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# Semivariation in $L^p$-spaces

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*Abstract.* Suppose that $X$ and $Y$ are Banach spaces and that the Banach space $X\hat{\otimes}_\tau Y$ is their complete tensor product with respect to some tensor product topology $\tau$. A uniformly bounded $X$-valued function need not be integrable in $X\hat{\otimes}_\tau Y$ with respect to a $Y$-valued measure, unless, say, $X$ and $Y$ are Hilbert spaces and $\tau$ is the Hilbert space tensor product topology, in which case Grothendieck's theorem may be applied.

In this paper, we take an index $1 \le p < \infty$ and suppose that $X$ and $Y$ are $L^p$-spaces with $\tau_p$ the associated $L^p$-tensor product topology. An application of Orlicz's lemma shows that not all uniformly bounded $X$-valued functions are integrable in $X\hat{\otimes}_{\tau_p} Y$ with respect to a $Y$-valued measure in the case $1 \le p < 2$. For $2 < p < \infty$, the negative result is equivalent to the fact that not all continuous linear maps from $\ell^1$ to $\ell^p$ are $p$-summing, which follows from a result of S. Kwapien.

*Keywords:* absolutely $p$-summing, bilinear integration, semivariation, tensor product

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## 1. Introduction

Bilinear integration arises in many areas of analysis, such as the representation of solutions of evolution equations [8]. Given a vector measure $m : \mathcal{E} \to Y$ with values in a Banach space $Y$ and defined over a measurable space $(\Sigma, \mathcal{E})$, an $\mathcal{E}$-measurable simple function $s = \sum_{j=1}^n x_j \chi_{E_j}$ with values in a Banach space $X$ has an indefinite integral $s \otimes m : \mathcal{E} \to X \otimes Y$ with respect to $m$ defined by

$$(s \otimes m)(E) = \sum_{j=1}^{n} x_j \otimes m(E_j \cap E), \qquad E \in \mathcal{E}. \tag{1.1}$$

If the tensor product $X \otimes Y$ of $X$ and $Y$ has a given locally convex topology $\tau$, then by a suitable limiting procedure, the integral (1.1) can be extended to more general functions $f : \Sigma \to X$ so that the indefinite integral $f \otimes m : \mathcal{E} \to X\hat{\otimes}_\tau Y$ takes values in the completion $X\hat{\otimes}_\tau Y$ of the tensor product $X \otimes_\tau Y$ endowed with the topology $\tau$.

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A general procedure of this nature is studied in [9] in the case that the tensor product topology $\tau$ satisfies the special condition that $X' \otimes Y'$ separates the space $X \hat{\otimes}_\tau Y$, see [9] for the relationship of this approach to other bilinear integrals ([1], [6]). It is a fact of bilinear life that not all uniformly bounded, strongly $\mathcal{E}$-measurable functions $f : \Sigma \to Y$ need be $m$-integrable in $X \hat{\otimes}_\tau Y$.

A simple example is given in [9, Proposition 4.2]. Take $X = Y = L^2([0,1])$ and let $\pi$ be the projective tensor product topology on $L^2([0,1]) \otimes L^2([0,1])$. For the Borel $\sigma$-algebra $\mathcal{B}([0,1])$ of $[0,1]$, the vector measure $m : \mathcal{B}([0,1]) \to L^2([0,1])$ is defined by $m(B) = \chi_B$ for every set $B \in \mathcal{B}([0,1])$. A function $f : [0,1] \to L^2([0,1])$ is $m$-integrable in $L^2([0,1]) \hat{\otimes}_\pi L^2([0,1])$ if and only if there exists a trace-class operator on $L^2([0,1])$ with kernel $(x,y) \mapsto k(x,y)$, $x, y \in [0,1]$, such that $f(x) = k(x,\cdot)$ for almost all $x \in [0,1]$. For $f$ to be $m$-integrable in the Banach space $L^2([0,1]) \hat{\otimes}_\pi L^2([0,1])$, it is simply not enough that there exists $M > 0$ such that $\|f(x)\|_2 \le M$ for almost all $x \in [0,1]$.

A key consideration here is whether or not there exists a bound $C > 0$ such that

$$\|(s \otimes m)(\Sigma)\|_\tau \le C\|s\|_\infty \tag{1.2}$$

for every $X$-valued $\mathcal{E}$-measurable simple function $s$. Here we suppose that the tensor product topology $\tau$ is actually given by a norm $\|\cdot\|_\tau$ and $\|s\|_\infty = \max_j \|x_j\|_X$ for $s = \sum_{j=1}^n x_j \chi_{E_j}$ and $\{E_j\}_{j=1}^n$ pairwise disjoint. If the bound (1.2) holds, then we can hope to approximate a bounded $X$-valued function by the pointwise limit of uniformly bounded sequence of $X$-valued simple functions.

To be more precise, the *$X$-semivariation of $m$ in $X \hat{\otimes}_\tau Y$* is the set function $\beta_X(m) : \mathcal{E} \to [0,\infty]$ defined by

$$\beta_X(m)(E) = \sup \left\{ \left\| \sum_{j=1}^k x_j \otimes m(E_j) \right\|_\tau \right\} \tag{1.3}$$

for every $E \in \mathcal{E}$; the supremum is taken over all pairwise disjoint sets $E_1, \dots, E_k$ from $\mathcal{E} \cap E$ and vectors $x_1, \dots, x_k$ from $X$, such that $\|x_j\|_X \le 1$ for all $j = 1, \dots, k$ and $k = 1, 2, \dots$ . The bound (1.2) therefore holds exactly when $\beta_X(m)(\Sigma) < \infty$. If $\beta_X(m)(\Sigma) < \infty$ and the Banach space $X \hat{\otimes}_\tau Y$ contains no copy of $c_0$, then the $X$-semivariation $\beta_X(m)$ is *continuous* in the sense of Dobrakov, namely, $\beta_X(m)(A_k) \to 0$ whenever $\{A_k\}_{k=1}^\infty$ is a sequence in $\mathcal{E}$ decreasing to the empty set; see [6, *-Theorem]. This suffices to deduce that bounded strongly measurable $X$-valued functions are $m$-integrable in $X \hat{\otimes}_\tau Y$, see [7, Theorem 5] and [9, Theorem 2.7]. For the converse statement, see [13, Theorem 6]. If, in particular, $\|x \otimes y\|_\tau = \|x\| \cdot \|y\|$ for all $x \in X$ and $y \in Y$ (that is, $\|\cdot\|_\tau$ is a cross norm), then

$$\|m\|(E) \le \beta_X(m)(E), \qquad E \in \mathcal{E}. \tag{1.4}$$

Here $\|m\| : \mathcal{E} \to [0,\infty)$ denotes the usual semivariation of the vector measure $m$, [4, Definition I.1.4].

This note is concerned with the natural situation in which $1 \le p < \infty$, $\mu$ and $\nu$ are $\sigma$-finite measures, $X = L^p(\mu)$, $Y = L^p(\nu)$ and $\tau$ is the relative tensor product topology of the space $L^p(\mu \otimes \nu)$ of functions $p$th-integrable with respect to the product measure $\mu \otimes \nu$. The completion $L^p(\mu)\hat{\otimes}_\tau L^p(\nu)$ may be identified with any of the spaces $L^p(\mu \otimes \nu)$, $L^p(\mu, L^p(\nu))$ or $L^p(\nu, L^p(\mu))$ and in the case $p = 1$, the tensor product topology $\tau$ is just the projective tensor product topology $\pi$, [4, Example VIII.1.10].

In the main result of this work, Theorem 3.3, we show that for every $2 < p < \infty$, there is some vector measure $m : \mathcal{E} \to L^p([0,1])$ whose $L^p([0,1])$-semivariation in $L^p([0,1]^2)$ is infinite. We prove this by reducing the problem to determining whether or not any continuous linear mapping from $\ell^1$ into $\ell^p$ is $p$-summing. That this is false follows from a result of S. Kwapien [10, Theorem 7, $2^0$] and some standard Banach space arguments. The proof does not obviously give an explicit example of a continuous linear map from $\ell^1$ into $\ell^p$ that is not $p$-summing when $2 < p < \infty$. It is a well-known consequence of Grothendieck's inequality that any continuous linear map from $\ell^1$ into $\ell^2$ is absolutely summing and so $p$-summing for all $1 \le p < \infty$.

Some background on semivariation in $L^p$-spaces is provided in Section 2. Many of the basic facts given in Section 2 were proved by the authors prior to the publication of [8], where they were needed for the representation of evolutions. The connection between absolutely $p$-summing maps and semivariation in $L^p$-spaces is explained in Section 3, where the main result Theorem 3.3 is stated. The short argument that reduces the search for a non-$p$-summing map from $\ell^1$ into $\ell^p$ to Kwapien's result is given in Lemma 4.1 in Section 4.

## 2. Semivariation

An example of an $L^p([0,1])$-valued measure without finite $L^p([0,1])$-semivariation in $L^p([0,1]^2)$ was given in [9, Example 2.2], for any $1 \le p < 2$, as a consequence of Orlicz's Theorem [11, Theorem 1.c.2]; see Example 2.3 below.

In the case $p = 2$, let $X = L^2(\mu)$ and $Y = L^2(\nu)$ for $\sigma$-finite measures $\mu$ and $\nu$. The inner product is denoted by $(\,\cdot\,|\,\cdot\,)$. Then with $(s \otimes m)(E)$ given by formula (1.1) and $\|x_j\|_2 = 1$ for $j = 1, \ldots, n$, we note that

$$\begin{aligned} \|(s \otimes m)(E)\|_2^2 &= \big((s \otimes m)(E)|(s \otimes m)(E)\big) \\ &= \sum_{j,k=1}^{n} (x_j|x_k) \cdot \big(m(E_j \cap E) \,\big|\, m(E_k \cap E)\big) \end{aligned}$$

$$\leq K_G \sup \left| \sum_{k,j=1}^{n} s_j t_k \big( m(E_j \cap E) \,\big|\, m(E_k \cap E) \big) \right|$$
$$= K_G \big( \|m\|(E) \big)^2.$$

Here the supremum on the right is over all complex numbers $s_j$, $t_k$ with $j,k = 1,\dots,n$, such that $|s_j| \leq 1$ and $|t_k| \leq 1$ for all $j,k = 1,\dots,n$, $K_G$ is Grothendieck's constant [11, Theorem 2.b.5] and the bound is uniform in $n = 1,2,\dots$ . The $L^2(\mu)$-semivariation in $L^2(\mu \otimes \nu)$ of any $L^2(\nu)$-valued vector measure $m$ is therefore finite and (1.4) gives

$$\|m\|(E) \leq \beta_X(m)(E) \leq \sqrt{K_G}\|m\|(E), \qquad E \in \mathcal{E}.$$

We note this in the following statement.

**Proposition 2.1** ([8, Proposition 4.5.3]). *Let $H$ be a Hilbert space and $m : \mathcal{E} \to L^2(\nu)$ a measure. Let $\|m\| : \mathcal{E} \to [0,\infty)$ be the semivariation of $m$ in $L^2(\nu)$. Then the measure $m$ has finite $H$-semivariation $\beta_H(m)$ in $L^2(\nu, H)$. Moreover, there exists a constant $C > 0$, independent of $H$ and $m$, and a finite measure $\eta$ with $0 \leq \eta \leq \|m\|$ such that $\lim_{\eta(E)\to 0} \|m\|(E) = 0$ and $\beta_H(m)(E) \leq C\|m\|(E)$, for all $E \in \mathcal{E}$, and hence $\beta_H(m)$ is continuous in the sense of Dobrakov.*

On the positive side, by [8, Proposition 4.5.1], for every $1 \leq p < \infty$ and any Banach space $X$, an $L^p(\nu)$-valued measure $m$ with *order bounded* range has finite $X$-semivariation in $L^p(\nu, X)$ and $\beta_X(m)$ is continuous.

Now consider the case $p = \infty$, every $L^\infty(\nu)$-valued measure $m$ automatically has order bounded range because its range is bounded ([4, Corollary I.2.7]). So, $m$ admits $\sigma$-additive modulus $|m| : \mathcal{E} \to L^\infty(\nu)_+$, [12, Theorem 5]. The same argument as in the proof of [8, Proposition 4.5.1] shows that

$$\beta_X(m)(A) \leq \big\| |m| \big\|(A), \qquad A \in \mathcal{E}$$

and hence, $m$ has finite $X$-semivariation for every Banach space $X$. So it is the oscillatory nature of vector measures that is of concern in this note.

Let $Y$ be a Banach space and $1 \leq p < \infty$. A vector measure $m : \mathcal{E} \to Y$ is said to have *finite $p$-variation* if there exists $C > 0$ such that for every $n = 1,2,\dots$ and every finite family of pairwise disjoint sets $E_j, j = 1,\dots,n$, the inequality $\sum_{j=1}^{n} \|m(E_j)\|_Y^p \leq C$ holds.

According to the following observation, for any $1 \leq p < \infty$, the property of having finite $L^p(\mu)$-semivariation in $L^p(\mu \otimes \nu)$ is stronger than having finite $p$-variation.

**Proposition 2.2** ([8, Proposition 4.5.5]). *Let $1 \leq p < \infty$ and let $m : \mathcal{E} \to L^p(\nu)$ be a measure. Let $\mathcal{F}$ be a $\sigma$-algebra of subsets of a set $\Lambda$ and $\mu : \mathcal{F} \to [0,\infty)$*

*a finite measure for which $\mathcal{F}$ contains infinitely many, pairwise disjoint non-$\mu$-null sets. If the measure $m$ has finite $L^p(\mu)$-semivariation $\beta_{L^p(\nu)}(m)$ in $L^p(\mu \otimes \nu)$, then $m$ has finite $p$-variation.*

We use this observation to construct, for $1 \le p < 2$, an example of an $L^p(\nu)$-valued measure without finite $L^p(\mu)$-semivariation in $L^p(\mu \otimes \nu)$.

**Example 2.3.** Let $Y$ be an infinite-dimensional Banach space. If $\{\lambda_j\}_{j=1}^\infty$ is a sequence of positive numbers such that $\sum_{j=1}^\infty \lambda_j^2 < \infty$, then there exists an unconditionally summable sequence $\{y_j\}_{j=1}^\infty$ in $Y$ such that $\|y_j\| = \lambda_j$, ([11, Theorem 1.c.2]). Let $1 \le p < 2$. We can choose $\{\lambda_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty \lambda_j^2 < \infty$ and $\sum_{j=1}^\infty \lambda_j^p = \infty$. It follows that there exists an unconditionally summable sequence $\{y_j\}_{j=1}^\infty$ in $Y$ such that $\sum_{j=1}^\infty \|y_j\|^p = \infty$. For $Y = L^p(\nu)$, the vector measure $m : 2^{\mathbb{N}} \to Y$ defined by $m(E) = \sum_{j \in E} y_j$, $E \subseteq \mathbb{N}$, therefore has infinite $p$-variation, and so it has infinite $L^p(\mu)$-semivariation in $L^p(\mu \otimes \nu)$ by Proposition 2.2.

We show in Theorem 3.3 below, that for every $2 < p < \infty$, there is some vector measure $m : \mathcal{E} \to L^p([0,1])$ whose $L^p([0,1])$-semivariation in $L^p([0,1]^2)$ is infinite. Nevertheless, for $2 \le p < \infty$, every vector measure $m : \mathcal{E} \to L^p([0,1])$ does have finite $p$-variation as will be shown in the following proposition, and therefore it is not possible to adapt the arguments in Example 2.3.

**Proposition 2.4.** *Let $2 \le p < \infty$ and let $\nu$ be a $\sigma$-finite measure. Then every vector measure $m : \mathcal{E} \to L^p(\nu)$ has finite $p$-variation.*

Proof: According to [5, Corollary 10.7], every weak $\ell^1$-sequence is a strong $\ell^p$-sequence and there exists $C > 0$ such that

$$\left( \sum_{j=1}^n \|x_j\|_p^p \right)^{\frac{1}{p}} \le C \sup_{\|x'\|_q \le 1} \sum_{j=1}^n |\langle x_j, x' \rangle|,$$

for all $\{x_j\}_{j=1}^n \subset L^p(\nu)$ and all $n = 1, 2, \dots$ . In particular, the bound

$$\left( \sum_{j=1}^n \big\|m(E_j)\big\|_p^p \right)^{\frac{1}{p}} \le C \sup_{\|x'\|_q \le 1} \sum_{j=1}^n \big|\langle m(E_j), x' \rangle\big| \le C\|m\|(\Sigma) < \infty,$$

holds for all finite $\mathcal{E}$-partitions $E_1, \dots, E_n$ of $\Sigma$. $\square$

## 3. Absolutely $p$-summing maps and semivariation

Let $X$ and $Y$ be Banach spaces. Let $1 \le p < \infty$. A continuous linear map $u : X \to Y$ is called *absolutely $p$-summing* if there exists $C > 0$ such that

$$\left(\sum_{j=1}^{k} \|u(x_j)\|_Y^p\right)^{\frac{1}{p}} \le C \sup_{\|x'\|_{X'} \le 1} \left(\sum_{j=1}^{k} |\langle x_j, x'\rangle|^p\right)^{\frac{1}{p}} \tag{3.1}$$

for all $x_j \in X$, $j = 1, \dots, k$ and $k = 1, 2, \dots$ . The set of all absolutely $p$-summing maps from $X$ into $Y$ is denoted by $\Pi_p(X, Y)$. An absolutely summing map (for $p = 1$) is characterised by the fact that it maps unconditionally summable sequences into absolutely summable sequences.

To see how $p$-summing maps relate to semivariation, let us start with the following general result.

**Lemma 3.1.** *Let $\mathcal{M}(2^{\mathbb{N}}, Y)$ denote the vector space of all $Y$-valued vector measures on the $\sigma$-algebra $2^{\mathbb{N}}$. Let $\tau$ be a cross norm on the tensor product $X \otimes Y$ and assume that $\beta_X(m)(\mathbb{N}) < \infty$ for every $m \in \mathcal{M}(2^{\mathbb{N}}, Y)$. Then there exists a constant $C > 0$ such that*

$$\beta_X(m)(\mathbb{N}) \le C\|m\|(\mathbb{N}), \qquad m \in \mathcal{M}(2^{\mathbb{N}}, Y).$$

Proof: It is clear that the vector space $\mathcal{M}(2^{\mathbb{N}}, Y)$ is complete in the norm $\|\cdot\|_{\mathrm{sv}} : m \mapsto \|m\|(\mathbb{N})$. Define another norm by $\|m\|_{\mathrm{bsv}} = \beta_X(m)(\mathbb{N})$ for $m \in \mathcal{M}(2^{\mathbb{N}}, Y)$. By (1.4) this new norm $\|\cdot\|_{\mathrm{bsv}}$ is stronger than $\|\cdot\|_{\mathrm{sv}}$. From this we can deduce that $\mathcal{M}(2^{\mathbb{N}}, Y)$ is complete even in the new norm. Hence, it follows from the open mapping theorem that these two norms $\|\cdot\|_{\mathrm{sv}}$ and $\|\cdot\|_{\mathrm{bsv}}$ are equivalent, which completes the proof. □

Now, let $n = 1, 2, \dots$ and suppose that $\mathcal{F}_n = (f_1, \dots, f_n)$ is a finite ordered subset of $L^p([0,1])$ with $n$ elements. The norm of $L^p([0,1])$ is denoted by $\|\cdot\|_p$. Set $m_{\mathcal{F}_n}(A) = \sum_{j \in A} f_j$ for every subset $A$ of the finite set $\{1, \dots, n\}$. Then, this $L^p([0,1])$-valued vector measure $m_{\mathcal{F}_n}$ satisfies

$$\big(\beta_{L^p}(m_{\mathcal{F}_n})\big)([0,1]) = \sup_{\|x_j\|_p \le 1} \left\|\sum_{j=1}^{n} x_j \otimes f_j\right\|_{L^p([0,1]^2)}. \tag{3.2}$$

Here $x \otimes f$ is the element of $L^p([0,1]^2)$ defined for functions $x$ and $f$ in $L^p([0,1])$ by the function $(s,t) \longmapsto x(s)f(t)$, for almost all $s, t \in [0,1]$. If the $L^p$-semivariation of every $L^p$-valued measure were finite in $L^p([0,1]^2)$, then Lemma 3.1 would imply that there exists $C > 0$ such that

$$\big(\beta_{L^p}(m_{\mathcal{F}_n})\big)([0,1]) \le C \sup_{|a_j| \le 1} \left\|\sum_{j=1}^{n} a_j f_j\right\|_p \tag{3.3}$$

for any finite set $\mathcal{F}_n \subset L^p([0,1])$ and $n = 1, 2, \ldots$ .

Let $\ell_n^1 = \mathbb{C}^n$ with the $\ell^1$-norm and then denote the standard basis vectors by $e_j$, $j = 1, \ldots, n$. For any finite ordered subset $\mathcal{X}_n = (x_1, \ldots, x_n)$ of the closed unit ball of $L^p([0,1])$ with $n$ elements, let $U_{\mathcal{X}_n} : \ell_n^1 \to L^p([0,1])$ denote the linear map such that $U_{\mathcal{X}_n}(e_j) = x_j$ for $j = 1, \ldots, n$.

For any finite ordered subset $\mathcal{F}_n = (f_1, \ldots, f_n)$ of $L^p([0,1])$ with $n$ elements, let $F_{\mathcal{F}_n}(t) = \sum_{k=1}^n f_k(t)e_k \in \ell_n^1$ for almost all $t \in [0,1]$. Then the bound (3.3) can be rewritten as

$$\left(\int_0^1 \left\|U_{\mathcal{X}_n} \circ F_{\mathcal{F}_n}(t)\right\|_p^p dt\right)^{\frac{1}{p}} \le C \sup_{\|\xi\|_{\ell^\infty} \le 1} \left\|\langle F_{\mathcal{F}_n}(\cdot), \xi\rangle\right\|_p \tag{3.4}$$

for any choice of the finite $n$-tuples $\mathcal{X}_n, \mathcal{F}_n$ and $n = 1, 2, \ldots$ .

**Lemma 3.2.** *Suppose that the linear map $u : \ell^1 \to L^p([0,1])$ maps the closed unit ball of $\ell^1$ into the closed unit ball of $L^p([0,1])$. For each $n = 1, 2, \ldots$, let $\mathcal{X}_n = \big(u(e_1), \ldots, u(e_n)\big)$ with $e_j$, $j = 1, 2, \ldots$, being the standard basis vectors of $\ell^1$.*

*Then there exists $C > 0$ (which depends on $u$) such that the bound (3.4) holds for every finite ordered subset $\mathcal{F}_n$ of $L^p([0,1])$ with $n$ elements and every $n = 1, 2, \ldots$ if and only if the map $u$ is absolutely $p$-summing.*

PROOF: Suppose first that (3.4) holds for every finite subset $\mathcal{F}_n$ of $L^p([0,1])$ with $n$ elements and every $n = 1, 2, \ldots$ . Let $N = 1, 2, \ldots$ and let $y_j$, $j = 1, \ldots, N$, be elements of $\ell^1$. For each $n = 1, 2, \ldots$, denote the projection onto the first $n$ coordinates by $P_n : \ell^1 \to \ell^1$ and identify $\ell_n^1$ with the finite-dimensional subspace $P_n(\ell^1)$ of $\ell^1$. Let $E_j$, $j = 1, \ldots, N$, be pairwise disjoint intervals in $[0,1]$ with positive length $|E_j|$, $j = 1, \ldots, N$, such that $\bigcup_{j=1}^N E_j = [0,1]$. Define $F_{\mathcal{F}_n} : [0,1] \to \ell_n^1$ by

$$F_{\mathcal{F}_n}(t) = \sum_{j=1}^N |E_j|^{-1/p} \cdot \chi_{E_j}(t) \cdot P_n(y_j), \qquad t \in [0,1]. \tag{3.5}$$

Here, the $n$-tuple $\mathcal{F}_n = (f_1, \ldots, f_n)$ of elements of $L^p([0,1])$ consists of the functions

$$f_k = \sum_{j=1}^N |E_j|^{-1/p} \cdot \chi_{E_j}(\cdot) \cdot y_{j,k}, \qquad k = 1, \ldots, n,$$

where $y_j = \big(y_{j,k}\big)_{k=1}^\infty \in \ell^1$. For each $\xi \in \ell^\infty$, we have

$$\begin{aligned}\|\langle F_{\mathcal{F}_n}(\cdot), \xi\rangle\|_p^p &= \int_0^1 |\langle F_{\mathcal{F}_n}(t), \xi\rangle|^p \, dt \\ &= \sum_{j=1}^N \left|\big\langle P_n(y_j), \xi\big\rangle\right|^p\end{aligned}$$

and on the other hand,

$$\int_0^1 \|U_{\mathcal{X}_n} \circ F_{\mathcal{F}_n}(t)\|_p^p \, dt = \sum_{j=1}^N \|u(P_n(y_j))\|_p^p ,$$

so that by (3.4), we have

$$\sum_{j=1}^N \big\|u(P_n(y_j))\big\|_p^p \le C^p \sup_{\|\xi\|_{\ell^\infty} \le 1} \sum_{j=1}^N \big| \langle P_n(y_j), \xi \rangle \big|^p. \tag{3.6}$$

For each $j = 1, \ldots, N$, the vectors $P_n(y_j)$ converge to $y_j$ in $\ell^1$ as $n \to \infty$. The continuity of $u$ ensures that we can take $n \to \infty$ in the estimate (3.6) to obtain the bound (3.1) for every $N = 1, 2, \ldots$, so that $u$ is absolutely $p$-summing.

Conversely, suppose that $u : \ell^1 \to L^p([0,1])$ is absolutely $p$-summing. By the Pietsch Domination Theorem [5, Theorem 2.12], there exist $C > 0$ and a weak*-regular Borel probability measure $\mu$ on the closed unit ball $B(\ell^\infty)$ of $\ell^\infty$ such that

$$\|u(x)\|_p \le C \left( \int_{B(\ell^\infty)} \big|\langle x, \xi \rangle\big|^p \, d\mu(\xi) \right)^{\frac{1}{p}}, \qquad x \in \ell^1.$$

Then for any $n$-tuple $\mathcal{F}_n$ of elements of $L^p([0,1])$, the operator $U_{\mathcal{X}_n}$ being the restriction of $u$ to $P_n(\ell^1)$ gives

$$\begin{aligned}
\int_0^1 \|U_{\mathcal{X}_n} \circ F_{\mathcal{F}_n}(t)\|_p^p \, dt &= \int_0^1 \|u \circ F_{\mathcal{F}_n}(t)\|_p^p \, dt \\
&\le C^p \int_0^1 \left( \int_{B(\ell^\infty)} \big|\langle F_{\mathcal{F}_n}(t), \xi \rangle\big|^p \, d\mu(\xi) \right) dt \\
&= C^p \int_{B(\ell^\infty)} \left( \int_0^1 \big|\langle F_{\mathcal{F}_n}(t), \xi \rangle\big|^p \, dt \right) d\mu(\xi) \\
&\le C^p \sup_{\|\xi\|_{\ell^\infty} \le 1} \big\|\langle F_{\mathcal{F}_n}(\cdot), \xi \rangle\big\|_p^p
\end{aligned}$$

by Fubini's theorem. It follows that the bound (3.4) is valid. □

For each $2 < p < \infty$, once we know the existence of a continuous linear map $u : \ell^1 \to L^p([0,1])$ which is not absolutely $p$-summing, then there exists no constant $C$ for which the bound (3.3) holds uniformly for any choice of $\mathcal{F}_n$ and $n = 1, 2, \ldots$ . Then it follows that not every $L^p$-valued measure has finite $L^p$-semivariation in $L^p([0,1]^2)$.

The space $\ell^p$ embeds isometrically onto a closed subspace of $L^p([0,1])$ by choosing pairwise disjoint intervals $E_j$ in $[0,1]$ with positive length $|E_j|$, $j = 1, 2, \ldots$, and mapping $\alpha = (\alpha_j)_{j=1}^\infty \in \ell^p$ to the function $\sum_{j=1}^\infty \alpha_j |E_j|^{-1/p} \chi_{E_j}$. Therefore, if $2 < p < \infty$, the existence of a continuous linear map $u : \ell^1 \to \ell^p$ which is not absolutely $p$-summing also implies that not every $L^p$-valued measure has finite $L^p$-semivariation in $L^p([0,1]^2)$. Moreover, such a measure $m$ is constructed explicitly in the following fashion. The construction is best motivated by the discussion preceding Lemma 3.2.

Let $2 < p < \infty$ and suppose that the continuous linear map $u : \ell^1 \to \ell^p$ is *not* absolutely $p$-summing. Choose a sequence $\{y_j\}_{j=1}^\infty$ in $\ell^1$ such that

$$\sum_{j=1}^\infty |\langle y_j, \xi\rangle|^p < \infty, \quad \text{for every } \ \xi \in \ell^\infty, \tag{3.7}$$

but $\sum_{j=1}^\infty \|u(y_j)\|_{\ell^p}^p = \infty$. Choosing pairwise disjoint intervals $E_j$ in $[0,1]$ with positive length $|E_j|$, $j = 1, 2, \ldots$ , the function $F : [0,1] \to \ell^1$ is defined in the same manner as in (3.5) by

$$F(t) = \sum_{j=1}^\infty |E_j|^{-1/p} \cdot \chi_{E_j}(t) \cdot y_j, \qquad t \in [0,1]. \tag{3.8}$$

Then

$$\int_0^1 |\langle F(t), \xi\rangle|^p dt = \sum_{j=1}^\infty |\langle y_j, \xi\rangle|^p, \tag{3.9}$$

that is, $\langle F(\cdot), \xi\rangle \in L^p([0,1])$ for all $\xi \in \ell^\infty$.

For each $k = 1, 2, \ldots$, the evaluation functional $\delta_k$ at the $k$'th coordinate is an element of $(\ell^1)' = \ell^\infty$, and set $f_k(t) = \langle F(t), \delta_k\rangle$ for each $t \in [0,1]$. Then, $F(t) = \sum_{k=1}^\infty f_k(t) e_k$ pointwise on $[0,1]$. Let $x_k = u(e_k)$ for each $k = 1, 2, \ldots$ . Now $u$ is continuous and linear, so $\sum_{k=1}^\infty f_k(t) x_k = u(F(t)) \in \ell^p$ for all $t \in [0,1]$. Furthermore,

$$\begin{aligned}
\int_0^1 \left\| \sum_{k=1}^\infty f_k(t) x_k \right\|_{\ell^p}^p dt &= \int_0^1 \|u(F(t))\|_{\ell^p}^p \, dt \\
&= \sum_{j=1}^\infty \int_{E_j} \frac{1}{|E_j|} \|u(y_j)\|_{\ell^p}^p \, dt \\
&= \sum_{j=1}^\infty \|u(y_j)\|_{\ell^p}^p = \infty.
\end{aligned}$$

Consequently, Fatou's lemma gives

$$\liminf_{n\to\infty} \int_0^1 \left\| \sum_{k=1}^n f_k(t) x_k \right\|_{\ell^p}^p dt = \infty. \tag{3.10}$$

Next we claim that the sequence $\{f_k\}_{k=1}^\infty$ is unconditionally summable in $L^p([0,1])$. To this end, let $p' = p/(p-1)$ and we shall show that

$$\sup_{\|\phi\|_{p'}\le 1} \sum_{k=1}^\infty \left|\langle f_k, \phi\rangle\right| \le \sup_{\|\xi\|_{\ell^\infty}\le 1} \left\|\langle F(\cdot), \xi\rangle\right\|_p < \infty. \tag{3.11}$$

Fix $n \in \mathbb{N}$. Apply [4, Proposition I.1.11] to the $L^p([0,1])$-valued vector measure $m_n : A \mapsto \sum_{k\in A} f_k$ on $2^{\{1,2,\dots,n\}}$, in order to deduce that

$$\sup_{\|\phi\|_{p'}\le 1} \sum_{k=1}^n \left|\langle f_k, \phi\rangle\right| = \sup_{|\epsilon_k|\le 1} \left\| \sum_{k=1}^n \epsilon_k f_k \right\|_p . \tag{3.12}$$

Given scalars $\epsilon_k$ with $|\epsilon_k| \le 1$ for $k = 1, 2, \dots, n$, since $\| \sum_{k=1}^n \epsilon_k \delta_k \|_{\ell^\infty} \le 1$, it follows that $\left\| \sum_{k=1}^n \epsilon_k f_k \right\|_p \le \sup_{\|\xi\|_{\ell^\infty}\le 1} \left\|\langle F(\cdot), \xi\rangle\right\|_p$. This and (3.12) establish the first inequality of (3.11). Now the linear map $v : \xi \mapsto \big(\langle y_j, \xi\rangle\big)_{j=1}^\infty$ from $\ell^\infty$ into $\ell^p$ is continuous by the closed graph theorem. So, it follows from (3.9) that $\sup_{\|\xi\|_{\ell^\infty}\le 1} \left\|\langle F(\cdot), \xi\rangle\right\|_p = \|v\| < \infty$, which establishes (3.11). In particular, $\sum_{k=1}^\infty \left|\langle f_k, \phi\rangle\right| < \infty$ for every $\phi \in L^{p'}([0,1]) = \big(L^p([0,1])\big)'$. The Bessaga-Pelczynski theorem [3, Theorem V.8] implies that $\{f_k\}_{k=1}^\infty$ is unconditionally summable in $L^p([0,1])$.

We can now define the vector measure $m : 2^{\mathbb{N}} \to L^p([0,1])$ by $m(A) = \sum_{k\in A} f_k$ for every subset $A$ of $\mathbb{N}$. With $\|u\|$ denoting the operator norm of $u$, we have, from the definition of $\beta_{\ell^p}(m)$ and (3.10), that

$$\beta_{\ell^p}(m)([0,1]) \ge \frac{1}{\|u\|} \sup_{n\in\mathbb{N}} \left( \int_0^1 \left\| \sum_{k=1}^n f_k(t) x_k \right\|_{\ell^p}^p dt \right)^{1/p} = \infty$$

because $x_k/\|u\|$ belongs to the unit ball of $\ell^p$. So, the $L^p$-semivariation of $m$ in $L^p([0,1]^2)$ is also infinite.

The same argument will work for any $\sigma$-finite measures $\mu$ and $\nu$ for which $L^p(\mu)$ and $L^p(\nu)$ are infinite-dimensional vector spaces, that is, they have infinitely many essentially distinct non-null sets. We now state the main result of the paper.

**Theorem 3.3.** *Let $2 < p < \infty$ and let $\mu$, $\nu$ be $\sigma$-finite measures for which $L^p(\mu)$ and $L^p(\nu)$ are infinite-dimensional vector spaces. Then there exists a vector measure $m : 2^{\mathbb{N}} \to L^p(\mu)$ with infinite $L^p(\nu)$-semivariation in $L^p(\mu \otimes \nu)$.*

**Corollary 3.4.** *Let $2 < p < \infty$ and let $\mu$, $\nu$ be $\sigma$-finite measures for which $L^p(\mu)$ and $L^p(\nu)$ are infinite-dimensional vector spaces. Then there exists a vector measure $m : 2^{\mathbb{N}} \to L^p(\nu)$ and a bounded function $f : \mathbb{N} \to L^p(\mu)$ such that the sequence $\{f(k) \otimes m(\{k\})\}_{k=1}^{\infty}$ is unbounded in $L^p(\mu \otimes \nu)$.*

The proof of these statements will follow from the preceding discussion once we show that for $2 < p < \infty$, not every continuous linear map from $\ell^1$ into $\ell^p$ is $p$-summing.

## 4. A non-$p$-summing map from $\ell^1$ to $\ell^p$ for $p > 2$

Let $\mathcal{L}(X, Y)$ denote the space of all continuous linear maps from a Banach space $X$ into a Banach space $Y$. Let $2 < p < \infty$ be fixed throughout this section and let $p' = p/(p-1)$ as before.

**Lemma 4.1.** *One has $\Pi_p(\ell^1, \ell^p) \neq \mathcal{L}(\ell^1, \ell^p)$.*

Proof: We shall assume that $\Pi_p(\ell^1, \ell^p) = \mathcal{L}(\ell^1, \ell^p)$ and deduce that $\Pi_p(\ell^\infty, \ell^p) = \mathcal{L}(\ell^\infty, \ell^p)$, so contradicting [10, Theorem 7, $2^0$]. Hence, there exists $u \in \mathcal{L}(\ell^1, \ell^p)$ such that $u$ is not absolutely $p$-summing and the proof of Theorem 3.3 is then complete.

Let $u \in \mathcal{L}(\ell^\infty, \ell^p)$ and let $v \in \mathcal{L}(\ell^{p'}, \ell^\infty)$. Then $u \circ v \in \mathcal{L}(\ell^{p'}, \ell^p)$. Because $v$ is necessarily $\sigma(\ell^{p'}, \ell^p)$-$\sigma(\ell^\infty, \ell^1)$-continuous, there exists $w \in \mathcal{L}(\ell^1, \ell^p)$ such that $v = w'$. By assumption, $w \in \Pi_p(\ell^1, \ell^p)$, and hence, $v' = w'' \in \Pi_p((\ell^\infty)', \ell^p)$ by [5, Proposition 2.19]. Therefore, $(u \circ v)' = v' \circ u' \in \Pi_p(\ell^{p'}, \ell^p)$, and [5, Corollary 5.22] then implies that $u \circ v \in \Pi_p(\ell^{p'}, \ell^p)$, too. Since $v$ can be any continuous linear map from $\ell^{p'}$ to $\ell^\infty$, it follows from [5, Proposition 2.7] that $u \in \Pi_p(\ell^\infty, \ell^p)$. This contradicts [10, Theorem 7, $2^0$], so the assumption that $\Pi_p(\ell^1, \ell^p) = \mathcal{L}(\ell^1, \ell^p)$ must be false. $\square$

Continuous linear maps from $\ell^1$ to $\ell^p$ only just fail to be $p$-summing. We have

**Remark 4.2.** It follows from [2, Corollary 24.6] that $\Pi_q(\ell^1, \ell^p) = \mathcal{L}(\ell^1, \ell^p)$ whenever $q > p > 2$. This observation may be useful for obtaining conditions for a bounded $L^p$-valued function to be $m$-integrable in $L^p$ for $p > 2$.

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