Weak extent in normal spaces

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Abstract. If X is a space, then the *weak extent* $\mathrm{we}(X)$ of X is the cardinal $\min\{\alpha :$ If $\mathcal{U}$ is an open cover of X, then there exists $A \subseteq X$ such that $|A| = \alpha$ and $\mathrm{St}(A,\mathcal{U}) = X\}$. In this note, we show that if X is a normal space such that $|X| = \mathfrak{c}$ and $\mathrm{we}(X) = \omega$, then X does not have a closed discrete subset of cardinality $\mathfrak{c}$. We show that this result cannot be strengthened in ZFC to get that the extent of X is smaller than $\mathfrak{c}$, even if the condition that $\mathrm{we}(X) = \omega$ is replaced by the stronger condition that X is separable.

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In [M], the author showed that the extent of a Tychonoff space X having countable weak extent could be arbitrarily large, but that in normal spaces, the extent is at most $\mathfrak{c}$. (Definitions are given below.) Since then, alternative proofs have been given by Fleissner (unpublished) and G. Kozma ([K]). This result suggests the question of whether in a normal space, the extent could, in fact, equal $2^{\mathrm{we}(X)}$. In this note, we show that if a normal space has cardinality $\mathfrak{c}$ and weak extent ω, then it has no closed, discrete subset of size $\mathfrak{c}$ $(= 2^{\mathrm{we}(X)})$. On the other hand, even in this case, the extent can consistently equal $\mathfrak{c}$. We also show that there is a Tychonoff star-Lindelöf space X having a closed discrete subset D of cardinality $\mathfrak{c}$ such that every two disjoint subsets of D can be separated by open subsets of X.

Suppose X is a Hausdorff topological space. If $\mathcal{U}$ is an open cover of X and $A \subseteq X$, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$. The *extent* $\mathrm{e}(X)$ is $\sup\{|D| : D$ is a closed discrete susbet of $X\}$. The *weak extent* $\mathrm{we}(X)$ is $\min\{\alpha :$ If $\mathcal{U}$ is an open cover of X, then there exists $A \subseteq X$ such that $|A| = \alpha$ and $\mathrm{St}(A,\mathcal{U}) = X\}$. If $\mathrm{we}(X) = \omega$, then X is *star-Lindelöf*. (The weak extent is also called the *star-Lindelöf number*, or the **Lindelöf number*, and may also be denoted $\mathrm{st\text{-}l}(X)$ or $\mathrm{l}^*(X)$.) (See [H], [I], or [M].) It is clear that every separable space and every Lindelöf space is star-Lindelöf. Also, $\mathrm{we}(X) \leq \mathrm{e}(X)$. Since a countably compact Hausdorff space has countable extent, it follows that a countably compact Hausdorff space is star-Lindelöf.

Proposition 1. *Suppose that X is a normal space such that $|X|^{\mathrm{we}(X)} = |X|$. Then X has no closed discrete subset of size $|X|$.*

Proof: Denote $|X|$ by κ. Suppose that $D = \{d_\alpha : \alpha < \kappa\}$ were a closed discrete subset of X such that $|D| = \kappa$. List the subsets of X of size $\mathrm{we}(X)$ as $\{B_\lambda : \lambda < \kappa\}$. Since $2^\kappa > \kappa = \kappa^{\mathrm{we}(X)} \geq 2^{\mathrm{we}(X)}$, by Jones' Lemma, the normal space $\mathrm{Cl}_X B_0$ cannot have a closed discrete subset of size κ. Let α_0 be the smallest index such that $d_{\alpha_0} \notin \mathrm{Cl}_X B_0$ and let U_0 be an open neighborhood of d_{α_0} which does not intersect $B_0 \cup (D \setminus \{d_{\alpha_0}\})$. Now suppose that $\lambda < \kappa$ and that indices α_δ and open sets U_δ have been defined for each $\delta < \lambda$. By Jones' Lemma the normal space $\mathrm{Cl}_X B_\lambda$ does not have a closed discrete subset of cardinality κ, and since the closed discrete set $\{d_{\alpha_\delta} : \delta < \lambda\}$ has cardinality smaller than κ, there exists $\alpha_\lambda < \kappa$ such that $d_{\alpha_\lambda} \notin \mathrm{Cl}_X(B_\lambda) \cup \{d_{\alpha_\delta} : \delta < \lambda\}$, and we may assume that α_λ is the smallest such index. Let U_λ be an open neighborhood of d_{α_λ} which does not intersect $B_\lambda \cup (D \setminus \{d_{\alpha_\lambda}\})$. Let $D' = \{d_{\alpha_\lambda} : \lambda < \kappa\}$. Let $\mathcal{U} = \{U_\lambda : \lambda < \kappa\} \cup \{(X \setminus D')\}$. Then $\mathcal{U}$ is an open cover of X so by assumption, there exists $\lambda < \kappa$ such that $\mathrm{St}(B_\lambda, \mathcal{U}) = X$. But U_λ is the only element of $\mathcal{U}$ containing d_{α_λ}, and this open set does not intersect B_λ, a contradiction. $\square$

Corollary 2. *If X is a normal star-Lindelöf space such that $|X| \leq \mathfrak{c}$, then X does not have a closed discrete subset of cardinality $\mathfrak{c}$.*

Note that because of the distinction between sup and max, Corollary 2 does not assert that the extent of a star-Lindelöf normal space of size $\mathfrak{c}$ is smaller than $\mathfrak{c}$. In fact, we show below that this need not be true.

The following result uses the argument in [T2]. (See [T1, Theorem 2.2]).

Proposition 3. *Let Y be a closed discrete subspace of a normal space X. Denote $k_1 = \sup\{|Z| : Z \subset Y,\ Z$ can be separated by open sets of $X\}$ and $k_2 = \max\{|Y|, \sup\{\chi(y, X) : y \in Y\}\}$. Then $(k_2)^{k_1} \geq 2^{|Y|}$.*

Proof: For each $y \in Y$ pick a neighborhood base $\mathcal{B}_y$ of cardinality $\chi(y, X)$. Using normality, for each $Z \subset Y$ pick an open subset U_Z of X such that $U_Z \supseteq Z$ and $\overline{U_Z} \cap (Y \setminus Z) = \emptyset$. Further, for each $Z \subset Y$ pick a maximal disjoint subfamily $\mathcal{S}_Z$ of $\{U \in \cup\{\mathcal{B}_y : y \in Z\} : U \subseteq U_Z\}$. Then the families $\mathcal{S}_Z$ are distinct for distinct Z, and each $\mathcal{S}_Z$ has cardinality at most k_1. Thus $Z \to \mathcal{S}_Z$ is an injection from the family of all subsets of Y to the family of all subsets of cardinality $\leq k_1$ of the set $\bigcup\{\mathcal{B}_y : y \in Y\}$ which has cardinality $\leq k_2$. $\square$

Corollary 4. *If a normal star-Lindelöf space of character at most $\mathfrak{c}$ contains an infinite closed discrete subset of cardinality κ, then $2^\kappa = \mathfrak{c}$.*

Proof: Suppose that X is a normal star-Lindelöf space of character at most $\mathfrak{c}$ and Y is an infinite closed discrete subset of X such that $|Y| = \kappa$. Let k_1 and k_2 be the cardinal numbers defined in Proposition 3. In [M] it is proved that the extent of a normal star-Lindelöf space is at most $\mathfrak{c}$. Therefore, $\kappa \leq \mathfrak{c}$, so $k_2 \leq \mathfrak{c}$. Furthermore, $k_1 = \omega$, because if there were an uncountable closed discrete subset D of X whose elements were separated by open sets $\{U_d : d \in D\}$, then $\{U_d : d \in D\} \cup \{X \setminus D\}$ would be an open cover witnessing that X is not star-Lindelöf. Therefore, by Proposition 3, $2^\kappa = 2^{|Y|} \leq \mathfrak{c}^\omega = \mathfrak{c}$. $\square$

The next corollary follows from Proposition 3, and its proof is similar to that of Corollary 4.

Corollary 5. *Suppose that X is normal. Then* $(\mathrm{e}(X) \cdot \chi(X))^{\mathrm{we}(X)} \geq \sup\{2^\lambda : \lambda < \mathrm{e}(X)\}$.

Remark 6. The result in Corollary 5 cannot be strengthened to give the inequality $(\mathrm{e}(X) \cdot \chi(X))^{\mathrm{we}(X)} \geq 2^{\mathrm{e}(X)}$. An example is provided by the space constructed in Lemma 8 below.

We will use the following observation.

Lemma 7. *If Y is normal and the weight of βY is at most $\mathfrak{c}$, then Y can be embedded as a closed subset of a normal separable space.*

Proof: Since the weight of βY is at most $\mathfrak{c}$, βY is homeomorphic to a subspace of $[0,1]^{\mathfrak{c}}$, which is separable. Let $B\omega$ be a compactification of ω such that $B\omega \setminus \omega$ is $[0,1]^{\mathfrak{c}}$, and let $X = \omega \cup Y$ with the subspace topology from $B\omega$. Then X is clearly separable, and Y is a closed subspace of X. It remains to show that X is normal. Suppose that H and K are disjoint closed subsets of X. Since elements of ω are isolated, we may assume that H and K are subsets of Y. Since Y is normal, there exists a continuous function $f\colon Y \to [0,1]$ such that $f \restriction H \equiv 0$ and $f \restriction K \equiv 1$. Y is C*-embedded in βY which is compact and, therefore, C*-embedded in $B\omega$. Therefore, f extends to a continuous function $F\colon X \to [0,1]$. Then F completely separates H and K. $\square$

Lemma 8. *The following statements are equivalent:*

(1) *there exists a separable normal space X such that* $\mathrm{e}(X) = |X| = \mathfrak{c}$;
(2) *there exists a separable normal space X such that* $\mathrm{e}(X) = \mathfrak{c}$;
(3) $\mathfrak{c}$ *is a limit cardinal satisfying* $2^{<\mathfrak{c}} = \mathfrak{c}$.

Proof: (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Suppose that X is a separable normal space such that $\mathrm{e}(X) = \mathfrak{c}$. If $\mathfrak{c}$ were not a limit cardinal, then X would have a closed discrete subset of size $\mathfrak{c}$, contradicting Jones' Lemma. If $2^{<\mathfrak{c}}$ were larger than $\mathfrak{c}$, there would be $\kappa < \mathfrak{c}$ such that $2^\kappa > \mathfrak{c}$, and X would have a closed discrete subset of size κ, again violating Jones' Lemma.

(iii) $\Rightarrow$ (i): Assume $\mathfrak{c}$ is a limit cardinal satisfying $2^{<\mathfrak{c}} = \mathfrak{c}$. We construct a normal separable space X satisfying $|X| = \mathfrak{c}$ such that for each $\kappa < \mathfrak{c}$, X has a closed discrete subset of cardinality κ. Let S be a set such that $|S| = \mathfrak{c}$ and let Y be the set $S \cup \{\infty\}$ topologized so that elements of S are isolated and neighborhoods of ∞ have the form $\{\infty\} \cup (S \setminus A)$ where $|A| < \mathfrak{c}$. It follows from König's Lemma that Y is a P-space. Since Y has only one non-isolated point, it is normal. Furthermore, the assumption on $\mathfrak{c}$ implies that there are only $\mathfrak{c}$ continuous real-valued functions on Y, so βY has weight $\mathfrak{c}$. Now Lemma 7 applies. $\square$

We wish to thank Bill Fleissner for the following observations about the consistency status of the conditions in Lemma 8. If the existence of an inaccessible cardinal is consistent, then there exists a model of GCH in which there is a strongly inaccessible cardinal α. If α Cohen reals are added to such a model, then the condition in Lemma 8(iii) is satisfied. On the other hand, if the condition in Lemma 8(iii) is satisfied, then it follows from König's Lemma that $\mathfrak{c}$ is inaccessible, and, hence, the existence of a strongly inaccessible cardinal is consistent. Therefore, the consistency of the conditions in Lemma 8 are equivalent to the consistency of the existence of an inaccessible cardinal.

Corollary 9. (a) *If CH holds, then every normal star-Lindelöf space of cardinality at most $\mathfrak{c}$ has extent smaller than $\mathfrak{c}$.*

(b) *The consistency of the existence of an inaccessible cardinal implies the consistency of the existence of a star-Lindelöf normal space of cardinality $\mathfrak{c}$ having extent $\mathfrak{c}$.*

PROOF: If CH holds, then by Corollary 2 every closed discrete subset of a star-Lindelöf normal space of cardinality $\mathfrak{c}$ is countable so the extent of any such space is ω. On the other hand, by Lemma 8, in a model where $\mathfrak{c}$ is a limit cardinal satisfying $2^{<\mathfrak{c}} = \mathfrak{c}$ there is a separable, and, therefore, star-Lindelöf, normal space with extent $\mathfrak{c}$. $\square$

We close with an example of a space which comes close to being normal and star-Lindelöf with a large closed discrete subset. A collection of subsets of a set is *linked* if every two elements of the collection have non-empty intersection. A collection which is a union of countably many linked collections is said to be *σ-linked*.

Proposition 10. *There is a Tychonoff space X and a closed subspace $Z \subset X$ such that:*

(1) *Z is discrete;*
(2) *$|Z| = \mathfrak{c}$;*
(3) *Z is normal in X, that is every two disjoint subsets of Z can be separated by open subsets of X;*
(4) $\mathrm{we}(X) = \omega$.

PROOF: Denote $K = 2^{(2^{\mathfrak{c}})}$. Let Z' be a discrete subspace of K such that $|Z'| = \mathfrak{c}$ and $\mathrm{Cl}_K(Z')$ is homeomorphic to $\beta Z'$. Put $X = K \times \omega \cup Z' \times \{\omega\} \subset K \times (\omega + 1)$ and $Z = Z' \times \{\omega\} \subset X$.

Conditions (1), (2) and (3) hold trivially. Condition (4) follows from the two facts (that hold for every τ):

(a) every family of no more than $\mathfrak{c}$ nonempty open sets in 2^τ is σ-linked;

(b) every linked subfamily of the standard base of 2^τ has non empty intersection. □

We close with some questions.

(1) Does there exist in ZFC a normal star-Lindelöf space having uncountable extent?
(2) Does there exist in ZFC a normal star-Lindelöf space having extent $\mathfrak{c}$?
(3) Is the existence of a normal star-Lindelöf space having extent $\mathfrak{c}$ equivalent to the conditions in Lemma 8?
(4) Is there, even consistently, a normal star-Lindelöf space having a closed discrete subset of cardinality $\mathfrak{c}$?

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