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# Some relative properties on normality and paracompactness, and their absolute embeddings

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*Dedicated to Professor Takao Hoshina on his 60th birthday.*

*Abstract.* Paracompactness (= 2-paracompactness) and normality of a subspace $Y$ in a space $X$ defined by Arhangel'skii and Genedi [4] are fundamental in the study of relative topological properties ([2], [3]). These notions have been investigated by primary using of the notion of weak $C$- or weak $P$-embeddings, which are extension properties of functions defined in [2] or [18]. In fact, Bella and Yaschenko [8] characterized Tychonoff spaces which are normal in every larger Tychonoff space, and this result is essentially implied by their previous result in [8] on a corresponding case of weak $C$-embeddings. In this paper, we introduce notions of 1-normality and 1-collectionwise normality of a subspace $Y$ in a space $X$, which are closely related to 1-paracompactness of $Y$ in $X$. Furthermore, notions of quasi-$C^*$- and quasi-$P$-embeddings are newly defined. Concerning the result of Bella and Yaschenko above, by characterizing absolute cases of quasi-$C^*$- and quasi-$P$-embeddings, we obtain the following result: a Tychonoff space $Y$ is 1-normal (or equivalently, 1-collectionwise normal) in every larger Tychonoff space if and only if $Y$ is normal and almost compact. As another concern, we also prove that a Tychonoff (respectively, regular, Hausdorff) space $Y$ is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space if and only if $Y$ is compact. Finally, we construct a Tychonoff space $X$ and a subspace $Y$ such that $Y$ is 1-paracompact in $X$ but not 1-subparacompact in $X$. This is a negative answer to a question of Qu and Yasui in [25].

*Keywords:* 1-paracompactness of $Y$ in $X$, 2-paracompactness of $Y$ in $X$, 1-collectionwise normality of $Y$ in $X$, 2-collectionwise normality of $Y$ in $X$, 1-normality of $Y$ in $X$, 2-normality of $Y$ in $X$, quasi-$P$-embedding, quasi-$C$-embedding, quasi-$C^*$-embedding, 1-metacompactness of $Y$ in $X$, 1-subparacompactness of $Y$ in $X$

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## 1. Introduction

Throughout this paper all spaces are assumed to be $T_1$ and the symbol $\gamma$ denotes an infinite cardinal.

As central notions in the study of relative topological properties which has been posed by Arhangel'skii and Genedi [4], and also in the subsequent articles [2], [3] by Arhangel'skii, we can mention those on relative normality and relative paracompactness.

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Let $X$ be a space and $Y$ a subspace of $X$. A subspace $Y$ is said to be *normal* (respectively, *strongly normal*) *in* $X$ if for each disjoint closed subsets $F_0$, $F_1$ of $X$ (respectively, of $Y$), there exist disjoint open subsets $G_0$, $G_1$ of $X$ such that $F_i \cap Y \subset G_i$ for $i = 0, 1$. A subspace $Y$ is said to be 1- (respectively, 2-) *paracompact in* $X$ if for every open cover $\mathcal{U}$ of $X$, there exists a collection $\mathcal{V}$ of open subsets of $X$ with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of $Y$. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing $V$. The term "2-paracompact" is often shortened to "paracompact". In the definition of 2-paracompactness of $Y$ in $X$ above, when we replace "open cover of $X$" by "collection of open subsets of $X$ with $Y \subset \bigcup \mathcal{U}$", $Y$ is said to be *Aull-paracompact in* $X$ ([3], [5]). The 1-paracompactness and Aull-paracompactness of $Y$ in $X$ need not imply each other ([5]), but each of them clearly implies 2-paracompactness of $Y$ in $X$.

On the other hand, Hoshina and Yamazaki ([18]) introduced two relative notions called *collectionwise normality* and *strong collectionwise normality of* $Y$ *in* $X$ (see Section 2 for the definition), which imply normality and strong normality of $Y$ in $X$, respectively. It is known that in case $X$ is regular (more strictly, $Y$ is strongly regular in $X$ ([2])), 2-paracompactness of $Y$ in $X$ implies normality of $Y$ in $X$ ([4]); in fact, 2-paracompactness of $Y$ in $X$ implies collectionwise normality of $Y$ in $X$ (see Section 2). It will be also shown that in case $Y$ is Hausdorff in $X$, Aull-paracompactness of $Y$ in $X$ implies strong collectionwise normality of $Y$ in $X$ (Lemma 2.6).

In view of these results, it is suggested to define suitable notions on relative normality and relative collectionwise normality which are closely related to 1-paracompactness. For this purpose, in Section 2 we define the notions of 1-normality and 1-collectionwise normality of $Y$ in $X$, and prove several results including relations between these notions and those mentioned above. In particular, we show that in case of $Y$ being strongly regular in $X$, 1- (respectively, 2-) paracompactness of $Y$ in $X$ implies 1- (respectively, 2-) collectionwise normality of $Y$ in $X$ (Proposition 2.5).

In Section 3, we are concerned to describe spaces that are 1-normal or 1-collectionwise normal in every larger space (that is, in every space containing $Y$ as a subspace). Bella and Yaschenko [8] and Matveev et al. [24] proved a related theorem, where a Tychonoff space $Y$ being normal in every larger Tychonoff (or every regular) space was characterized (Theorem 2.10). This result was implied by another theorem in [8] which describes a Tychonoff space $Y$ being weakly $C$-embedded in every larger Tychonoff space (Theorem 2.8), where weak $C$-embedding is due to Arhangel'skii [2]. In [18], a characterization of weak $C$-embedding was given by extending collections of subsets.

Being motivated by this result, we define a new extension property called *quasi-*$C^*$*-embedding* for a subspace $Y$ of a space $X$. We characterize Tychonoff spaces $Y$

which are quasi-$C^*$-embedded in every larger Tychonoff space (Theorem 3.11). By virtue of this result, we establish a theorem that a Tychonoff space $Y$ is 1-normal in every larger Tychonoff space if and only if $Y$ is normal and almost compact. We also introduce "quasi-$P$-embedding" and give a similar characterization for 1-collectionwise normality of $Y$ in $X$ (Corollary 3.12).

In Section 4, we consider 1-metacompactness defined by Kočinac [20]. Extending [16, Corollary 27] essentially, we obtain the following theorems; a Tychonoff (respectively, regular, Hausdorff) space $Y$ is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space $X$ if and only if $Y$ is compact (Theorems 4.2 and 4.3).

In the final section, on 1-subparacompactness defined by Qu and Yasui [25], we obtain a Tychonoff space $X$ and a subspace $Y$ such that $Y$ is 1-paracompact, but not 1-subparacompact in $X$. This is a negative answer to a question in [25].

Other undefined notations and terminology are used as in [12].

## 2. Preliminaries and 1- or 2- (collectionwise) normality of a subspace in a space

Throughout this paper symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{I}$ denote the set of real numbers, the set of natural numbers and the closed unit interval, respectively.

First let us recall some preliminary notions and facts. Let $Y$ be a subspace of a space $X$. As is known, $Y$ is said to be $C^*$- (respectively, $C$-) *embedded in* $X$ if every bounded real-valued (respectively, real-valued) continuous function on $Y$ is continuously extended over $X$. A subspace $Y$ is said to be $P^\gamma$- (respectively, $P$-) *embedded in* $X$ if every continuous $\gamma$-separable (respectively, continuous) pseudo-metric on $Y$ is extended to a continuous pseudo-metric on $X$ ([1]); a pseudo-metric $d$ on $Y$ is *$\gamma$-separable* if the pseudo-metric space $(Y, d)$ has weight $\leq \gamma$. It is known that $P^\omega$-embedding is equal to $C$-embedding ([1]).

By [2], $Y$ is said to be *weakly $C$-embedded* in $X$ if for every real-valued continuous function $f$ on $Y$ there exists a real-valued function on $X$ which is an extension of $f$ and continuous at each point of $Y$. By [18], $Y$ is said to be *weakly* $P^\gamma$- (respectively, *weakly* $P$-) embedded in $X$ if every continuous $\gamma$-separable (respectively, continuous) pseudo-metric on $Y$ is extended to a pseudo-metric on $X$ which is continuous at each point of $Y \times Y$. Weak $P^\omega$-embedding is equal to weak $C$-embedding ([18]). A space $X$ is *$\gamma$-collectionwise normal* if for every discrete collection $\{E_\alpha \,|\, \alpha < \gamma\}$ of closed subsets there exists a pairwise disjoint collection $\{G_\alpha \,|\, \alpha < \gamma\}$ of open subsets such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Clearly, $X$ is collectionwise normal if $X$ is $\gamma$-collectionwise normal for every $\gamma$.

A subspace $Y$ is said to be *Hausdorff in* $X$ if for every two distinct points $y_1$, $y_2$ of $Y$, there are disjoint open subsets $U_1$, $U_2$ of $X$ such that $y_i \in U_i$ for $i = 0, 1$. A subspace $Y$ is said to be *strongly regular in* $X$ if for each $x \in X$ and each closed subset $F$ of $X$ with $x \notin F$, there exist disjoint open subsets $U$, $V$ of $X$ such that $x \in U$ and $F \cap Y \subset V$.

Following [18], $Y$ is said to be *$\gamma$-collectionwise normal* (respectively, *strongly $\gamma$-collectionwise normal*) *in* $X$ if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets of $X$ (respectively, $Y$), there is a pairwise disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of open subsets of $X$ such that $E_\alpha \cap Y \subset U_\alpha$ (respectively, $E_\alpha \subset U_\alpha$) for every $\alpha < \gamma$. As is easily seen, in case $\gamma = \omega$ it is equivalent to say that $Y$ is normal (respectively, strongly normal) in $X$. When $Y$ is $\gamma$-collectionwise normal (respectively, strongly $\gamma$-collectionwise normal) in $X$ for every $\gamma$, we say $Y$ is *collectionwise normal* (respectively, *strongly collectionwise normal*) *in* $X$; we see that collectionwise normality (respectively, strongly collectionwise normality) of $Y$ in $X$ is equal to being $\alpha - CN$ (respectively, $\gamma - CN$) of $Y$ in the sense of Aull [7].

Let $X_Y$ denote the space obtained from the space $X$, with the topology generated by a subbase $\{U \mid U$ is open in $X$ or $U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and $Y$ is closed in $X_Y$. Moreover, $X$ and $X_Y$ generate the same topology on $Y$ ([12]). As is seen in [2], the space $X_Y$ is often useful in discussing several relative topological properties. It is easy to see that $Y$ is Hausdorff in $X$ if and only if $X_Y$ is Hausdorff. The following results given in [2], [18] are fundamental in the present paper; (a)$\Leftrightarrow$(c)$\Leftrightarrow$(e) in Lemma 2.1 have been already shown in [2].

**Lemma 2.1** ([2], [18]). *For a subspace $Y$ of a space $X$ the following statements are equivalent.*

(a) *$Y$ is strongly normal in $X$.*
(b) *$Y$ is normal in $G$ for every open subset $G$ of $X$ with $Y \subset G$.*
(c) *$X_Y$ is normal.*
(d) *$Y$ is normal in $X_Y$.*
(e) *$Y$ is normal itself and weakly $C$-embedded in $X$.*

**Lemma 2.2** ([18]). *For a subspace $Y$ of a space $X$ the following statements are equivalent.*

(a) *$Y$ is strongly $\gamma$-collectionwise normal in $X$.*
(b) *$Y$ is $\gamma$-collectionwise normal in $G$ for every open subset $G$ of $X$ with $Y \subset G$.*
(c) *$X_Y$ is $\gamma$-collectionwise normal.*
(d) *$Y$ is $\gamma$-collectionwise normal in $X_Y$.*
(e) *$Y$ is $\gamma$-collectionwise normal itself and weakly $P^\gamma$-embedded in $X$.*

We now introduce notions of 1- or 2- (collectionwise) normality of $Y$ in $X$. We say that a subspace $Y$ of a space $X$ is 1- (respectively, 2-) *normal in* $X$ if for each disjoint closed subsets $F_0$, $F_1$ of $X$ there exist open subsets $G_0$, $G_1$ of $X$ such that $F_i \cap Y \subset G_i$ for $i = 0, 1$ and $\{G_0, G_1\}$ is discrete in $X$ (i.e. $\overline{G_0} \cap \overline{G_1} = \emptyset$) (respectively, discrete at each point of $Y$ in $X$ (i.e. $\overline{G_0} \cap \overline{G_1} \cap Y = \emptyset$)).

A subspace $Y$ of a space $X$ is 1-$\gamma$- (respectively, 2-$\gamma$-) *collectionwise normal in* $X$ if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of $X$ there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of $X$ such that $F_\alpha \cap Y \subset G_\alpha$ for each

$\alpha < \gamma$ and $\{G_\alpha \,|\, \alpha < \gamma\}$ is discrete in $X$ (respectively, discrete at each point of $Y$ in $X$). If $Y$ is 1- (respectively, 2-) $\gamma$-collectionwise normal in $X$ for every $\gamma$, $Y$ is said to be 1- (respectively, 2-) *collectionwise normal in* $X$.

In the above definitions of 2-normality and 2-$\gamma$-collectionwise normality of $Y$ in $X$, it is easy to see that both $\{G_1, G_2\}$ and $\{G_\alpha \,|\, \alpha < \gamma\}$ can be taken to be disjoint. Therefore, 2- (collectionwise) normality of $Y$ in $X$ implies (collectionwise) normality of $Y$ in $X$.

As was mentioned in the introduction, we have

**Proposition 2.3.** *Suppose $Y$ is strongly regular in $X$. If $Y$ is 1-paracompact in $X$, then $Y$ is 1-collectionwise normal in $X$.*

PROOF: Assume $Y$ is 1-paracompact in $X$. Let $\{F_\alpha \,|\, \alpha \in \Omega\}$ be a discrete collection of closed subsets of $X$. Since $Y$ is strongly regular in $X$, for each $x \in X$ we can choose an open neighborhood $U_x$ of $x$ in $X$ such that $\left|\{\alpha \in \Omega \,|\, \overline{U_x} \cap Y \cap F_\alpha \neq \emptyset\}\right| \leq 1$. Set $\mathcal{U} = \{U_x \,|\, x \in X\}$. Since $\mathcal{U}$ is an open cover of $X$ and $Y$ is 1-paracompact in $X$, there exists an open cover $\mathcal{V}$ of $X$ such that $\mathcal{V}$ refines $\mathcal{U}$ and is locally finite at each point of $Y$ in $X$. We put for $\alpha \in \Omega$

$$G_\alpha = X \setminus \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}}.$$

Since $Y \cap \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}} = Y \cap (\bigcup\{\overline{V} \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\})$, we have $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha \in \Omega$. Note that for $V \in \mathcal{V}$, $\overline{V} \cap Y \cap F_\alpha \neq \emptyset$ if $G_\alpha \cap V \neq \emptyset$. Hence $\{G_\alpha \,|\, \alpha \in \Omega\}$ is discrete in $X$. Thus, $Y$ is 1-collectionwise normal in $X$. This completes the proof. □

**Proposition 2.4.**[†] *Suppose $Y$ is strongly regular in $X$. If $Y$ is 2-paracompact in $X$, then $Y$ is 2-collectionwise normal in $X$.*

PROOF: Assume $Y$ is 2-paracompact in $X$. Let $\{F_\alpha \,|\, \alpha \in \Omega\}$ and $\mathcal{U}$ be the same as in the proof of Proposition 2.3. Take a collection $\mathcal{V}$ of open subsets of $X$ such that $\mathcal{V}$ partially refines $\mathcal{U}$, $\mathcal{V}$ is locally finite at each point of $Y$ in $X$ and $Y \subset \bigcup \mathcal{V}$. Put

$$G_\alpha = \mathrm{St}(F_\alpha \cap Y, \mathcal{V}) \setminus \overline{\bigcup\{V \in \mathcal{V} \,|\, \overline{V} \cap Y \cap F_\alpha = \emptyset\}}$$

for each $\alpha \in \Omega$. Then $\{G_\alpha \,|\, \alpha \in \Omega\}$ is the desired collection. Hence $Y$ is 2-collectionwise normal in $X$, which completes the proof. □

These propositions and definitions above admit the following result; for brevity "cw-normal" means collectionwise normal. Moreover, the symbols "H" and "SR" mean the assumptions that "$Y$ is Hausdorff in $X$" and "$Y$ is strongly regular in $X$", respectively. "$T_2$" means "Hausdorff".

[†]This was independently proved by E. Grabner, G. Grabner, K. Miyazaki and J. Tartir, assuming that all spaces are Hausdorff.

**Proposition 2.5.** *For a subspace $Y$ of a space $X$ the following implications hold.*

$$
\begin{tikzcd}
X\text{ is paracompact} \arrow[r, "X\text{ is }T_2"] \arrow[d] & X\text{ is cw-normal} \arrow[rr] \arrow[d] & & X\text{ is normal} \arrow[d] & \\
\begin{array}{c} Y\text{ is}\\ \text{1-paracompact}\\ \text{in }X \end{array} \arrow[r, "\mathsf{SR}"] \arrow[d] & \begin{array}{c} Y\text{ is}\\ \text{1-cw-normal}\\ \text{in }X \end{array} \arrow[rr] \arrow[d] & & \begin{array}{c} Y\text{ is}\\ \text{1-normal}\\ \text{in }X \end{array} \arrow[d] & \\
\begin{array}{c} Y\text{ is}\\ \text{2-paracompact}\\ \text{in }X \end{array} \arrow[r, "\mathsf{SR}"] & \begin{array}{c} Y\text{ is}\\ \text{2-cw-normal}\\ \text{in }X \end{array} \arrow[rr] \arrow[dr] & & \begin{array}{c} Y\text{ is}\\ \text{2-normal}\\ \text{in }X \end{array} \arrow[dr] & \\
 & & \begin{array}{c} Y\text{ is}\\ \text{cw-normal}\\ \text{in }X \end{array} \arrow[rr] & & \begin{array}{c} Y\text{ is}\\ \text{normal}\\ \text{in }X \end{array} \\
\begin{array}{c} Y\text{ is}\\ \text{Aull-paracompact}\\ \text{in }X \end{array} \arrow[uu] \arrow[r, "\mathsf{H}"] \arrow[d, leftrightarrow] & \begin{array}{c} Y\text{ is}\\ \text{strongly cw-normal}\\ \text{in }X \end{array} \arrow[uu] \arrow[rr] \arrow[d, leftrightarrow] & & \begin{array}{c} Y\text{ is}\\ \text{strongly normal}\\ \text{in }X \end{array} \arrow[uu] \arrow[d, leftrightarrow] & \\
X_Y\text{ is paracompact} \arrow[r, "X_Y\text{ is }T_2"] \arrow[d] & X_Y\text{ is cw-normal} \arrow[rr] \arrow[d] & & X_Y\text{ is normal} \arrow[d] & \\
Y\text{ is paracompact} \arrow[r, "Y\text{ is }T_2"] & Y\text{ is cw-normal} \arrow[rr] & & Y\text{ is normal} &
\end{tikzcd}
$$

Proof: The implication "$Y$ is 1- (respectively, 2-) paracompact in $X \Rightarrow Y$ is 1- (respectively, 2-) collectionwise normal in $X$" is Proposition 2.3 (respectively, Proposition 2.4).

The facts "$Y$ is Aull-paracompact in $X \Leftrightarrow X_Y$ is paracompact", "$Y$ is strongly collectionwise normal in $X \Leftrightarrow X_Y$ is collectionwise normal" and "$Y$ is strongly normal in $X \Leftrightarrow X_Y$ is normal" were proved in [29], [18] and [2], respectively; the last two equivalences are available to prove immediately "$Y$ is strongly collectionwise normal in $X \Rightarrow Y$ is 2-collectionwise normal in $X$" and "$Y$ is strongly normal in $X \Rightarrow Y$ is 2-normal in $X$", respectively.

For the implication "$Y$ is Aull-paracompact in $X \Rightarrow Y$ is strongly collectionwise normal", see Proposition 2.7 below.

Other implications are obvious. □

The reverse implications in Proposition 2.5 will be discussed later (see Remark 3.5).

Corresponding to Lemmas 2.1 and 2.2 we have the following lemma: (a)⇔(c) was recently obtained in [29], and (c)⇔(e) for $Y$ being Hausdorff in $X$ was proved in [18, Lemma 4.6]. Other equivalences are easy to prove.

**Lemma 2.6.** *For a subspace $Y$ of a space $X$, the following statements from* (a) *to* (d) *are equivalent. If $Y$ is Hausdorff in $X$, these are equivalent to* (e).

(a) *$Y$ is Aull-paracompact in $X$.*
(b) *$Y$ is 2-paracompact in $G$ for every open subset $G$ of $X$ with $Y \subset G$.*
(c) *$X_Y$ is paracompact.*
(d) *$Y$ is 2-paracompact in $X_Y$.*
(e) *$Y$ is paracompact itself and weakly $P$-embedded in $X$.*

Combining Lemmas 2.2 and 2.6, we have

**Proposition 2.7.** *Suppose $Y$ is Hausdorff in $X$. If $Y$ is Aull-paracompact in $X$, then $Y$ is strongly collectionwise normal in $X$.*

A space $X$ is *almost compact* if for every pair of disjoint zero-sets $Z_0$, $Z_1$ in $X$, either $Z_0$ or $Z_1$ is compact. Note that a Tychonoff space $X$ is almost compact if and only if $|\beta X \setminus X| \leq 1$, where $\beta X$ is the Stone-Čech compactification of $X$. As was mentioned in the introduction, Bella and Yaschenko [8] proved the following theorem.

**Theorem 2.8** ([8]). *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *$Y$ is weakly $C$-embedded in every larger Tychonoff* (*or equivalently, regular*) *space.*
(b) *$Y$ is weakly $C$-embedded in every larger Tychonoff* (*or equivalently, regular*) *space containing $Y$ as a closed subspace.*
(c) *$Y$ is either Lindelöf or almost compact.*

Theorem 2.8 was improved to the following.

**Theorem 2.9** ([18]). *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *$Y$ is weakly $P^\gamma$-embedded in every larger Tychonoff* (*or equivalently, regular*) *space.*
(b) *$Y$ is weakly $P^\gamma$-embedded in every larger Tychonoff* (*or equivalently, regular*) *space containing $Y$ as a closed subspace.*
(c) *$Y$ is either Lindelöf or almost compact.*

Using Theorem 2.8, Bella and Yaschenko [8] further proved the following theorem, which was independently proved by Matveev et al. [24]. As was pointed out in [18], with Lemma 2.1, Theorem 2.10 directly follows from Theorem 2.8 for the case when $Y$ is Tychonoff. For the case when $Y$ is regular, since each of conditions from (a) to (d) induces normality of $Y$ itself, Theorem 2.10 also follows from Lemma 2.1 and Theorem 2.8.

**Theorem 2.10** ([8], [24]). *For a Tychonoff* (*respectively, regular*) *space $Y$, the following statements are equivalent.*

(a) *$Y$ is strongly normal in every larger Tychonoff* (*respectively, regular*) *space.*
(b) *$Y$ is normal in every larger Tychonoff* (*respectively, regular*) *space.*
(c) *$Y$ is normal in every larger Tychonoff* (*respectively, regular*) *space containing $Y$ as a closed subspace.*
(d) *$Y$ is either Lindelöf or normal and almost compact.*

Similarly, Theorem 2.9 and Lemma 2.2 imply the following theorem. Notice that spaces satisfying (d) are collectionwise normal.

**Theorem 2.11** ([18]). *For a Tychonoff* (*respectively, regular*) *space $Y$, the following statements are equivalent.*

(a) *$Y$ is strongly collectionwise normal in every larger Tychonoff* (*respectively, regular*) *space.*
(b) *$Y$ is collectionwise normal in every larger Tychonoff* (*respectively, regular*) *space.*
(c) *$Y$ is collectionwise normal in every larger Tychonoff* (*respectively, regular*) *space containing $Y$ as a closed subspace.*
(d) *$Y$ is either Lindelöf or normal and almost compact.*

**Remark 2.12.** Combining Proposition 2.5 and Theorems 2.10, 2.11, it is clear that "strongly normal" (respectively, "strongly collectionwise normal") can be replaced by "2-normal" (respectively, "2-collectionwise normal") in Theorem 2.10 (respectively, Theorem 2.11).

Moreover, Theorem 2.9 and Lemma 2.6 imply the following theorem: (b)$\Leftrightarrow$(c)$\Leftrightarrow$(d) was actually obtained by Arhangel'skii and Genedi [4] and Gordienko [15] (see also [2, Theorems 52 and 53] or [3, Theorem 7.10]), (a)$\Leftrightarrow$(d) was pointed out by Yamazaki in [29]. Note that almost compact paracompact Hausdorff spaces are Lindelöf.

**Theorem 2.13** ([4], [15], [29]). *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *$Y$ is Aull-paracompact in every larger Tychonoff* (*or equivalently, regular*) *space.*
(b) *$Y$ is 2-paracompact in every larger Tychonoff* (*or equivalently, regular*) *space.*

(c) *$Y$ is 2-paracompact in every larger Tychonoff (or equivalently, regular) space containing $Y$ as a closed subspace.*
(d) *$Y$ is Lindelöf.*

**Remark 2.14.** Yamazaki [28] showed that the following are equivalent for a Hausdorff space $Y$:

(a) $Y$ is weakly $C$-embedded (or equivalently, weakly $P$-embedded) in every larger Hausdorff space.
(b) $Y$ is weakly $C$-embedded (or equivalently, weakly $P$-embedded) in every larger Hausdorff space containing $Y$ as a closed subspace.
(c) $Y$ is either compact or every continuous real-valued function on $Y$ is constant.

Hence, applying Lemmas 2.1 and 2.2, if we replace all "Tychonoff" in Theorems 2.10, 2.11 and 2.13 by "Hausdorff", the conditions (d) of each theorems are replaced by "$Y$ is compact" (see also [28], [29]).

## 3. Quasi-C$^*$-, C- and P$^\gamma$-embeddings

Let us introduce extension properties called quasi-$C^*$-, $C$- and $P$-embeddings, which will play basic roles in study of 1- (collectionwise) normality.

Let $X$ be a space and $\mathcal{E} = \{E_\alpha \mid \alpha \in \Omega\}$ a collection of subsets of $X$. Then $\mathcal{E}$ is said to be *uniformly discrete* in $X$ if there exist a collection $\{Z_\alpha \mid \alpha \in \Omega\}$ of zero-sets of $X$ and a discrete collection $\{G_\alpha \mid \alpha \in \Omega\}$ of cozero-sets of $X$ such that $E_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$ ([9]).

Let us now define that a subspace $Y$ of a space $X$ is *quasi-$C^*$-embedded in $X$* if for each pair $Z_0$, $Z_1$ of disjoint zero-sets of $Y$, there exist open subsets $G_0$, $G_1$ of $X$ such that $\{G_0, G_1\}$ is discrete in $X$ and $Z_i \subset G_i$ for $i = 0, 1$.

A subspace $Y$ of a space $X$ is said to be *quasi-$P^\gamma$-embedded in $X$* if for each uniformly discrete collection $\{Z_\alpha \mid \alpha < \gamma\}$ of zero-sets of $Y$, there exists a discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of $X$ such that $Z_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. A subspace $Y$ is *quasi-$P$-embedded in $X$* if $Y$ is quasi-$P^\gamma$-embedded in $X$ for every $\gamma$. In case $\gamma = \omega$, quasi-$P^\omega$-embedding is called *quasi-$C$-embedding*.

Definitions of quasi-$C^*$-embedding and quasi-$P^\gamma$-embedding should be compared with the following results obtained in [9], [18] and [19].

**Lemma 3.1** ([9]). *A subspace $Y$ of a space $X$ is $P^\gamma$-embedded in $X$ if and only if every uniformly discrete collection of subsets of $Y$ of cardinality $\leq \gamma$ is also uniformly discrete in $X$.*

**Lemma 3.2** ([18]). *A subspace $Y$ of a space $X$ is weakly $C$-embedded in $X$ if and only if for each pair $Z_0$, $Z_1$ of disjoint zero-sets of $Y$, there exist disjoint open subsets $G_0$, $G_1$ of $X$ such that $Z_i \subset G_i$ for $i = 0, 1$.*

**Lemma 3.3** ([19]). *A subspace $Y$ of a space $X$ is weakly $P^\gamma$-embedded in $X$ if and only if for each uniformly discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of zero-sets of $Y$ there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of $X$ such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$.*

By Lemmas 3.1, 3.2 and 3.3, we have the following implications.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C^* \arrow[d] \\
\text{quasi-}P \arrow[r] \arrow[d] & \text{quasi-}C \arrow[r] & \text{quasi-}C^* \arrow[d] \\
\text{weak } P \arrow[rr] & & \text{weak } C
\end{tikzcd}$$

The following examples show that none of reverse implications above is true.

**Example 3.4.** (1) Let $X$ be the Tychonoff plank $(\omega_1+1)\times(\omega+1)\setminus\{\langle\omega_1,\omega\rangle\}$ and $Y$ the right edge $\{\langle\omega_1,n\rangle \mid n<\omega\}$ of $X$. Then $Y$ is weakly $P$-embedded, closed, but not quasi-$C^*$-embedded in $X$.

(2) Let $X_i = \big((\omega_1+1)\times(\omega_1+1)\setminus\{\langle\omega_1,\omega_1\rangle\}\big)\times\{i\}$ for $i=0,1$. Let $X$ be the space obtained from $X_0\oplus X_1$ by identifying two points $\langle\omega_1,\alpha,0\rangle$ and $\langle\omega_1,\alpha,1\rangle$ for each $\alpha<\omega_1$, and let $q : X_0\oplus X_1 \to X$ be the resulting quotient map. Let $Y = q(\omega_1\times\{\omega_1\}\times\{0,1\})$.

To prove that $Y$ is quasi-$P$-embedded in $X$, let $\{E_\alpha \mid \alpha\in\Omega\}$ be a uniformly discrete collection of zero-sets of $Y$. Let $E_\alpha^i = \{t\in\omega_1 \mid \langle t,\omega_1,i\rangle \in E_\alpha\}$ for each $\alpha\in\Omega$ and $i=0,1$. Since $Y$ is countably compact, we may assume $\Omega$ is finite. Since $\omega_1$ is normal, for $i=0,1$, there is a discrete collection $\{V_\alpha^i \mid \alpha\in\Omega\}$ of open subsets of $\omega_1$ such that $E_\alpha^i \subset V_\alpha^i$ for every $\alpha\in\Omega$ and $i=0,1$. Set for every $\alpha\in\Omega$,

$$U_\alpha = q(\textstyle\bigcup_{i=0,1}((V_\alpha^i\times(\omega_1+1)\times\{i\})\cap\{\langle\beta_1,\beta_2,i\rangle \mid \beta_1<\beta_2\le\omega_1\})).$$

It is easy to see that $\{U_\alpha \mid \alpha\in\Omega\}$ is a discrete collection of open subsets of $X$ satisfying $E_\alpha\subset U_\alpha$ for every $\alpha\in\Omega$. This shows $Y$ is quasi-$P$-embedded in $X$.

On the other hand, $Y$ is not $C^*$-embedded in $X$.

(3) Let $\Lambda = \beta\mathbb{R}\setminus(\beta\mathbb{N}\setminus\mathbb{N})$. It is well-known that the subspace $\mathbb{N}$ of the space $\Lambda$ is $C^*$-embedded closed, but not $C$-embedded in $\Lambda$ (see [13]). In fact, $\mathbb{N}$ is not quasi-$C$-embedded in $\Lambda$.

(4) Bing's example $G$ [10] gives a normal space $X$ containing a closed discrete subset $F = \{f_\alpha \mid \alpha<\omega_1\}$ which admits no disjoint collection $\{U_\alpha \mid \alpha<\omega_1\}$ of open subsets of $X$ such that $f_\alpha\in U_\alpha$ for every $\alpha<\omega_1$. Hence, $F$ is $C$-embedded, but not weakly $P$-embedded in $X$ ([18]).

**Remark 3.5.** Using Example 3.4, let us return to discuss reverse implications given in Proposition 2.5. First, observe Lemmas 2.1, 2.2 and also Proposition 3.6 below. A non-normal subspace $Y$ of a paracompact Hausdorff space $X$ is 1-paracompact, but not strongly normal in $X$.

In (1) of Example 3.4, $Y$ is Aull-paracompact, but not 1-normal in $X$. Hence, $Y$ is 2-paracompact, but not 1-collectionwise normal in $X$. In (3) of Example 3.4, $\mathbb{N}$ is 1-normal but not 1-$\omega$-collectionwise normal in $\Lambda$. In (4) of Example 3.4, $F$ is 1-normal and strongly normal, but not collectionwise normal in $X$. Examples for other reverse implications are easy to see.

Yamazaki [30] constructed a $T_1$-space $X$ and a subspace $Y$ such that $Y$ is normal in $X$, but not 2-normal in $X$. We do not know similar examples under higher separation axioms. Furthermore, it is unknown whether 2-normality implies 2-$\omega$-collectionwise normality, or whether collectionwise normality implies 2-collectionwise normality.

**Proposition 3.6.** *For a subspace $Y$ of a space $X$, the following statements hold. If $Y$ is closed in $X$, each of them reverses.*

(a) *If $Y$ is itself $\gamma$-collectionwise normal and quasi-$P^{\gamma}$-embedded in $X$, then $Y$ is 1-$\gamma$-collectionwise normal in $X$.*
(b) *If $Y$ is itself normal and quasi-$C^*$-embedded in $X$, then $Y$ is 1-normal in $X$.*

The proof of Proposition 3.6 is easy and omitted. Trivially, the reverse implications need not be true unless $Y$ is closed.

In [6], Aull defined that a subspace $Y$ of a space $X$ is *$\alpha$-paracompact in $X$* if for every collection $\mathcal{U}$ of open subsets of $X$ with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of $X$ such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in $X$. Note that $\alpha$-paracompactness of $Y$ in $X$ implies Aull-paracompactness of $Y$ in $X$, and the converse does not necessarily hold.

Related to $\alpha$-paracompactness, let us recall the following results in [21] and [22, Theorem 1.3].

**Theorem 3.7** ([21])**.** *A Hausdorff* (*respectively, regular, Tychonoff*) *space $Y$ is $\alpha$-paracompact in every larger Hausdorff* (*respectively, regular, Tychonoff*) *space containing $Y$ as a closed subspace if and only if $Y$ is compact.*

**Theorem 3.8** ([22])**.** *For a closed subspace $Y$ of a regular space $X$, $Y$ is 1-paracompact in $X$ if and only if $Y$ is $\alpha$-paracompact in $X$.*

Theorems 3.7 and 3.8 immediately induce the following:

**Corollary 3.9.** *For a Tychonoff* (*respectively, regular*) *space $Y$, the following statements are equivalent.*

(a) *$Y$ is 1-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*

(b) *$Y$ is $\alpha$-paracompact in every larger Tychonoff (respectively, regular) space.*
(c) *$Y$ is 1-paracompact in every larger Tychonoff (respectively, regular) space containing $Y$ as a closed subspace.*
(d) *$Y$ is $\alpha$-paracompact in every larger Tychonoff (respectively, regular) space containing $Y$ as a closed subspace.*
(e) *$Y$ is compact.*

**Lemma 3.10.** *Let $Y$ be an almost compact Tychonoff space. If $Y$ is contained in a regular space $X$, then $Y$ is quasi-$P$-embedded in $X$.*

Proof: Let $\{Z_\alpha \mid \alpha \in \Omega\}$ be a uniformly discrete collection of zero-sets of $Y$. We may assume that $\Omega = \{0,1\}$ and $Z_0$ is compact since $Y$ is almost compact. Since $X$ is regular, there exist open subsets $G$, $H$ of $X$ such that $Z_0 \subset H \subset \overline{H} \subset G$ and $\overline{G} \cap Z_1 = \emptyset$. Then $\{H, X \setminus \overline{G}\}$ is discrete in $X$. □

**Theorem 3.11.** *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *$Y$ is quasi-$P$-embedded in every larger Tychonoff space.*
(a$'$) *$Y$ is quasi-$P$-embedded in every larger Tychonoff space containing $Y$ as a closed subspace.*
(b) *$Y$ is quasi-$C$-embedded in every larger Tychonoff space.*
(b$'$) *$Y$ is quasi-$C$-embedded in every larger Tychonoff space containing $Y$ as a closed subspace.*
(c) *$Y$ is quasi-$C^*$-embedded in every larger Tychonoff space.*
(c$'$) *$Y$ is quasi-$C^*$-embedded in every larger Tychonoff space containing $Y$ as a closed subspace.*
(d) *$Y$ is almost compact.*

*In the above conditions from* (a) *to* (c$'$), *"Tychonoff" can be replaced by "regular".*

Proof: Assume that $Y$ is Tychonoff. Then, the implications (a)$\Rightarrow$(a$'$)$\Rightarrow$(b$'$)$\Rightarrow$(c$'$) and (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(c$'$) are obvious. To complete the proof, we show the implications (c$'$)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a).

(c$'$)$\Rightarrow$(c). Let $X$ be a Tychonoff space containing $Y$ as a subspace, and let $Z_0$, $Z_1$ be disjoint zero-sets of $Y$. Let $D(2^{|X|})$ denote the discrete space of cardinality $2^{|X|}$, and $A(2^{|X|})$ the one-point compactification of $D(2^{|X|})$. Put $W = \big(X \times A(2^{|X|})\big) \setminus \big((X \setminus Y) \times \{\infty\}\big)$, where $\infty$ is the non-isolated point of $A(2^{|X|})$. Then $Y$ is homeomorphic to $Y \times \{\infty\}$ which is a closed subset of $W$. By (c$'$), there exist open subsets $G_0, G_1 \subset W$ with $\{G_0, G_1\}$ discrete such that $Z_i \subset G_i$ for $i = 0,1$. For every $a \in Z_0$ and $b \in Z_1$, take finite subsets $E_a$, $F_b$ of $D(2^{|X|})$ such that

$$\{a\} \times (D(2^{|X|}) \setminus E_a) \subset G_0 \ \text{ and } \ \{b\} \times (D(2^{|X|}) \setminus F_b) \subset G_1.$$

Since the cardinality of $(\bigcup_{a\in Z_0} E_a)\cup(\bigcup_{b\in Z_1} F_b)$ is less than $2^{|X|}$, we can take $\alpha^*\in D(2^{|X|})\setminus(\bigcup_{a\in Z_0} E_a\cup\bigcup_{b\in Z_1} F_b)$ so that $Z_i\times\{\alpha^*\}\subset(X\times\{\alpha^*\})\cap G_i$ for $i=0,1$. Then, the collection $\{(X\times\{\alpha^*\})\cap G_0,\ (X\times\{\alpha^*\})\cap G_1\}$ is discrete in $W$. Define $G'_i=\{x\in X\,|\,\langle x,\alpha^*\rangle\in G_i\}$ for $i=0,1$. Then, $\{G'_0,G'_1\}$ is a discrete collection of open subsets in $X$ and $Z_i\subset G'_i$ for $i=0,1$.

(c)⇒(d). Assume that $Y$ is not almost compact. There exist disjoint zero sets $Z_0$, $Z_1$ in $Y$ such that neither $Z_0$ nor $Z_1$ is compact. Pick $x_i\in\overline{Z_i}^{\beta Y}\setminus Z_i$ for $i=0,1$. Let $X$ be the space obtained from $\beta Y$ by identifying two points $x_0$ and $x_1$, and let $q:\beta Y\to X$ be the resulting quotient map. Then $X$ is compact Hausdorff and $Y$ is obviously a subspace of $X$. By assumption (c), there exist open subsets $G_0$, $G_1$ of $X$ such that $\overline{G_0}^X\cap\overline{G_1}^X=\emptyset$ and $Z_i=q(Z_i)\subset G_i$ for $i=0,1$. We may assume $q(x_0)=q(x_1)\notin\overline{G_0}^X$. Then $X\setminus\overline{G_0}^X$ is an open neighborhood of $q(x_0)$. Hence, $x_0\notin\overline{Z_0}^{\beta Y}$, a contradiction.

(d)⇒(a) follows from Lemma 3.10.

In the case $Y$ is embedded in every larger regular space, it suffices to show (d)⇒(a), and this is obvious. □

By Proposition 3.6 and Theorem 3.11, we have

**Corollary 3.12.** *For a Tychonoff (respectively, regular) space $Y$, the following statements are equivalent.*

(a) *$Y$ is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space.*
(a′) *$Y$ is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space containing $Y$ as a closed subspace.*
(b) *$Y$ is 1-normal in every larger Tychonoff (respectively, regular) space.*
(b′) *$Y$ is 1-normal in every Tychonoff (respectively, regular) space containing $Y$ as a closed subspace.*
(c) *$Y$ is normal and almost compact.*

In Corollary 3.12, (b′)⇔(c) also follows from the following result in [24].

**Theorem 3.13** ([24]). *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *Any disjoint closed subsets of $Y$ are completely separated in every larger Tychonoff space.*
(b) *For any larger Tychonoff space $X$ containing $Y$ as a closed subspace and any disjoint closed subsets $F_0$, $F_1$ of $Y$, there exist disjoint open subsets $G_0$, $G_1$ of $X$ such that $F_i\subset G_i$ for $i=0,1$ and $\overline{G_0}^X\cap\overline{G_1}^X=\emptyset$.*
(c) *$Y$ is normal and almost compact.*

For the Hausdorff case, we have the following.

**Theorem 3.14.** *For a Hausdorff space $Y$, the following statements are equivalent.*

(a) *$Y$ is quasi-$C^*$-embedded in every larger Hausdorff space.*
(b) *$Y$ is quasi-$C^*$-embedded in every larger Hausdorff space containing $Y$ as a closed subspace.*
(c) *Every continuous real-valued function on $Y$ is constant.*

*In* (a) *and* (b), *"quasi-$C^*$-embedded" can be replaced by "quasi-$P$-embedded" or "quasi-$C$-embedded".*

Proof: We only prove (b)⇒(c). Suppose that there exists a continuous real-valued function $f$ on $Y$ such that $f$ is not constant. Then we choose distinct points $y_0$, $y_1$ in $Y$ such that $f(y_0) \neq f(y_1)$.

Let $X_i = \big((\omega_1+1)\times(\omega+1)\setminus\{(\omega_1,\omega)\}\big)\times\{i\}$ and $q_i : X_i \to X_i/(\{\omega_1\}\times\omega\times\{i\})$ be the quotient map obtained by collapsing $\{\omega_1\}\times\omega\times\{i\}$ to one point, for $i = 0, 1$. Let $X$ be the space obtained from $\big(X_0/(\{\omega_1\}\times\omega\times\{0\})\big) \oplus \big(X_1/(\{\omega_1\}\times\omega\times\{1\})\big) \oplus Y$ by identifying two points $\langle\alpha,\omega,0\rangle$ and $\langle\alpha,\omega,1\rangle$ for each $\alpha < \omega_1$ and by identifying two points $q_i(\{\omega_1\}\times\omega\times\{i\})$ and $y_i$ for $i = 0, 1$. Then $X$ is Hausdorff and $Y$ is closed in $X$. Put $Z_i = f^{-1}(f(y_i))$ for $i = 0, 1$. Then $Z_0$, $Z_1$ are disjoint zero-sets of $Y$. By (b), there exist open subsets $G_0, G_1 \subset X$ with $\{G_0, G_1\}$ discrete such that $Z_i \subset G_i$ for $i = 0, 1$. But, by the construction of $X$, $\overline{G_0}^X \cap \overline{G_1}^X \neq \emptyset$, a contradiction. □

By Theorem 3.14 and Proposition 3.6, we have

**Corollary 3.15.** *For a Hausdorff space $Y$, the following statements are equivalent.*

(a) *$Y$ is* 1*-collectionwise normal* (*or equivalently,* 1*-normal*) *in every larger Hausdorff space.*
(b) *$Y$ is* 1*-collectionwise normal* (*or equivalently,* 1*-normal*) *in every larger Hausdorff space containing $Y$ as a closed subspace.*
(c) $|Y| \leq 1$.

Finally we consider a condition under which 2-paracompactness implies 1-paracompactness. We say a subspace $Y$ of a space $X$ is $T_4$- (respectively, $T_3$-) *embedded in* $X$ if for every closed subset $F$ of $X$ disjoint from $Y$ (respectively, $z \in X\setminus Y$), $F$ (respectively, $z$) and $Y$ are separated by disjoint open subsets of $X$. The idea of these notions already appeared in Aull [6].

The following result refines Theorem 3.8; the implication "(b) ⇒ $Y$ is $T_4$-embedded in $X$" is due to Aull [6, Theorem 6]. By using this fact, Lupiañez and Outerelo [22, Lemma 1.2 and Theorem 1.3] essentially proved the implications (a)⇒(c)⇒(b)⇒(c)⇒(a).

**Proposition 3.16.** *Let $X$ be a space and $Y$ a subspace. Then the following statements are equivalent.*

(a) *$Y$ is 1-paracompact in $X$ and $T_3$-embedded in $X$.*
(b) *$Y$ is $\alpha$-paracompact in $X$ and for every $y \in Y$ and every closed subset $F$ of $X$ with $F \cap Y = \emptyset$, there exists an open subset $U$ of $X$ such that $y \in U \subset \overline{U}^X \subset X \setminus F$.*
(c) *$Y$ is 2-paracompact in $X$ and $T_4$-embedded in $X$.*

**Corollary 3.17.** *A closed subspace $Y$ of a regular space $X$ is 1-paracompact in $X$ if and only if $Y$ is 2-paracompact in $X$ and $T_4$-embedded in $X$.*

**Proposition 3.18.** *For a Tychonoff space $Y$, the following statements are equivalent.*

(a) *$Y$ is $T_4$-embedded in every larger Tychonoff space.*
(b) *$Y$ is $T_4$-embedded in every larger Tychonoff space containing $Y$ as a closed subspace.*
(c) *$Y$ is compact.*

*In* (a) *and* (b), *"Tychonoff" can be replaced by "regular".*

Proof: We only prove (b)$\Rightarrow$(c).

Assume that $Y$ is $T_4$-embedded in every larger Tychonoff space containing $Y$ as a closed subspace and $Y$ is not compact. Then pick $x_0 \in \beta Y \setminus Y$. Let $D(2^{|\beta Y|})$ denote the discrete space of cardinality $2^{|\beta Y|}$, and $A(2^{|\beta Y|})$ the one-point compactification of $D(2^{|\beta Y|})$. Put $X = (\beta Y \times A(2^{|\beta Y|})) \setminus ((\beta Y \setminus Y) \times \{\infty\})$, where $\infty$ is the non-isolated point of $A(2^{|\beta Y|})$. Then $Y$ is homeomorphic to $Y \times \{\infty\}$ which is a closed subset of $X$. Put $F = \{x_0\} \times D(2^{|\beta Y|})$. Then $F$ is a closed subset of $X$ disjoint from $Y$. By (b), there exist disjoint open subsets $G_0$, $G_1$ of $X$ such that $Y \subset G_0$ and $F \subset G_1$. For each $y \in Y$, take a finite subset $E_y$ of $D(2^{|\beta Y|})$ such that $\{y\} \times (D(2^{|\beta Y|}) \setminus E_y) \subset G_0$. Since the cardinality of $\bigcup_{y \in Y} E_y$ is not greater than $2^{|\beta Y|}$, we can take $\alpha^* \in D(2^{|\beta Y|}) \setminus \bigcup_{y \in Y} E_y$ so that $Y \times \{\alpha^*\} \subset (\beta Y \times \{\alpha^*\}) \cap G_0$ and $\langle x_0, \alpha^* \rangle \in (\beta Y \times \{\alpha^*\}) \cap G_1$. Then, $(\beta Y \times \{\alpha^*\}) \cap G_0$ and $(\beta Y \times \{\alpha^*\}) \cap G_1$ are disjoint. Define $G_i' = \{x \in \beta Y \mid \langle x, \alpha^* \rangle \in G_i\}$ for $i = 0, 1$. Then, $G_0'$, $G_1'$ are disjoint open subsets of $\beta Y$ with $Y \subset G_0'$ and $x_0 \in G_1'$. This is a contradiction. □

Theorem 2.13, Propositions 3.17 and 3.18 give an alternative proof of Corollary 3.9.

In case $Y$ is Hausdorff, we have

**Proposition 3.19.** *For a Hausdorff space $Y$, the following statements are equivalent.*

(a) *$Y$ is $T_4$-embedded in every larger Hausdorff space.*

(b) *$Y$ is $T_4$-embedded in every larger Hausdorff space containing $Y$ as a closed subspace.*
(c) $Y = \emptyset$.

Proof: We only prove (b)$\Rightarrow$(c). Assume $Y \neq \emptyset$. Take $y \in Y$ and let $T = (\omega_1+1)\times(\omega+1)\setminus\{(\omega_1,\omega)\}$. Consider the quotient space $Z$ obtained from $T \oplus Y$ by identifying the right edge $\{\omega_1\}\times\omega$ of $T$ and $y$ to one point. Let $q : T\oplus Y \to Z$ be the natural quotient map. Then, $Z$ is Hausdorff and $Y$ is homeomorphic to $q(Y)$ which is a closed subset of $Z$. But, $Y$ is not $T_4$-embedded in $Z$. Hence (b) implies (c). □

A similar proof provides the following proposition, and this should be compared with Theorem 3.7 and Corollary 3.9.

**Proposition 3.20.** *For a Hausdorff space $Y$, the following statements are equivalent.*

(a) *$Y$ is 1-paracompact in every larger Hausdorff space.*
(b) *$Y$ is 1-paracompact in every larger Hausdorff space containing $Y$ as a closed subspace.*
(c) $Y = \emptyset$.

## 4. On 1-metacompactness of a subspace in a space

In this section, we prove an extension of Corollary 3.9.

A subspace $Y$ of a space $X$ is said to be 1-*metacompact in* $X$ if for every open cover $\mathcal{U}$ of $X$, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is point-finite at every $y \in Y$ ([20]).

A space $X$ satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open subsets of $X$ is finite (see [23], for example). Recall that a Tychonoff space $X$ is pseudocompact if and only if $X$ satisfies the DFCC. It is also known that a Tychonoff space $X$ is compact if and only if $X$ is pseudocompact and metacompact ([26], [27]). Furthermore, a regular space $X$ is compact if and only if $X$ satisfies the DFCC and is metacompact ([26]).

According to [2], in [4], Arhangel'skii and Genedi remarked the following fact: *Let $Y$ be a countable dense subset of a regular space $X$. Then $Y$ is* 1*-metacompact (or equivalently,* 1*-paracompact) in $X$ if and only if $X$ is Lindelöf.*

The following result is based on this fact.

**Lemma 4.1.** *Let $Z$ be an arbitrary separable space and $Y$ an arbitrary non-DFCC space.*

*Let $D = \{d_n \,|\, n \in \mathbb{N}\}$ be a countable dense subset of $Z$, $\{U_n \,|\, n \in \mathbb{N}\}$ a countable discrete collection of non-empty open subsets of $Y$ and $A = \{y_n \,|\, n \in \mathbb{N}\}$*

*a countable subset of $Y$ such that $y_n \in U_n$ for each $n \in \mathbb{N}$. Let $X$ be the quotient space obtained from $Y \oplus Z$ by identifying $y_n$ with $d_n$ for each $n \in \mathbb{N}$.*

*If $Y$ is 1-metacompact in $X$, then $Z$ is Lindelöf.*

*Moreover, if $Y$ and $Z$ are Tychonoff (respectively, regular), then $X$ is also Tychonoff (respectively, regular).*

Proof: Let $q : Y \oplus Z \to X$ be the quotient map. Since $Y$ is homeomorphic to $q(Y)$, $Y$ is viewed as a subspace of $X$. Put $p_n = q(y_n) = q(d_n)$ for each $n \in \mathbb{N}$, and let $P = \{p_n \,|\, n \in \mathbb{N}\}$. Note that $A$ is a closed discrete subset of $Y$.

Suppose $Y$ is 1-metacompact in $X$. Let $\mathcal{G}$ be an open cover of $Z$. For each $G \in \mathcal{G}$, put $V_G = q\left(G \cup \bigcup\{U_n \,|\, d_n \in G\}\right)$. Define $\mathcal{V} = \{V_G \,|\, G \in \mathcal{G}\} \cup \{q(Y \setminus A)\}$. Then $\mathcal{V}$ is an open cover of $X$. Since $Y$ is 1-metacompact in $X$, $\mathcal{V}$ has an open refinement $\mathcal{W}$ which is point-finite at every $x \in q(Y)$. For each $W \in \mathcal{W}$ with $W \cap q(Z) \neq \emptyset$, $W$ must contain some element of $P$ since $D$ is dense in $Z$. Then, $\mathcal{U}$ has a countable subcover because $\mathcal{W}$ is point-finite at $p_n$ for every $n \in \mathbb{N}$. Hence, $Z$ is Lindelöf.

Let us prove that if $Y$ and $Z$ are Tychonoff, then $X$ is also Tychonoff. Assume $Y$ and $Z$ are Tychonoff. Let $x \in X$ and $U$ be a neighborhood of $x$ in $X$.

If $x \in q(Z)$, pick $z \in q^{-1}(x) \cap Z$. Then, there exists a neighborhood $W$ of $z$ in $Z$ such that $q(W) \subset U$. For every $m \in \mathbb{N}$ with $d_m \in W$, there exists a neighborhood $V_m$ of $y_m$ in $Y$ such that $q(V_m) \subset U$. Here, we may assume that $V_m \subset U_m$. Define $G = q\left(\bigcup\{V_m \,|\, d_m \in W\} \cup W\right)$. Since $Z$ is Tychonoff, there exists a continuous function $f : Z \to \mathbb{I}$ such that $f(z) = 1$ and $f(Z \setminus W) \subset \{0\}$. Because $Y$ is also Tychonoff, for each $m \in \mathbb{N}$ with $d_m \in W$, there exists a continuous function $g_m : Y \to \mathbb{I}$ such that $g_m(y_m) = f(d_m)$ and $g_m(Y \setminus V_m) \subset \{0\}$. Then, define a function $h : X \to \mathbb{I}$ by

$$h(a) = \begin{cases} \sum_{d_m \in W} g_m(q^{-1}(a)), & \text{if } a \in q(Y) \setminus P, \\ f(q^{-1}(a)), & \text{if } a \in q(Z) \setminus P, \\ f(d_n), & \text{if } a = p_n \in P. \end{cases}$$

Then, $h$ is continuous and $h(x) = 1$, $h(X \setminus G) \subset \{0\}$.

In case $x \notin q(Z)$, that is $x \in q(Y) \setminus P$, since $A$ is closed in $Y$, it is easy to find a continuous function $g : X \to \mathbb{I}$ such that $g(x) = 1$ and $g(X \setminus U) \subset \{0\}$. Therefore, $X$ is Tychonoff.

If $Y$ and $Z$ are regular, we can similarly prove that $X$ is regular. □

**Theorem 4.2.** *A Tychonoff (respectively, regular) space $Y$ is 1-metacompact in every larger Tychonoff (respectively, regular) space if and only if $Y$ is compact.*

Proof: The "if" part is obvious.

To prove the "only if" part, assume that $Y$ is 1-metacompact in every larger Tychonoff (respectively, regular) space but not compact. Since $Y$ itself is metacompact, $Y$ does not satisfy the DFCC. Thus, there exists a countable discrete collection $\{U_n \mid n \in \mathbb{N}\}$ of non-empty open subsets of $Y$. For each $n \in \mathbb{N}$, take $y_n \in U_n$, and let $A = \{y_n \mid n \in \mathbb{N}\}$. Then $A$ is a discrete closed subset of $Y$.

Let $Z$ be an arbitrary separable, non-Lindelöf Tychonoff (respectively, regular) space, for example, $Z$ is the $\Psi$-space in [13, 5I]. Let $D = \{d_n \mid n \in \mathbb{N}\}$ be a countable dense subset of $Z$. Now, let $X$ be the space constructed in Lemma 4.1. Then, $X$ is Tychonoff (respectively, regular) and $Y$ is not 1-metacompact in $X$ by Lemma 4.1. □

Theorem 4.2 extends the following result due to E. Grabner et al. [16]: *A normal space $Y$ is* 1*-metacompact in every larger regular space if and only if $Y$ is compact.*

To prove the Hausdorff case, we need the following well-known fact: *A Hausdorff space $X$ is compact if and only if $X$ is countably compact and metacompact.*

**Theorem 4.3.** *Let $Y$ be a Hausdorff space. Then $Y$ is* 1*-metacompact in every larger Hausdorff space if and only if $Y$ is compact.*

Proof: The "if" part is obvious.

To prove "only if" part, assume that $Y$ is 1-metacompact in every larger Hausdorff space but not compact. Then, $Y$ is not countably compact. Hence, there exists a countable discrete closed subset $A = \{a_n \mid n \in \mathbb{N}\}$ of $Y$. Let $Z$ be an arbitrary separable, non-Lindelöf Hausdorff space with a countable dense subset $D = \{d_n \mid n \in \mathbb{N}\}$. Define $X$ be the quotient space obtained from $Y \oplus Z$ by identifying $a_n$ with $d_n$ for each $n \in \mathbb{N}$.

Then, it is easy to see that $X$ is Hausdorff. Define $U_n = (Y \setminus A) \cup \{y_n\}$ for each $n \in \mathbb{N}$. Then, similarly to the proof of Lemma 4.1, $Y$ is not 1-metacompact in $X$. □

**Remark 4.4.** In contrast with (a)$\Leftrightarrow$(c) in Corollary 3.9, consider an analogous statement that a Tychonoff (respectively, regular, Hausdorff) space $Y$ is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space containing $Y$ as a closed subspace. This means, however, nothing but that $Y$ is metacompact.

## 5. On 1-subparacompactness of a subspace in a space

It was defined in [25] that a subspace $Y$ of a space $X$ is 1-*subparacompact in* $X$ if for every open cover $\mathcal{U}$ of $X$, there exists a $\sigma$-discrete collection $\mathcal{P}$ of closed subsets of $X$ with $Y \subset \bigcup \mathcal{P}$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$.

In [25], Qu and Yasui asked a question as follows: *Let $X$ be a regular space and $Y$ a subspace of $X$. Is it true that if $Y$ is* 1*-paracompact in $X$, then $Y$ is* 1*-subparacompact in $X$?*

The following theorem is a negative answer to this question.

**Theorem 5.1.** *There exist a Tychonoff space $X$ and a subspace $Y$ of $X$ such that $Y$ is* 1*-paracompact in $X$ but not* 1*-subparacompact in $X$.*

Proof: Let $X$ be the set $(\omega_2+1)\times(\omega_1+1)\setminus\{\langle\omega_2,\omega_1\rangle\}$. For $\alpha\in\omega_1$ and $\beta\in\omega_2$, define $G_\alpha=(\omega_2+1)\times\{\alpha\}$ and $H_\beta=\{\beta\}\times(\omega_1+1)$, respectively. Define a topology on $X$ as follows. For $\alpha\in\omega_1$, a neighborhood base at $\langle\omega_2,\alpha\rangle$ is the family of all sets of the form $G_\alpha\setminus E$, where $E$ is a finite subset of $\omega_2\times\{\alpha\}$. For $\beta\in\omega_2$, a neighborhood base at $\langle\beta,\omega_1\rangle$ is the family of all sets of the form $H_\beta\setminus F$, where $F$ is a finite subset of $\{\beta\}\times\omega_1$. All other points of $X$ are isolated in $X$. The construction of $X$ is based on a example in [11]. Let $Y=\omega_2\times\omega_1\subset X$.

Let us prove that $Y$ is 1-paracompact in $X$. To prove this, let $\mathcal{U}$ be any open cover of $X$. For every $\alpha\in\omega_1$, take a finite subset $E_\alpha$ of $\omega_2\times\{\alpha\}$ such that $\langle\omega_2,\alpha\rangle\in G_\alpha\setminus E_\alpha$ and $G_\alpha\setminus E_\alpha$ is contained some $U\in\mathcal{U}$. Similarly, for every $\beta\in\omega_2$, take a finite subset $F_\beta$ of $\{\beta\}\times\omega_1$ such that $\langle\beta,\omega_1\rangle\in H_\beta\setminus F_\beta$ and $H_\beta\setminus F_\beta$ is contained some $U\in\mathcal{U}$. Then, the collection

$$\{G_\alpha\setminus E_\alpha\,|\,\alpha\in\omega_1\}\cup\{H_\beta\setminus F_\beta\,|\,\beta\in\omega_2\}\cup\{\{\langle\beta,\alpha\rangle\}\,|\,\alpha\in\omega_1,\beta\in\omega_2\}$$

is an open refinement of $\mathcal{U}$ which is locally finite at every $y\in Y$ in $X$. Hence, $Y$ is 1-paracompact in $X$.

Now, we shall show that $Y$ is not 1-subparacompact in $X$. Let $\mathcal{U}=\{G_\alpha\,|\,\alpha\in\omega_1\}\cup\{H_\beta\,|\,\beta\in\omega_2\}$. Then, $\mathcal{U}$ is an open cover of $X$. Assume that there is a collection $\mathcal{P}=\bigcup_{n\in\mathbb{N}}\mathcal{P}_n$ of closed subsets of $X$ with $\bigcup\mathcal{P}\supset Y$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{P}_n$ is discrete in $X$ for every $n\in\mathbb{N}$. For any $n$, let

$$\mathcal{P}_n^0=\{P\in\mathcal{P}_n\,|\,(\exists\beta\in\omega_2)\ P\subset H_\beta\},\ \mathcal{P}_n^1=\{P\in\mathcal{P}_n\,|\,(\exists\alpha\in\omega_1)\ P\subset G_\alpha\}\setminus\mathcal{P}_n^0.$$

Note that $\mathcal{P}_n=\mathcal{P}_n^0\cup\mathcal{P}_n^1$. For each $n\in\mathbb{N}$ and each $\alpha\in\omega_1$, $\{P\in\mathcal{P}_n^0\,|\,P\cap G_\alpha\neq\emptyset\}$ is finite. Since $|P\cap G_\alpha|\le 1$ for each $P\in\mathcal{P}_n^0$, $(\bigcup\mathcal{P}_n^0)\cap G_\alpha$ is finite. Therefore, $(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^0)\cap G_\alpha$ is countable, and hence $V_\alpha^0=\{\lambda\in\omega_2\,|\,\langle\lambda,\alpha\rangle\in(\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^0)\cap G_\alpha\}$ is also countable. Therefore, we can take $\beta^*\in\omega_2\setminus\bigcup_{\alpha\in\omega_1}V_\alpha^0$. Since $H_{\beta^*}\cap Y\subset\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{P}_n^1$, we can take $n\in\mathbb{N}$ so that $(\bigcup\mathcal{P}_n^1)\cap H_{\beta^*}$ is uncountable.

On the other hand, $\{P\in\mathcal{P}_n^1\,|\,P\cap H_{\beta^*}\neq\emptyset\}$ is finite and $|P\cap H_{\beta^*}|\le 1$ for each $P\in\mathcal{P}_n^1$. This is a contradiction. Hence, $Y$ is not 1-subparacompact in $X$. This completes the proof. □

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