I = \{o\}$, then $a + a^2 \neq o$ for every $a \in S$, $a \neq o$. But $a^2 + a^4 = a(a + a^3) = ao = o$ and $a^2 \in I$. It follows that $a^2 = o$ for every $a \in S$, a contradiction with 4.2. Thus $I \neq \{o\}$ and we get $I = S$ and $x + x^2 = o$ for every $x \in S$. Further, $x + x = o$ by the zp-property and $x + x^n = o$ for every $n \geq 3$, since $x + xSx = o$.

If $2 \leq n \leq m$, then $x^n + x^m = x^{n-1}(x + x^{m-n+1}) = x^{n-1}o = o$. □

9.5 Proposition. *Denote by A the set of maximal elements of the ordered set $(S \setminus \{o\}, \leq)$. If A is non-empty, then $x + xSx = o$ for every $x \in S$ (i.e., the case 9.4(1) takes place).*

Proof: Combine 8.1 and 9.4. □

10. An open problem

10.1. No example of a non-trivial zp-semiring S with $S + S = S$ (see 8.11) is known (at least to the authors of the present brief note).

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(*Received* October 12, 2004, *revised* March 9, 2005)