A nice class extracted from C_p-theory

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Abstract. We study systematically a class of spaces introduced by Sokolov and call them Sokolov spaces. Their importance can be seen from the fact that every Corson compact space is a Sokolov space. We show that every Sokolov space is collectionwise normal, ω-stable and ω-monolithic. It is also established that any Sokolov compact space X is Fréchet-Urysohn and the space $C_p(X)$ is Lindelöf. We prove that any Sokolov space with a G_δ-diagonal has a countable network and obtain some cardinality restrictions on subsets of small pseudocharacter lying in Σ-products of cosmic spaces.

Keywords: Corson compact space, Sokolov space, extent, ω-monolithic space, Σ-products

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0. Introduction

It is Gul'ko's result [Gu1] that, for any Corson compact space X, the family $\mathcal{R}$ of retractions in X is very rich and, in some sense, determines the topology of X. Gul'ko proved, using the properties of $\mathcal{R}$ that, for any Corson compact X, the odd iterated function spaces (with the topology of pointwise convergence) are Lindelöf and the even ones are normal.

This result was strengthened by Sokolov [So1] who established that all iterated function spaces of a Corson compact space are Lindelöf. His method of proof also used functions from a space to itself. He did not require them to be retractions but the family $C(X, X)$ of continuous functions from X to itself must be also rich enough to allow a general construction for generating dual properties.

Sokolov's method could be resumed as follows: given a cardinal invariant φ and an infinite cardinal κ, let $\mathcal{P}(\varphi, \kappa)$ be the class of spaces X such that, for any sequence $\{F_n : n \in \mathbb{N}\}$, where every F_n is a closed subset of X^n, there exists a continuous map $f : X \to X$ such that $\varphi(f(X)) \leq \kappa$ and $f^n(F_n) \subset F_n$ for any $n \in \mathbb{N}$. Sokolov was mainly interested in the case when φ is either hereditary density or hereditary Lindelöf degree of all finite powers. He established, in particular, that if φ is finitely multiplicative and η is a dual for φ (i.e., $\varphi(X) = \eta(C_p(X))$ for any Tychonoff space X) then $X \in \mathcal{P}(\varphi, \kappa)$ implies $C_p(X) \in \mathcal{P}(\eta, \kappa)$. He also proved

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that, under some general assumptions on X, if φ is the supremum of hereditary densities of finite powers and $X \in \mathcal{P}(\varphi, \omega)$ then $C_p(X)$ is Lindelöf.

The paper [So1] contains quite a few excellent results; however, the class $\mathcal{P}(nw, \omega)$ is only mentioned briefly, always in the context that the results proved for hereditary density and hereditary Lindelöf degree are also valid for the spaces from $\mathcal{P}(nw, \omega)$. In the paper [So2] Sokolov introduces a class $\mathcal{S}$ as follows: a space X belongs to $\mathcal{S}$ if and only if $X \in \mathcal{P}(nw, \omega)$ and $t(X^n) \le \omega,\ l(X^n) \le \omega$ for any $n \in \mathbb{N}$. He proved that a space X belongs to $\mathcal{S}$ if and only if $C_p(X)$ is in $\mathcal{S}$ and constructed an example of a compact $X \in \mathcal{S}$ which is not Corson compact.

The purpose of this paper is to show that the class $\mathcal{P}(nw, \omega)$ has so nice and unexpected properties that it deserves its own name. So, we say that X is *a Sokolov space* (*or has the Sokolov property*) if X belongs to $\mathcal{P}(nw, \omega)$. The importance of this property becomes evident after we observe that every Corson compact space is Sokolov so, studying Sokolov compact spaces we actually obtain new information about Corson compacta.

We prove that every Sokolov space is normal, ω-monolithic and ω-stable. It follows from Sokolov's results that X is a Sokolov space if and only if $C_p(X)$ is a Sokolov space. We also show that any Sokolov space with a G_δ-diagonal is cosmic and establish that $|X| \le (\psi(X))^\omega$ for any X which embeds as a closed subspace in a Σ-product of cosmic spaces.

1. Notation and terminology

All spaces are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. A family $\mathcal{N}$ is a *network* of a space X if every $U \in \tau(X)$ is a union of a subfamily of $\mathcal{N}$. In other words, a network is like a base, only its elements need not be open. The cardinal $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network of $X\}$ is called *the network weight* of X. The spaces with a countable network weight are called *cosmic*. The *tightness* $t(X)$ of a space X is the minimal cardinal κ such that for any $A \subset X$ and $x \in \overline{A}$ there is a set $B \subset A$ such that $|B| \le \kappa$ and $x \in \overline{B}$. The *Lindelöf number* $l(X)$ of the space X is the minimal cardinal κ such that any open cover of X has a subcover of cardinality $\le \kappa$. The *pseudocharacter* $\psi(X)$ is the smallest cardinal κ such that every $x \in X$ is the intersection of at most κ-many open subsets of X. The cardinal $\mathrm{ext}(X) = \sup\{|D| : D$ is a closed discrete subset of $X\}$ is called *the extent* of the space X.

A space X is *Lindelöf* Σ if it is a continuous image of a space Y which can be perfectly mapped onto a second countable space. If κ is a cardinal then a space X is called *κ-monolithic* if $nw(\overline{A}) \le \kappa$ for any $A \subset X$ with $|A| \le \kappa$; the space X is *κ-stable* provided that, for any continuous image Y of the space X, if there is a continuous bijection of Y onto a space of weight $\le \kappa$ then $nw(Y) \le \kappa$. A space X has a *G_δ-diagonal* if the diagonal $\Delta = \{(x, x) : x \in X\}$ of the space X is the intersection of countably many open subsets of $X \times X$. The *diagonal of X is small* if, for any uncountable $A \subset (X \times X)\backslash\Delta$ there is an uncountable $B \subset A$

such that $\overline{B} \cap \Delta = \emptyset$. A regular uncountable cardinal κ is *a caliber* of X if, for any $\mathcal{U} \subset \tau^*(X)$ with $|\mathcal{U}| = \kappa$ there is $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| = \kappa$ and $\bigcap \mathcal{V} \neq \emptyset$.

Given spaces X, Y and $f : X \to Y$, the n-th power $f^n : X^n \to Y^n$ of the map f is defined by $f^n((x_1, \dots, x_n)) = (f(x_1), \dots, f(x_n))$ for any $(x_1, \dots, x_n) \in X^n$. A space X is *Sokolov* (*or, has the Sokolov property*) if, for any sequence $\{F_n : n \in \mathbb{N}\}$ with F_n closed in X^n for every $n \in \mathbb{N}$, there is a continuous map $f : X \to X$ such that $nw(f(X)) \leq \omega$ and $f^n(F_n) \subset F_n$ for any $n \in \mathbb{N}$. The iterated function spaces $C_{p,n}(X)$ are defined as follows: $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for any $n \in \omega$. If φ is a cardinal invariant then the cardinal invariant φ^* is defined by $\varphi^*(X) = \sup\{\varphi(X^n) : n \in \mathbb{N}\}$ for any space X. A map $f : X \to Y$ is called $\mathbb{R}$-*quotient* if, for any $g : Y \to \mathbb{R}$, the continuity of $g \circ f$ implies continuity of g. A space is *metacompact* if any open cover of X has a point-finite open refinement.

We say that a space S is a Σ-*product* of spaces from a class $\mathcal{C}$ if there is a family $\{X_t : t \in T\}$ of spaces from $\mathcal{C}$ and a point $a \in X = \prod\{X_t : t \in T\}$ such that $S = \{x \in X : \text{the set } \{t \in T : x(t) \neq a(t)\} \text{ is countable}\}$. Compact subspaces of Σ-products of real lines are called *Corson compact*.

The rest of our notation is standard and follows [En].

2. General facts about Sokolov spaces

We compile the known facts about Sokolov spaces in the following theorem. The respective statements were either proved by Sokolov in [So1] and [So2] or can be easily deduced from his results. Taking in consideration that the paper [So1] is less accessible and the results are often not explicitly given in [So1] and [So2], we will give "a proof" which will consist of some sketches and/or exact references to the respective passages in Sokolov's papers.

2.1 Theorem. (a) *Every closed subset of a Sokolov space is a Sokolov space and the countable power of a Sokolov spaces is a Sokolov space.*
(b) *If X is a Sokolov space and $f : X \to Y$ is an $\mathbb{R}$-quotient map then Y is a Sokolov space.*
(c) *Any closed subspace of a Σ-product of second countable spaces* (*and hence any Corson compact space*) *is Sokolov.*
(d) *A space X is Sokolov if and only if $C_p(X)$ is Sokolov. Thus, if X is Sokolov then $C_{p,n}(X)$ is also Sokolov for any $n \in \mathbb{N}$.*
(e) *If a space X is Sokolov and $t^*(X) \leq \omega$ then $C_{p,2n+1}(X)$ is Lindelöf for any $n \in \mathbb{N}$.*
(f) *If a space X is Sokolov and $l^*(X) \leq \omega$ then $C_{p,2n}(X)$ is Lindelöf for any $n \in \mathbb{N}$.*
(g) *If a space X is Sokolov and $l^*(X) \cdot t^*(X) \leq \omega$ then $C_{p,n}(X)$ is Lindelöf for any $n \in \mathbb{N}$.*
(h) *A space X is Sokolov if and only if, for any family $\{F_{mn} : m, n \in \mathbb{N}\}$ such that F_{mn} is a closed subset of X^n for any $n, m \in \mathbb{N}$, there is a continuous map*

$f : X \to X$ for which $nw(f(X)) \le \omega$ and $f^n(F_{mn}) \subset F_{mn}$ for any $m, n \in \mathbb{N}$.
(i) *A space with a unique non-isolated point is Sokolov iff it is Lindelöf.*
(j) *There is a scattered Sokolov compact space which is not Corson compact.*

Proof: The statement of (h) is formulated in [So1, Remark 1]. The idea of the proof is that we can situate every X^n as a face in some X^k for $k > n$ in such a way that every X^n occurs infinitely many times in this representation. Then every F_{mn} will be a closed subset of the respective X^k so the Sokolov property of X can be applied to find the promised map.

The assertion of (a) is an immediate consequence of (h). If we take $\varphi = nw$ in [So1, Theorem 3] then we can see that the Sokolov property of X implies the Sokolov property of $C_p(X)$. The same result shows that if $C_p(X)$ is Sokolov then so is $C_p(C_p(X))$ and hence X is also Sokolov being homeomorphic to a closed subspace of $C_p(C_p(X))$. This proves (d). A verification of (d) can also be extracted from the proof of [So2, Theorem 3.2] where Sokolov establishes the same for a class $\mathcal{S}$ which, in our terminology, consists of Sokolov spaces whose all finite powers are Lindelöf and have countable tightness.

To see that (b) holds observe that $C_p(Y)$ embeds in $C_p(X)$ as a closed subspace and apply (d) together with (a). The statement (c) can be obtained applying Theorem 1 of [So1] and observing that a subspace of a Σ-product of second countable spaces is separable if and only if it is second countable. Here, even a sketch of a proof requires a considerable space so we refer the reader, apart from [So1], to the papers [Gu1] and [Gu2] in which Gul'ko applies a technique of dealing with retractions which, once understood, can be easily transformed into a proof of (c).

The assertions (e) and (f) can be obtained from Theorem 4 and Theorem 5 of [So1] if we observe that their proofs, done for hereditary density and hereditary Lindelöf number, can give the same, with no change at all, for network weight. A little bit more difficult challenge is to analyze the proof of Theorem 3.2 of [So2] and see that its method can also be used to prove (e) and (f). The statement of (g) is an immediate consequence of (e) and (f) as well as of Theorem 3.2 of [So2]. The existence of the example mentioned in (j) follows from Proposition 4.2 and Corollary 4.3 of [So2].

The equivalence in (i) can be proved observing that Corollary 3 of [So1], although formulated for Lindelöf number and density, is actually true for network weight. The proof need not be changed at all but let us give here a sketch anyway.

The only non-trivial part is to show that if a Lindelöf space X has a unique non-isolated point a then X is Sokolov; let $Y = X \setminus \{a\}$. For any $A \subset Y$ let $r_A(x) = x$ if $x \in A$ and $r_A(x) = a$ if $x \notin A$. It is straightforward that $r_A : X \to A \cup \{a\}$ is a continuous retraction for any $A \subset Y$. It turns out that, for any sequence $\{F_n : n \in \mathbb{N}\}$ such that F_n is a closed subset of X^n for each $n \in \mathbb{N}$, there is a countable set $A \subset Y$ such that $(r_A)^n(F_n) \subset F_n$ for any $n \in \mathbb{N}$. This, of course, implies that X is Sokolov. The method of finding the promised set A actually belongs to Gul'ko; it is described in detail on page 140 of the book [Ar1] in the proofs of Proposition IV.3.10 and Lemma IV.3.11. $\square$

Theorem 2.1 shows that Sokolov spaces form a very nice class. Let us show that it is even nicer than this.

2.2 Proposition. *Every Sokolov space is collectionwise normal, ω-stable, ω-monolithic and has countable extent.*

PROOF: Let X be a Sokolov space; if F and G are disjoint closed subsets of X then, by Theorem 2.1(h), there is a continuous map $f : X \to X$ such that $Y = f(X)$ is cosmic while $F' = f(F) \subset F$ and $G' = f(G) \subset G$. Observe that $\mathrm{cl}_Y(F') \cap \mathrm{cl}_Y(G') \subset \overline{F'} \cap \overline{G'} \subset F \cap G = \emptyset$; since the cosmic space Y is normal, there is a continuous $g : Y \to \mathbb{R}$ such that $g(\mathrm{cl}_Y(F')) = 1$ and $g(\mathrm{cl}_Y(G')) = 0$. It is evident that $h = g \circ f$ is a continuous function on X which separates the sets F and G. Therefore X is normal.

Now take an arbitrary countable set $A \subset X$. The space $F = \overline{A}$ is Sokolov by Theorem 2.1(a) so we can apply Theorem 2.1(h) again to find a continuous map $r : F \to F$ for which $r(F)$ is cosmic and $r(a) = a$ for any $a \in A$. Since A is dense in F, we have $r(x) = x$ for any $x \in F$ and hence $F = r(F)$ is cosmic which proves that X is ω-monolithic.

To see that the space X is ω-stable it suffices to note that $C_p(X)$ is ω-monolithic by what we proved in the above paragraph and Theorem 2.1(d) so the space X has to be ω-stable by [Ar1, Theorem II.6.8].

Assume that $\mathrm{ext}(X) > \omega$ and fix a closed discrete uncountable set $D \subset X$. The space D is Sokolov by Theorem 2.1(a); since D is also first countable, we conclude that $C_p(D) = \mathbb{R}^D$ is Lindelöf (see Theorem 2.1(e)), which is a contradiction. Finally observe that any normal space of countable extent is collectionwise normal so X is collectionwise normal. $\square$

2.3 Corollary. *If X is a Sokolov space then $C_{p,n}(X)$ is normal and has countable extent for any $n \in \mathbb{N}$.*

2.4 Corollary. *A metrizable space is Sokolov if and only if it is separable.*

2.5 Corollary. *If X is a separable Sokolov space then $nw(X) \leq \omega$.*

2.6 Corollary. *Any Sokolov pseudocompact space is countably compact.*

2.7 Corollary. *Any Sokolov metacompact space is Lindelöf.*

PROOF: If X is a Sokolov metacompact space then X is paracompact because it is collectionwise normal (see Proposition 2.2 and [En, Theorem 5.3.3]). Since $\mathrm{ext}(X) = \omega$, the space X has to be Lindelöf (it is an easy exercise to see that any paracompact space of countable extent is Lindelöf). $\square$

It is well known that any Lindelöf Σ-space with a G_δ-diagonal is cosmic (see [Gr1, Theorem 4.15]). It turns out that, in this situation, Sokolov spaces behave similarly.

2.8 Theorem. *If X is a Sokolov space with a G_δ-diagonal then $nw(X) \leq \omega$.*

Proof: Let $\Delta = \{(x,x) : x \in X\}$ be the diagonal of the space X. Fix a sequence $\mathcal{F} = \{F_n : n \in \omega\}$ of closed subsets of $X \times X$ such that $\bigcup \mathcal{F} = (X \times X)\backslash\Delta$. By Theorem 2.1(h), there is a continuous map $f : X \to X$ such that $Y = f(X)$ is cosmic and $f^2(F_n) \subset F_n$ for any $n \in \omega$. The map f is injective for if we are given distinct $x, y \in X$ then $z = (x,y) \in F_n$ for some $n \in \omega$ and therefore $f^2(z) = (f(x), f(y)) \in F_n \subset (X \times X)\backslash\Delta$ which shows that $f(x) \neq f(y)$.

Since every cosmic space has a weaker second countable topology, there is a continuous bijection of X onto a second countable space. The space X being stable by Proposition 2.2, we conclude that $nw(X) = \omega$. □

The following two results show that spread restrictions have strong implications in Sokolov spaces. Recall that the spread $s(X)$ of a space X is the supremum of cardinalities of discrete subspaces of X.

2.9 Corollary. *If X is a Sokolov space and $s(X \times X) \leq \omega$ then X is cosmic.*

Proof: Recalling that the space $X \times X$ is ω-monolithic by Proposition 2.2, we can see that $hl(X \times X) \leq \omega$ (see [Ar2, Proposition 2]). In particular, $(X \times X)\backslash\Delta$ is a Lindelöf space which implies that the diagonal Δ is a G_δ-subset of $X \times X$. Now, apply Theorem 2.8 to complete the proof. □

2.10 Corollary. *If X is a Sokolov space and $s(C_p(X)) \leq \omega$ then X is cosmic.*

Proof: It suffices to notice that the inequality $s(C_p(X)) \leq \omega$ implies $s(X \times X) \leq \omega$ (see [Ar1, Corollary II.5.18]) and apply Corollary 2.9. □

2.11 Corollary. *If $X \times X$ is a hereditarily Sokolov space then X is cosmic.*

Proof: Any discrete Sokolov space is countable by Proposition 2.2 so $s(X \times X) \leq \omega$ and hence we can apply Corollary 2.9 to see that X is cosmic. □

2.12 Proposition. *If X is a Sokolov Lindelöf Σ-space then $t^*(X) = \omega$ and hence $C_{p,n}(X)$ is Lindelöf for any $n \in \mathbb{N}$.*

Proof: We have $\text{ext}(C_p(X)) = \omega$ by Theorem 2.1(d) and Proposition 2.2. Therefore Baturov's theorem [Ba] is applicable to see that $C_p(X)$ is Lindelöf and hence $t^*(X) = \omega$ by Asanov's theorem (see [Ar1, Theorem I.4.1]). Now apply Theorem 2.1(g) to finish the proof. □

2.13 Corollary. *If X is a Sokolov compact space then X is Fréchet-Urysohn, ω-monolithic and $C_{p,n}(X)$ is Lindelöf for any $n \in \mathbb{N}$.*

It is still an open problem whether a Lindelöf Σ-space with a small diagonal is cosmic. Recall that a space X *has a small diagonal* if, for any $D \subset (X \times X)\backslash\Delta$ such that $|D| > \omega$ there is an uncountable $E \subset D$ such that $\overline{E} \cap \Delta = \emptyset$. Gruenhage proved in [Gr2] that, under CH, every Lindelöf Σ-space with a small diagonal is cosmic. We will prove the same for Sokolov Lindelöf Σ-spaces in ZFC.

2.14 Theorem. *Assume that X is a Sokolov space with $l^*(X)\cdot t^*(X) \le \omega$. Then*

(a) *if X has a small diagonal then X is cosmic;*
(b) *if ω_1 is a caliber of X then X is cosmic.*

Proof: (a) If X is not cosmic then it is not separable by Corollary 2.5 so there is a left-separated subspace $Y = \{x_\alpha : \alpha < \omega_1\} \subset X$ (this means that $Y_\alpha = \{x_\beta : \beta < \alpha\}$ is closed in Y for any $\alpha < \omega_1$). Let $F = \bigcup\{\overline{Y}_\alpha : \alpha < \omega_1\}$; it follows from $t(X) = \omega$ that F is closed in X; the space X being ω-monolithic, we have $nw(F) = \omega_1$.

The small diagonal property is hereditary so F also has a small diagonal and hence ω_1 is a caliber of $C_p(F)$ (see [Tk1]). We have $nw(C_p(F)) = nw(F) = \omega_1$ so there is a dense subset $D = \{f_\alpha : \alpha < \omega_1\}$ of the space $C_p(F)$. If $D_\alpha = \{f_\beta : \beta < \alpha\}$ for any $\alpha < \omega_1$ then $P = \bigcup\{\overline{D}_\alpha : \alpha < \omega_1\}$ is closed in $C_p(F)$ because $t(C_p(F)) = \omega$. Since $P \supset D$ is dense in $C_p(F)$, we have $P = C_p(F)$ which shows that the family $\mathcal{U} = \{C_p(F)\backslash\overline{D}_\alpha : \alpha < \omega_1\}$ is point-countable.

The cardinal ω_1 being a caliber of $C_p(F)$ the family $\mathcal{U}$ cannot be uncountable so $\overline{D}_\alpha = C_p(F)$ for some $\alpha < \omega_1$ which shows that $C_p(F)$ is a separable Sokolov space. Applying Corollary 2.5 once more we conclude that $nw(F) = nw(C_p(F)) = \omega$; this contradiction shows that (a) is proved.

To prove (b) suppose that ω_1 is a caliber of X. Then $C_p(X)$ has a small diagonal (see [Tk1]) and hence $nw(C_p(X)) = \omega$ by (a). Therefore $nw(X) = nw(C_p(X)) = \omega$ and (b) is also settled. $\square$

2.15 Corollary. *Suppose that X is a Sokolov Lindelöf Σ-space. If either X has a small diagonal or ω_1 is a caliber of X then X is cosmic.*

2.16 Examples. (a) *The space ω_1 is Sokolov while ω_1+1 is not in spite of being a continuous image of ω_1. Thus Sokolov property is not preserved by continuous images.*
(b) *There are Sokolov spaces X and Y such that neither $X \times Y$ nor $X \oplus Y$ is Sokolov.*
(c) *If X is compact and $C_p(X)$ is Lindelöf Σ then X is Sokolov being a Corson compact by a Gul'ko's theorem [Gu2]. However, if X is not compact and $C_p(X)$ is Lindelöf Σ then X is not necessarily a Sokolov space.*

Proof: (a) It is a folklore (and easy to prove) that ω_1 embeds into a Σ-product of real lines as a closed subspace; by Theorem 2.1(c), ω_1 is a Sokolov space. On the other hand, the space ω_1+1 is not Sokolov since it is compact and has uncountable tightness (see Proposition 2.13).

(b) On the set ω_1+1 define a topology μ declaring all points of ω_1 isolated and let the family $\{\{\omega_1\} \cup \{\beta : \alpha < \beta\} : \alpha < \omega_1\}$ be a local base at the point ω_1. In other words, the space $X = (\omega_1+1, \mu)$ is the one-point lindelöfication of a discrete space of cardinality ω_1. The spaces X and $Y = \omega_1$ are what we are looking for.

We have already seen that Y is Sokolov; it follows from Theorem 2.1(i) that X is Sokolov as well. Consider the "diagonal" $D = \{(\alpha, \alpha) : \alpha < \omega_1\} \subset X \times Y$ of the space $X \times Y$. The projection onto the first factor maps D injectively onto the discrete space $X \backslash \{\omega_1\}$ so D is discrete. It is an easy exercise that D is closed in $X \times Y$ so $\mathrm{ext}(X \times Y) > \omega$ and hence $X \times Y$ is not a Sokolov space by Proposition 2.2.

If $X \oplus Y$ is Sokolov then $(X \oplus Y)^2$ is also Sokolov by Theorem 2.1(a); since $X \times Y$ embeds in $(X \oplus Y)^2$ as a closed subspace, we infer that $X \times Y$ is Sokolov which is a contradiction.

(c) Reznichenko proved (see [Fa, Section 8.4]) that there exists a compact space K such that $C_p(K)$ is Lindelöf Σ while there is a point $x \in K$ such that $X = K \backslash \{x\}$ is pseudocompact and C-embedded in K. The space K being Fréchet–Urysohn, X is not countably compact and hence not Sokolov (see Corollary 2.6). Since X is C-embedded in K, the image of $C_p(K)$ under the restriction map is the whole space $C_p(X)$. Therefore $C_p(X)$ is a Lindelöf Σ-space. □

Theorem 2.1(c) shows that the class of Sokolov spaces contains the class of all closed subsets of Σ-products of real lines. Therefore it is natural to deal with subspaces of "nice" Σ-products if we want to prove something which we suspect to be true for Sokolov spaces. This is the reason why the rest of our results is intended to show that some facts which we could not prove for Sokolov spaces, are true for subspaces of "nice" Σ-products. We first look at the situation with small diagonals and caliber ω_1.

2.17 Lemma. *Suppose that Z is a subspace of a Σ-product H of cosmic spaces. If κ is an infinite cardinal such that $\kappa = \kappa^\omega$ and $|Z| > \kappa$ then there is a point $y \in H$ and a subset $P \subset Z$ such that $|P| = \kappa^+$ and the set $\{y\} \cup P$ is homeomorphic to the one-point compactification $A(\kappa^+)$ of a discrete space of cardinality κ^+.*

Proof: Suppose that we are given a family $\{N_t : t \in T\}$ of cosmic spaces such that, for some point $a \in N = \prod\{N_t : t \in T\}$, our space H coincides with the subspace $\Sigma(N, a) = \{x \in N : \text{the set } \mathrm{supp}(x) = \{t \in T : x(t) \neq a(t)\} \text{ is countable}\}$ of the product N. For any $S \subset T$ the mapping $p_S : N \to N_S = \prod_{t \in S} N_t$ is the natural projection of N onto its face N_S.

Observe that it is impossible that the set $E = \bigcup\{\mathrm{supp}(x) : x \in Z\}$ have cardinality at most κ because then Z can be embedded in $\Sigma(N_E, p_E(a))$ and hence $|Z| \leq |\Sigma(N_E, p_E(a))| \leq \kappa^\omega = \kappa$ which is a contradiction.

Thus $|E| > \kappa$ and therefore we can choose a set $Y = \{y_\alpha : \alpha < \kappa^+\} \subset Z$ such that $\mathrm{supp}(y_\alpha)$ is not contained in $\bigcup\{\mathrm{supp}(y_\beta) : \beta < \alpha\}$ for any $\alpha < \kappa^+$. Since every $\mathrm{supp}(y_\alpha)$ is countable and $\mu^\omega \leq \kappa^\omega = \kappa < \kappa^+$ for any $\mu < \kappa^+$, there exists a set $A \subset \kappa^+$ such that $|A| = \kappa^+$ and there is $D \subset T$ for which $\mathrm{supp}(y_\alpha) \cap \mathrm{supp}(y_\beta) = D$ for any distinct $\alpha, \beta \in A$ (see [Ju, Fact 0.6]). Observe that $\mathrm{supp}(y_\alpha) \neq \mathrm{supp}(y_\beta)$ for distinct $\alpha, \beta < \kappa^+$ so the family $\{\mathrm{supp}(y_\alpha) \backslash D : \alpha \in A\}$ is disjoint and consists of non-empty subsets of T.

Since D is countable, the space N_D is cosmic and hence $|N_D| \leq \mathfrak{c} < \kappa^+$. Therefore we can choose a set $B \subset A$ and $h \in N_D$ such that $|B| = \kappa^+$ and $p_D(y_\alpha) = h$ for any $\alpha \in B$. Let $P = \{y_\alpha : \alpha \in B\}$ and define a point $y \in N$ by $y|D = h$ and $y(t) = a(t)$ for any $t \in T\backslash D$. It is evident that $y \in H$ and $\{y\} \cup P$ is homeomorphic to $A(\kappa^+)$. □

2.18 Proposition. *Suppose that X is a subspace of a Σ-product of second countable spaces.*

(a) *If ω_1 is a caliber of X then X is second countable.*
(b) *If X has a small diagonal and* $\operatorname{ext}(X) \leq \mathfrak{c}$ (*in particular, if X is closed in the respective Σ-product*) *then $|X| \leq \mathfrak{c}$.*

Proof: Let H be a Σ-product of second countable spaces such that $X \subset H$. It is a theorem of Shapirovsky [Sh, Corollary 11] that X has a point-countable π-base $\mathcal{B}$. If ω_1 is a caliber of X then $\mathcal{B}$ is countable whence X is separable and hence $w(X) = \omega$. This proves (a).

Now, if $|X| > \mathfrak{c}$ then we can apply Lemma 2.17 to find a set $P \subset X$ such that $|P| = \mathfrak{c}^+$ and there is a point $y \in H$ for which $\{y\} \cup P$ is homeomorphic to $A(\mathfrak{c}^+)$. If $y \notin X$ then P is a closed discrete subset of X of cardinality $\mathfrak{c}^+$ which contradicts $\operatorname{ext}(X) \leq \mathfrak{c}$. Therefore $y \in X$; the property of having a small diagonal is hereditary so the space $\{y\} \cup P \subset X$ has a small diagonal which is, evidently, false. Therefore $|X| \leq \mathfrak{c}$, i.e., we proved (b). □

The rest of our results deal with cardinality restrictions on subspaces of small pseudocharacter of "nice" Σ-products. The motivation here is to extend the famous theorem of Arhangel'skii's which says, in particular, that $|X| \leq \mathfrak{c}$ whenever X is a compact space of countable (pseudo)character. An easy consequence of the theorem of Arhangel'skii is that $|X| \leq \mathfrak{c}$ if X is a Lindelöf Σ-space of countable pseudocharacter because any such space is a union of $\leq \mathfrak{c}$-many of compact subspaces. Now, if we assume that $\psi(X) \leq \omega$ and $C_p(X)$ is a Lindelöf Σ-space then it is not clear at all whether $|X| \leq \mathfrak{c}$ even though, for such a space X, the space υX is Lindelöf Σ as well as $C_p(\upsilon X)$ (see [Ar1, Theorem IV.9.5] and [Tk2, Theorem 2.3]).

The author also believes (without being able to prove it) that countable pseudocharacter of a Sokolov space implies that its cardinality does not exceed $\mathfrak{c}$ so we prove analogous results for subspaces of "nice" Σ-products.

2.19 Theorem. *If X is an arbitrary subspace of a Σ-product of cosmic spaces then $|X| \leq (\operatorname{ext}(X) \cdot \psi(X))^\omega$.*

Proof: Let H be a Σ-product of cosmic spaces such that $X \subset H$. Consider the cardinal $\kappa = (\operatorname{ext}(X) \cdot \psi(X))^\omega$ and assume that $|X| > \kappa$. Since $\kappa^\omega = \kappa$, we can apply Lemma 2.17 to find a set $P \subset X$ such that $|P| = \kappa^+$ and there is a point $y \in H$ for which $\{y\} \cup P$ is homeomorphic to $A(\kappa^+)$. If $y \notin X$ then P is a closed discrete subset of X of cardinality κ^+ which contradicts $\operatorname{ext}(X) \leq \kappa$. Therefore

$y \in X$ which shows that $\psi(X) \geq \psi(\{y\} \cup P) = \kappa^+$ which is a contradiction with $\psi(X) \leq \kappa$. □

2.20 Corollary. *For any closed subspace X of a Σ-product of cosmic spaces we have $|X| \leq (\psi(X))^\omega$.*

Proof: It is not difficult to show that $\mathrm{ext}(X) = \omega$ so Theorem 2.19 completes the job. □

2.21 Corollary. *If X is homeomorphic to a closed subspace of a Σ-product of real lines and pseudocharacter of X is countable then $|X| \leq \mathfrak{c}$.*

3. Open problems

The following list of open questions contains the most interesting problems the author could not solve while working on this paper. As usual, the unsolved problems are more numerous than the solved ones; this shows, in particular, that the class of Sokolov spaces still offers quite a few challenges to a researcher.

3.1 Problem. *Is any hereditarily Sokolov space cosmic?*

3.2 Problem. *Suppose that $(X \times X)\backslash\Delta$ is a Sokolov space. Must X be cosmic? Here $\Delta = \{(x,x) : x \in X\}$ is the diagonal of the space X.*

3.3 Problem. *Suppose that X is a Sokolov space and ω_1 is a caliber of X. Must X be cosmic?*

3.4 Problem. *Suppose that X is a Sokolov space with a small diagonal. Must X be cosmic?*

3.5 Problem. *Let X be a Sokolov space with $\psi(X) = \omega$ (or even $\chi(X) = \omega$). Is it true that $|X| \leq \mathfrak{c}$?*

3.6 Problem. *Is every Sokolov space monolithic? (This is the same as asking whether every Sokolov space is stable.)*

3.7 Problem. *Suppose that X is a Sokolov compact space. Is it true that there exists an injective continuous map of $C_p(X)$ into a Σ-product of real lines?*

3.8 Problem. *Suppose that X and $C_p(X)$ are Lindelöf Σ-spaces. Must X be a Sokolov space? The answer is not clear even if X is σ-compact.*

3.9 Problem. *Must every Sokolov realcompact space be Lindelöf?*

3.10 Problem. *Suppose that X is a Sokolov Lindelöf space. Is it true that X^n is Lindelöf for any $n \in \mathbb{N}$?*

3.11 Problem. *Suppose that X is a Sokolov space with $t(X) = \omega$. Is it true that $t(X^n) = \omega$ for any $n \in \mathbb{N}$?*

3.12 Problem. *Is any Lindelöf P-space Sokolov?*

3.13 Problem. *Suppose that X is a Lindelöf P-space and K is a compact subspace of $C_p(X)$. Must K be a Sokolov space?*

3.14 Problem. *Suppose that X is a Sokolov compact space and $p(C_p(X)) = \omega$ (this means that every point-finite family of non-empty open subsets of $C_p(X)$ is countable). Must X be metrizable?*

3.15 Problem. *Is it true that any Sokolov space has a point-countable π-base?*

3.16 Problem. *Suppose that $C_p(X)$ is a Lindelöf Σ-space and $\psi(X) = \omega$ (or even $\chi(X) = \omega$). It is true that $|X| \leq \mathfrak{c}$?*

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