A note on operators extending partial ultrametrics

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Abstract. We consider the question of simultaneous extension of partial ultrametrics, i.e. continuous ultrametrics defined on nonempty closed subsets of a compact zero-dimensional metrizable space. The main result states that there exists a continuous extension operator that preserves the maximum operation. This extension can also be chosen so that it preserves the Assouad dimension.

Keywords: partial ultrametric, extension operator, Assouad dimension

Classification: 54E35, 54C20, 54E40

1. Introduction

C. Bessaga [6], [7] formulated a general problem of linear extensions of continuous (pseudo)metrics defined on a closed subset of a metrizable space and gave a partial solution of it. A complete solution was first obtained by T. Banakh [3], [4]; see also [5], [19], [23] for related results.

Recently, the authors [21] considered a problem of simultaneous linear extension of metrics with variable domains in a compact space. The obtained result on existence of linear extension operators is in some sense parallel to the corresponding result due to Künzi and Shapiro [11] on simultaneous linear extensions of partial functions. In the present paper we consider a problem of simultaneous extension of partial ultrametrics, i.e. ultrametrics defined on the nonempty closed subsets of a zero-dimensional compact metrizable space.

Recall that a metric ϱ on a set X is called an *ultrametric* (or *non-Archimedean metric*) if $\varrho(x,y) \le \max\{\varrho(x,z),\varrho(z,y)\}$ for all $x,y,z \in X$. It is well-known (see, e.g. [10]) that a metrizable space X admits an ultrametric compatible with its topology if and only if $\dim X = 0$. Obviously, in the case of ultrametrics, one cannot speak about linear extension operators, because the set of all ultrametrics is not, in general, closed with respect to linear operations (the sum of two ultrametrics need not be an ultrametric). However, the set of all ultrametrics is closed under the operation of pointwise maximum.

Identifying every ultrametric with its graph, one can topologize the set of all partial ultrametrics with the hyperspace topology. We show that there exists

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a continuous extension operator of partial ultrametrics that preserves the operation of maximum of two ultrametrics. In addition, the constructed operators preserve the so-called Assouad dimension of the ultrametric spaces. The results on bi-Lipschitz embeddings of ultrametric spaces ([14], [16]) allow us to derive from this that the extended ultrametric space is bi-Lipschitzely embeddable into $\mathbb{R}^n$ if so is the initial ultrametric space.

2. Preliminaries

2.1 Space of partial ultrametrics. By $\exp X$ we denote the *hyperspace* of X, i.e. the set of all nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1,\dots,U_k\rangle = \{A\in \exp X \mid A\subset \bigcup_{i=1}^k U_i,\ A\cap U_i\neq\emptyset \ \text{ for all } \ i\},$$

where $\{U_1,\dots,U_k\}$ run over all finite families of open subsets in X.

If d is a compatible metric on X, then the Vietoris topology is generated by the Hausdorff metric d_H,

$$d_H(A,B)=\inf\{\varepsilon>0 \mid A\subset O_\varepsilon(B),\ B\subset O_\varepsilon(A)\}.$$

Given a nonempty compact subset A of X, we denote by $\mathcal{UM}(A)$ the set of continuous ultrametrics on A. Set

$$\mathcal{UM}=\bigcup\{\mathcal{UM}(A)\mid A\in\exp X,\ |A|\geq 2\}.$$

Identifying every ultrametric $d\in\mathcal{UM}$ with its graph, which is a compact subset of $X\times X\times\mathbb{R}$, we consider the set $\mathcal{UM}$ as a subset of $\exp(X\times X\times\mathbb{R})$ and endow $\mathcal{UM}$ with the subspace topology. Note that such a topologization of functional spaces traces back to Kuratowski [12], [13] and is extensively used in the topological theory of differential equations (see e.g. [9]).

If $\varrho\in\mathcal{UM}$, then $\operatorname{dom}\varrho = A$ means $\varrho\in\mathcal{UM}(A)$. Note that the map $\operatorname{dom}\colon\mathcal{UM}\to\exp X$, being the restriction of the projection onto the first coordinate, is continuous. For $\varrho\in\mathcal{UM}$ let $\|\varrho\|=\max\{\varrho(x,y)\mid x,y\in\operatorname{dom}\varrho\}$.

For every $A\in\exp X$ the set $\mathcal{UM}(A)$ is a continuous $\vee$-semilattice with respect to the operation $\varrho\vee\varrho'=\max\{\varrho,\varrho'\}$. Also, the set $\mathcal{UM}$ is closed under pointwise multiplication by positive numbers.

2.2 Assouad dimension. Let $c,s\geq 0$. We say that a (pseudo)metric space (X,ϱ) is (c,s)-*homogeneous* if the inequality $|X_0|\leq c(b/a)^s$ holds for $a>0$, $b>0$ and $X_0\subset X$ provided that $b\geq a$ and that $a\leq\varrho(x,y)\leq b$ holds for every pair of distinct points x and y of X_0.

The space (X, ϱ) is *s-homogeneous* if it is (c, s)-homogeneous for some $c \geq 0$.

The *Assouad dimension*, $\dim_A(X, \varrho)$ of a (pseudo)metric space (X, ϱ) is defined as follows:

$$\dim_A(X, \varrho) = \inf\{s \geq 0 \mid (X, \varrho) \text{ is } s\text{-homogeneous}\}$$

(see [1], [2]).

Proposition 2.1. *Let ϱ, ϱ_1, ϱ_2 be (pseudo)metrics on a set X and $\varrho = \varrho_1 \vee \varrho_2$. Then $\dim_A(X, \varrho) \leq \dim_A(X, \varrho_1) + \dim_A(X, \varrho_2)$.*

PROOF: Let m denote the max-(pseudo)metric on $(X, \varrho_1) \times (X, \varrho_2)$, i.e.

$$m((x, y), (x', y')) = \max\{\varrho_1(x, x'), \varrho_2(y, y')\}.$$

It is proved in [15] (see Theorem 5.A therein; note that this theorem is formulated for metric spaces, its extension onto pseudometric spaces is straightforward) that $\dim_A(X \times X, m) \leq \dim_A(X, \varrho_1) + \dim_A(X, \varrho_2)$. Since the diagonal map $\Delta\colon (X, \varrho) \to (X \times X, m)$ is an isometric embedding and the Assouad dimension is monotonic, the result follows. □

Proposition 2.2. *Let (X, ϱ) be a compact (pseudo)metric space and $c > 0$. For the truncated (pseudo)metric ϱ', $\varrho'(x, y) = \min\{\varrho(x, y), c\}$, we have $\dim_A(X, \varrho) = \dim_A(X, \varrho')$.*

PROOF: The result follows from the fact that (X, ϱ) and (X, ϱ') are Lipschitz equivalent and from Theorem 5.A. (1) in [15]. □

Proposition 2.3. *Let $f\colon X \to Y$ be a map into a metric space (Y, ϱ). Denote by $f_*(\varrho)$ the pseudometric on X defined by the formula $f_*(\varrho)(x_1, x_2) = \varrho(f(x_1), f(x_2))$. Then $\dim_A(X, f_*(\varrho)) \leq \dim_A(Y, \varrho)$.*

PROOF: The result follows from the fact that, for any a, b, $0 < a < b$, and any subset $X_0 \subset X$ satisfying $a \leq f_*(\varrho)(x_1, x_2) \leq b$ whenever $x_1, x_2 \in X_0$, $x_1 \neq x_2$, the set $f(X_0)$ satisfies the property: $a \leq \varrho(y_1, y_2) \leq b$ whenever $y_1, y_2 \in f(X_0)$, $y_1 \neq y_2$. □

Proposition 2.4. *There exists an ultrametric d on the Cantor set C with the following properties:*

(i) *d takes only binary rational values;*
(ii) *$\dim_A(C, d) = 0$.*

PROOF: Identify C with the set $2^{\mathbb{N}}$ and define d by the formula

$$d((x_i), (y_i)) = \sup\{2^{-2^j} \mid x_j \neq y_j\},\ (x_i), (y_i) \in 2^{\mathbb{N}},\ (x_i) \neq (y_i).$$

Obviously, d is an ultrametric on C that takes only binary-rational values.

We are going to show that $\dim_A(C,d) = 0$. Let $s > 0$. We have to demonstrate that C is s-homogeneous.

Given a, b, $0 < a \le b$, find the minimal natural number m and the maximal natural number n such that $a \le 2^{-2^n} \le 2^{-2^m} \le b$ (without loss of generality we may suppose that such m, n exist). Suppose that $X_0 \subset C$ has the property that $a \le d(x,y) \le b$, for every $x, y \in X_0$, $x \ne y$. Then

$$2^{-2^n} \le d(x,y) \le 2^{-2^m} \tag{2.1}$$

for every $x, y \in X_0$, $x \ne y$. Suppose that $x = (x_i) \in X_0$. For arbitrary $y = (y_i) \in X_0$, $x \ne y$, it easily follows from condition (2.1) and the definition of the metric d, that

$$\{i \mid x_i \ne y_i\} \cap \{m, m+1, \dots, n\} \ne \emptyset.$$

Therefore, the projection map $\mathrm{pr}\colon 2^{\mathbb{N}} \to 2^{\{m,m+1,\dots,n\}}$ separates the points of the set X_0. We conclude that $|X_0| \le 2^{n-m+1}$.

There exists $N \in \mathbb{N}$ such that for every $p > N$ we have $p \le 2^{p-1}s$. Let $c = 2^{N+1}$. If $n = m$, then $|X_0| = 2 \le c(b/a)^s$.

Suppose now that $n > m$. If $n \le N$, then $|X_0| \le 2^{N+1} \le c(b/a)^s$.

If $n > N$, then

$$\begin{aligned}\log_2(c(b/a)^s) &\ge N + s\log_2(b/a) \ge N + s(2^n - 2^m) \ge N + s2^{n-1}\\ &\ge n \ge n - m + 1 \ge \log_2|X_0|\end{aligned}$$

i.e. $|X_0| \le c(b/a)^s$.

Since C is s-homogeneous for every $s > 0$, we conclude that $\dim_A C = 0$. □

3. Extension of partial ultrametrics

The following is the main result of this note.

Theorem 3.1. *Let X be a zero-dimensional compact metrizable space. There exists a map $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ that satisfies the following properties for every $\varrho, \varrho' \in \mathcal{UM}$:*

1. *u is continuous;*
2. *$u(\varrho)$ is an extension of ϱ for every $\varrho \in \mathcal{UM}$;*
3. *$\|u(\varrho)\| = \|\varrho\|$;*
4. *$u(\varrho \vee \varrho') = u(\varrho) \vee u(\varrho')$;*
5. *if $\varrho \in \mathcal{UM}$ takes only (binary) rational values then so does $u(\varrho)$;*
6. *$\dim_A(X, u(\varrho)) = \dim_A(\operatorname{dom}\varrho, \varrho)$.*

Proof: The set $K = \{(x, A) \in X \times \exp X \mid x \in A\}$ is closed in the space $X \times \exp X$. Since the space X is zero-dimensional compact metrizable, so is $\exp X$ and therefore $X \times \exp X$ (see, e.g. [18]) as well as the quotient space $(X \times \exp X)/K$. By the classical embedding theorem, there exists an embedding $f'\colon (X\times \exp X)/K \to C$ into the standard Cantor set C (see, e.g. [8, Theorem 6.2.16]) and, as C is topologically homogeneous, one may additionally assume that $f'(K) = 0 \in C$. Let $f = f'q\colon X \times \exp X \to C$, where $q\colon X \times \exp X \to (X \times \exp X)/K$ is the quotient map. Then $f(K) = \{0\}$ and $f|((X \times \exp X) \setminus K)$ is an embedding.

Define a multivalued map $F\colon X \times \exp X \to X$ by the formula

$$F(x,A) = \begin{cases} A & \text{if } x \notin A, \\ \{x\} & \text{if } x \in A. \end{cases}$$

We show that the map F is lower semicontinuous, i.e. the set $U^\sharp = \{(x,A) \in X \times \exp X \mid F(x,A) \cap U \neq \emptyset\}$ is open for every open subset U of X. Let $(x_0, A_0) \in U^\sharp$.

Case 1). $x_0 \notin A_0$. Then $F(x_0,A_0) = A_0$ and $A_0 \cap U \neq \emptyset$. There exist disjoint neighborhoods V of x_0 and W of A in X respectively. Then for every $(x,A) \in \langle W, W\cap U\rangle$ we have $x \notin A$ and thus $F(x,A) = A$. Since $A \cap (W \cap U) \neq \emptyset$, we see that $(x,A) \in U^\sharp$.

Case 2). $x_0 \in A_0$. Then $F(x_0,A_0) = \{x_0\}$ and $x_0 \in U$. Obviously, $U \times \langle X, U\rangle$ is a neighborhood of (x_0,A_0) and for every $(x,A) \in U \times \langle X,U\rangle$ we have $F(x,A) \cap U \neq \emptyset$, i.e. $(x,A) \in U^\sharp$.

As we already remarked, the space $X \times \exp X$ is a zero-dimensional compact metrizable space. We can apply the zero-dimensional Michael Selection Theorem [17] to find a continuous selection of F, i.e. a continuous map $g\colon X \times \exp X \to X$ such that $g(x,A) \in A$ for every $(x,A) \in X \times \exp X$.

Let d be an ultrametric on C generating its topology. We may suppose, by Proposition 2.4, that d takes only binary rational values and $\dim_A(C,d) = 0$.

Define the map $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ by the formula

$$\begin{aligned} u(\varrho)(x,y) = \ & \max\big\{\varrho(g(x,\operatorname{dom}\varrho), g(y,\operatorname{dom}\varrho)), \\ & \qquad \min\{d(f(x,\operatorname{dom}\varrho), f(y,\operatorname{dom}\varrho)), \|\varrho\|\}\big\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

As the maximum of two continuous ultrapseudometrics, $u(\varrho)$ is a continuous ultrapseudometric on X for every ultrametric $\varrho \in \mathcal{UM}$. Since $\|\varrho\| > 0$ due to our assumptions, one can easily see that $u(\varrho)$ is in fact an ultrametric.

We are going to verify properties (1)–(6).

(1) We show that the map u is continuous. Let (ϱ_n) be a sequence in $\mathcal{UM}$ that converges to $\varrho \in \mathcal{UM}$. Then, obviously, $\operatorname{dom}\varrho_n \to \operatorname{dom}\varrho$. As we have remarked, there exists a continuous ultrametric $\tilde\varrho$ on X that extends ϱ over $X \times X$ (take, e.g., $\tilde\varrho = u(\varrho)$). Let $\tilde\varrho_n = \tilde\varrho|(\operatorname{dom}\varrho_n \times \operatorname{dom}\varrho_n)$. Arguing like in the proof of Lemma 3 in [11], we can show that $\tilde\varrho_n \to \varrho$ in $\mathcal{UM}$.

Fix $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that for every $n \geq N$ we have

(a) $\|\varrho_n - \tilde{\varrho}_n\| < \varepsilon/2$;

(b) $|d(f(x, \operatorname{dom} \varrho), f(y, \operatorname{dom} \varrho)) - d(f(x, \operatorname{dom} \varrho_n), f(y, \operatorname{dom} \varrho_n))| < \varepsilon$ for every $x, y \in X \times X$ (this is a consequence of uniform continuity of the maps f and d and compactness of X);

(c) $|\varrho(g(x, \operatorname{dom} \varrho), g(y, \operatorname{dom} \varrho)) - \tilde{\varrho}(g(x, \operatorname{dom} \varrho_n), g(y, \operatorname{dom} \varrho_n))| < \varepsilon/2$ for every $x, y \in X \times X$.

Then for every $n \geq N$ we have

$$\begin{aligned}
&|\varrho(g(x, \operatorname{dom} \varrho), g(y, \operatorname{dom} \varrho)) - \varrho_n(g(x, \operatorname{dom} \varrho_n), g(y, \operatorname{dom} \varrho_n))| \\
&\quad \leq |\varrho(g(x, \operatorname{dom} \varrho), g(y, \operatorname{dom} \varrho)) - \tilde{\varrho}(g(x, \operatorname{dom} \varrho_n), g(y, \operatorname{dom} \varrho_n))| \\
&\qquad + |\tilde{\varrho}(g(x, \operatorname{dom} \varrho_n), g(y, \operatorname{dom} \varrho_n)) - \varrho_n(g(x, \operatorname{dom} \varrho_n), g(y, \operatorname{dom} \varrho_n))| \\
&\quad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned} \tag{3.2}$$

for every $x, y \in X \times X$.

Inequalities (b) and (3.2) together imply $|u(\varrho)(x, y) - u(\varrho_n)(x, y)| < \varepsilon$ for every $n \geq N$ and every $x, y \in X \times X$. This means that the sequence $(u(\varrho_n))$ converges to $u(\varrho)$ in $\mathcal{UM}$. Because of arbitrariness of (ϱ_n), the map u is continuous.

(2) If $x, y \in \operatorname{dom} \varrho$, then $f(x, \operatorname{dom} \varrho) = f(y, \operatorname{dom} \varrho)$, $g(x, \operatorname{dom} \varrho) = x$, $g(y, \operatorname{dom} \varrho) = y$ and hence $u(\varrho)(x, y) = \varrho(x, y)$, i.e. u is an extension operator.

(3) Let $c = \|\varrho\|$. Then

$$u(\varrho)(x, y) \leq \max\{c, \min\{d(f(x, \operatorname{dom} \varrho), f(y, \operatorname{dom} \varrho)), c\}\} = c$$

for every $x, y \in X$, i.e. $\|u(\varrho)\| \leq c$. Since $u(\varrho)$ is an extension of ϱ, we see that $\|u(\varrho)\| = c$.

(4) If $\varrho = \varrho_1 \vee \varrho_2$, then $\operatorname{dom} \varrho_1 = \operatorname{dom} \varrho_2$ and so

$$\begin{aligned}
(u(\varrho_1) \vee u(\varrho_2))(x, y) &= \max\Big\{ \max\{\varrho_1(g(x, \operatorname{dom} \varrho_1), g(y, \operatorname{dom} \varrho_1)), \\
&\qquad\qquad \min\{d(f(x, \operatorname{dom} \varrho_1), f(y, \operatorname{dom} \varrho_1)), \|\varrho_1\|\}\}, \\
&\qquad \max\{\varrho_2(g(x, \operatorname{dom} \varrho_2), g(y, \operatorname{dom} \varrho_2)), \\
&\qquad\qquad \min\{d(f(x, \operatorname{dom} \varrho_2), f(y, \operatorname{dom} \varrho_2)), \|\varrho_2\|\}\}\Big\} \\
&= \max\Big\{\varrho_1(g(x, \operatorname{dom} \varrho_1), g(y, \operatorname{dom} \varrho_1)), \varrho_2(g(x, \operatorname{dom} \varrho_2), g(y, \operatorname{dom} \varrho_2)), \\
&\qquad \min\{d(f(x, \operatorname{dom} \varrho_1), f(y, \operatorname{dom} \varrho_1)), \|\varrho_1\|\}, \\
&\qquad \min\{d(f(x, \operatorname{dom} \varrho_2), f(y, \operatorname{dom} \varrho_2)), \|\varrho_2\|\}\Big\} \\
&= \max\{(\varrho_1 \vee \varrho_2)(g(x, \operatorname{dom} \varrho_1), g(y, \operatorname{dom} \varrho_1)), \\
&\qquad \min\{d(f(x, \operatorname{dom} \varrho_1), f(y, \operatorname{dom} \varrho_1)), \max\{\|\varrho_1\|, \|\varrho_2\|\}\}\} \\
&= \max\Big\{(\varrho_1 \vee \varrho_2)(g(x, \operatorname{dom} \varrho_1), g(y, \operatorname{dom} \varrho_1)), \\
&\qquad \min\{d(f(x, \operatorname{dom} \varrho_1), f(y, \operatorname{dom} \varrho_1)), \|\varrho_1 \vee \varrho_2\|\}\Big\} \\
&= u(\varrho_1 \vee \varrho_2)(x, y),
\end{aligned}$$

i.e. u is a homomorphism of $\vee$-semilattices.

(5) Follows from formula (3.1).

(6) Note that the formula $\varrho'(x,y) = \min\{d(f(x, \operatorname{dom}\varrho), f(y, \operatorname{dom}\varrho)), \|\varrho\|\}$ determines a pseudometric ϱ' on X. It follows from Proposition 2.4 (applied to the truncated metric $\min(d, \|\varrho\|)$), Proposition 2.3, and the properties of the space (C,d) that $\dim_A(X,\varrho') = \dim_A(C,d) = 0$. Similarly, the formula $\varrho''(x,y) = \varrho(g(x,\operatorname{dom}\varrho), g(y,\operatorname{dom}\varrho))$ determines a pseudometric ϱ'' on X. By Proposition 2.3, $\dim_A(X,\varrho'') = \dim_A(\operatorname{dom}\varrho, \varrho)$.

By the definition, $u(\varrho) = \varrho'' \vee \varrho'$, whence, by Proposition 2.1 applied to the pseudometrics ϱ'' and ϱ', we see that

$$\begin{aligned}\dim_A(X,u(\varrho)) &\le \dim_A(X,\varrho'') + \dim_A(X,\varrho') \\ &= \dim_A(\operatorname{dom}\varrho,\varrho) + \dim_A(C,d) = \dim_A(\operatorname{dom}\varrho,\varrho).\end{aligned}$$

□

Corollary 3.2. *The operator u from Theorem 3.1 has the following property: if $\varrho \in \mathcal{UM}$ and the space $(\operatorname{dom}\varrho, \varrho)$ can be bi-Lipschitz embedded into the Euclidean space $\mathbb{R}^n$, for some n, then $(X, u(\varrho))$ can also be bi-Lipschitz embedded into $\mathbb{R}^n$.*

Proof: It is proved in [14] that if an ultrametric space can be bi-Lipschitz embedded in $\mathbb{R}^n$, then its Assouad dimension is less than n. Since $\dim_A(X,u(\varrho)) = \dim_A(\operatorname{dom}\varrho,\varrho) < n$, it follows from [16, Theorem 3.8] that $(X,u(\varrho))$ can also be bi-Lipschitz embedded into $\mathbb{R}^n$. □

4. Remarks and open questions

4.1 Homogeneous extension operators. Let X be a zero-dimensional compact metrizable space. A map $u\colon\mathcal{UM} \to \mathcal{UM}(X)$ is said to be *homogeneous* if $u(c\varrho) = cu(\varrho)$, for any $\varrho \in \mathcal{UM}$.

Question 4.1. In the assumptions of Theorem 3.1, is there a homogeneous map $u\colon\mathcal{UM} \to \mathcal{UM}(X)$ satisfying conditions (1)–(6) of this theorem?

Quite recently, I. Stasyuk constructed a homogeneous map $u\colon\mathcal{UM} \to \mathcal{UM}(X)$ that satisfies conditions (1)–(5) of Theorem 3.1.

4.2 Generalized ultrametric spaces. One can consider an extension problem also for generalized ultrametric spaces.

Let $(\Gamma, \le)$ be a partially ordered set with smallest element, denoted by 0. Let X be a non-empty set and $d : X \times X \to \Gamma$ be a mapping satisfying the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) if $d(x,y) \le \gamma$ and $d(x,z) \le \gamma$, then $d(y,z) \le \gamma$.

The pair (X, d, Γ) is then called a (generalized) *ultrametric space* ([22]). We leave to the reader the precise formulation of the problem of extension for partial generalized ultrametrics.

4.3 Extension of partial metrics with n-dimensional Nagata property. A metric space (X, d) satisfies the *n-dimensional Nagata property* if for every $r > 0$, every $x \in X$, and every collection of elements $y_1, \dots, y_{n+2}$ of the set

$$\{y \in X \mid \text{ there exists } \ z \in X \ \text{ with } \ d(x, z) < r, \ d(y, z) < 2r\}$$

there exist i, j, $i \neq j$ such that $d(y_i, y_j) < 2r$. It is proved in [1] that a separable metric space X admits a compatible metric with n-dimensional Nagata property if and only if $\dim X \leq n$. For every $A \in \exp X$ denote by $\mathrm{P}_n\mathcal{M}(A)$ the set of all compatible metrics on A with n-dimensional Nagata property. Set $\mathrm{P}_n\mathcal{M} = \bigcup\{\mathrm{P}_n\mathcal{M}(A) \mid A \in \exp X\}$.

The set of metrics with 0-dimensional Nagata property is easily seen to coincide with the set of all ultrametrics. Therefore, the set $\mathrm{P}_0\mathcal{M}(A)$ is closed under the operation max. There is no counterpart of this property for the spaces $\mathrm{P}_n\mathcal{M}(A)$ if $n \geq 1$. However, the sets $\mathrm{P}_n\mathcal{M}(A)$ are closed under multiplication by positive real numbers.

Question 4.2. Let X be an n-dimensional compact metrizable space, $n \geq 1$. Is there a continuous (homogeneous) extension operator $\mathrm{P}_n\mathcal{M} \to \mathrm{P}_n\mathcal{M}(X)$?

A version of this question can be formulated about the existence of a continuous extension operator $\mathrm{P}_n\mathcal{M} \to \mathrm{P}_n\mathcal{M}(X)$ that preserves the partial order relation $\leq$ on $\mathrm{P}_n\mathcal{M}$.

4.4 Non-metrizable case. If we replace the axiom $\varrho(x, y) = 0 \Leftrightarrow x = y$ by $x = y \Rightarrow \varrho(x, y) = 0$, we obtain the notion of ultrapseudometric. One can similarly formulate the problem of simultaneous extension of partial ultrapseudometrics. Denote by $\mathcal{UPM}(A)$ the set of all continuous ultrapseudometrics defined on a nonempty closed subset A of a compact zero-dimensional Hausdorff space X. Let $\mathcal{UPM} = \bigcup\{\mathcal{UPM}(A) \mid A \in \exp X\}$. As in [21], one can prove the following result.

Theorem 4.3. *For a compact zero-dimensional Hausdorff space X the following are equivalent:*

(1) *there exists a continuous extension operator $u\colon \mathcal{UPM} \to \mathcal{UPM}(X)$;*
(2) *there exists a continuous map $\Psi\colon (X \times X) \setminus \Delta_X \to \mathcal{UPM}(X)$, $(x, y) \mapsto \Psi_{(x,y)}$, with $\Psi_{(x,y)}(x, y) \neq 0$ for all $(x, y) \in X^2 \setminus \Delta_X$;*
(3) *X is metrizable.*

Proof: The argument coincides with that of the proof of [21, Theorem 6.1]. It is noted in [21] that implication (2) $\Rightarrow$ (3) is based on a result of Stepanova [20] on extension of partial continuous functions. $\square$

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