The Dirichlet problem for elliptic equations in the plane

Paola Cavaliere, Maria Transirico

Abstract. In this paper an existence and uniqueness theorem for the Dirichlet problem in $W^{2,p}$ for second order linear elliptic equations in the plane is proved. The leading coefficients are assumed here to be of class VMO.

Keywords: elliptic equations, VMO-coefficients

Classification: 35J25, 35R05

1. Introduction

Consider the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$, with a suitable regularity property, $p \in]1, +\infty[$ and L is the uniformly elliptic differential operator defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.2}$$

Suppose that the coefficients a_{ij} satisfy the requirement

$$a_{ij} = a_{ji} \in L^\infty(\Omega), \tag{1.3}$$

and that suitable summability conditions hold for the coefficients a_i and a. If $n \geq 3$, $p < n$ and Ω is bounded, it is well known that (1.3) is not enough to ensure the uniqueness for the problem (1.1). For this reason, problem (1.1) has been studied by several authors under various additional hypotheses on the a_{ij}'s. In particular, a relevant existence and uniqueness theorem has been obtained in [4] and [5], under the assumption that the a_{ij}'s are of class VMO and $a_i = a = 0$; this latter condition has been removed in [11] and [12]. Recently, these results have also been extended to the case of unbounded open sets (see [2] and [3]).

If $n = 2$ and Ω is bounded, a classical theorem by Talenti [9] shows that for $p = 2$ the condition (1.3) on the a_{ij}'s is enough to obtain an estimate for the solutions of (1.1) and so to prove an existence and uniqueness result. It is known that this theorem also holds when p lies in a certain neighborhood of 2 (see for instance [7]), and this interval has recently been determined in [1]. Observe that the lower critical exponent here is precisely the one conjectured by Pucci [8], who also proved that if p is smaller than this exponent, then the uniqueness of the solution of (1.1) cannot be proved.

The aim of this paper is to study problem (1.1) when $n = 2$, Ω is bounded and p is an arbitrary real number > 1. The above considerations show that some additional requirement on the a_{ij}'s is necessary, and it looks natural to impose that such coefficients belong to $VMO(\Omega)$.

2. Some notation

Let Ω be an open subset of $\mathbb{R}^n$, and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω. For each $A \in \Sigma(\Omega)$, we denote by $|A|$ the Lebesgue measure of A. Moreover, put

$$A(x,r) = A \cap B(x,r) \quad \forall\, x \in \mathbb{R}^n,\ \forall\, r \in \mathbb{R}_+,$$

where $B(x,r)$ is the open ball of $\mathbb{R}^n$ of radius r centered at x.

If Ω has the property

$$|\Omega(x,\rho)| \geq C\,\rho^n \quad \forall\, x \in \Omega,\ \forall\, \rho \in]0,1], \tag{2.1}$$

where C is some positive constant independent of x and ρ, one can consider the space $BMO(\Omega,t)$, $t \in \mathbb{R}_+$, consisting of all functions g in $L^1_{\mathrm{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x\in\Omega\\ r\in]0,t]}} \fint_{\Omega(x,r)} \Big|g - \fint_{\Omega(x,r)} g\Big| < +\infty, \tag{2.2}$$

where

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_C)$, with

$$t_C = \sup_{t\in\mathbb{R}_+} \Big(\sup_{\substack{x\in\Omega\\ r\in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{C} \Big),$$

we say that g is in $VMO(\Omega)$ if

$$\lim_{t\to 0^+} [g]_{BMO(\Omega,t)} = 0.$$

Moreover, a function $\eta[g] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$\eta[g](t) \geq [g]_{BMO(\Omega,t)} \quad \forall\, t \in \mathbb{R}_+, \qquad \lim_{t\to 0^+} \eta[g](t) = 0.$$

If $g \in L^p(\Omega)$, we put

$$\omega_p[g](t) = \sup_{\substack{E\in\Sigma(\Omega)\\ |E|\leq t}} \|g\|_{L^p(E)}, \;\; t \in R_+;$$

clearly, $\lim_{t\to 0^+} \omega_p[g](t) = 0$ and the function $\omega_p[g] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a modulus of continuity of g in $L^p(\Omega)$.

A more detailed account of properties of the above defined function spaces can be found in [10].

3. Statements and proofs

Let Ω be a bounded open subset of $\mathbb{R}^2$ with boundary of class $C^{1,1}$, and let $p \in]1, +\infty[$. Consider in Ω the differential operator

$$L = -\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a, \tag{3.1}$$

and suppose that the coefficients of L satisfy the following hypotheses:

$$(h_1) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO(\Omega), \quad i,j = 1,2, \\[2ex] \exists\, \nu \in \mathbb{R}_+ : \displaystyle\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \;\text{ a.e. in }\; \Omega, \;\; \forall\, \xi \in \mathbb{R}^2, \end{cases}$$

$$(h_2) \qquad \begin{cases} a_i \in L^r(\Omega), \;\; i = 1,2, \quad \text{where} \\[1ex] r = 2 \;\text{ if }\; p < 2, \;\; r > 2 \;\text{ if }\; p = 2, \;\; r = p \;\text{ if }\; p > 2, \\[1ex] a \in L^p(\Omega). \end{cases}$$

Observe that under the above assumptions, the operator

$$L : W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is bounded by Sobolev embedding theorem.

Our first result provides an a priori bound.

Lemma 3.1. *Suppose that conditions* (h_1) *and* (h_2) *are satisfied. Then there exists a positive constant* c *such that*

$$(3.2)\qquad \|u\|_{W^{2,p}(\Omega)} \le c\left(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) \quad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega),$$

where c *depends on* Ω, p, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$, $\|a_i\|_{L^r(\Omega)}$, $\|a\|_{L^p(\Omega)}$, $\omega_r[a_i]$, $\omega_p[a]$, *and* $p(a_{ij})$ *is an extension of* a_{ij} *to* $\mathbb{R}^2$ *of class* $L^\infty(\mathbb{R}^2) \cap VMO(\mathbb{R}^2)$.

Proof: Consider a function ζ such that

$$\zeta \in C_0^\infty(]-2,2[), \quad 0 \le \zeta \le 1, \quad \zeta_{|]-1,1[} = 1.$$

Put $\tilde\Omega = \Omega\times]-2,2[$ and fix an element u of $W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. It can be easily shown that the function

$$v(x,t) = u(x)\zeta(t)$$

is of class $W^{2,p}(\tilde\Omega) \cap \overset{\circ}{W}{}^{1,p}(\tilde\Omega)$. Let

$$L_0 = -\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

and consider the operator

$$\tilde L_0 = L_0 - \frac{\partial^2}{\partial t^2}\,.$$

An application of Theorem 5.1 in [10] (see also the proof of this result) yields that there exist extensions $p(a_{ij})$ of a_{ij} to $\mathbb{R}^2$ $(i,j=1,2)$ of class $L^\infty(\mathbb{R}^2) \cap VMO(\mathbb{R}^2)$. It follows that the functions $\tilde p(a_{ij})$, defined by the position

$$\tilde p(a_{ij})(x,t) = p(a_{ij})(x), \quad x \in \Omega, \quad t \in\,]-2,2[\,,$$

belong to $L^\infty(\mathbb{R}^3) \cap VMO(\mathbb{R}^3)$ and $\eta[\tilde p(a_{ij})] = \eta[p(a_{ij})]$. Thus by Theorem 4.2 of [5] there exists $c_1 \in \mathbb{R}_+$ such that

$$(3.3)\qquad \|v\|_{W^{2,p}(\tilde\Omega)} \le c_1\left(\|\tilde L_0 v\|_{L^p(\tilde\Omega)} + \|v\|_{L^p(\tilde\Omega)}\right),$$

where c_1 depends only on Ω, p, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$. By easy computation, it follows from (3.3) that there exists a positive constant c_2, depending on the same parameters as c_1 and on the function ζ, such that

$$(3.4)\qquad \|u\|_{W^{2,p}(\Omega)} \le c_2\left(\|L_0 u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

On the other hand, by using Theorem 3.2 in [6], we obtain that for any $\varepsilon \in \mathbb{R}_+$ there is $c(\varepsilon) \in \mathbb{R}_+$ such that

$$\Big\| \sum_{i=1}^{2} a_i u_{x_i} + au \Big\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}(\Omega)} + c(\varepsilon)\|u\|_{L^p(\Omega)}, \tag{3.5}$$

where the constant $c(\varepsilon)$ depends on $\|a_i\|_{L^r(\Omega)}$, $\|a\|_{L^p(\Omega)}$, $\omega_r[a_i]$ and $\omega_p[a]$. The statement follows now directly from (3.4) and (3.5). $\square$

Our next two lemmas are regularity results for the operators L_0 and L, respectively.

Lemma 3.2. *Under the hypothesis (h_1), if u is a solution of the problem*

$$\begin{cases} u \in W^{2,q}(\Omega) \cap \overset{\circ}{W}{}^{1,q}(\Omega), \\ L_0 u \in L^p(\Omega), \end{cases} \tag{3.6}$$

with $q \leq p$, then u belongs to $W^{2,p}(\Omega)$.

Proof: It follows from (3.6) that

$$\begin{cases} v(x,t) = u(x)\zeta(t) \in W^{2,q}(\tilde{\Omega}) \cap \overset{\circ}{W}{}^{1,q}(\tilde{\Omega}), \\ \tilde{L}_0 v \in L^p(\tilde{\Omega}), \end{cases} \tag{3.7}$$

where $\zeta, \tilde{\Omega}$ and $\tilde{L}_0$ are those introduced in the proof of Lemma 3.1. An application of Theorem 4.2 in [5] to the problem (3.7) yields now that v lies in $W^{2,p}(\tilde{\Omega})$ and hence $u \in W^{2,p}(\Omega)$. $\square$

In the following we shall denote with (h_2') the condition obtained from (h_2) requiring that $r > 2$ also when $p < 2$.

Lemma 3.3. *Suppose that the conditions (h_1) and (h_2') are satisfied. If u is a solution of the problem*

$$\begin{cases} u \in W^{2,q}(\Omega) \cap \overset{\circ}{W}{}^{1,q}(\Omega), \\ Lu \in L^p(\Omega), \end{cases} \tag{3.8}$$

with $q \leq p$, then u belongs to $W^{2,p}(\Omega)$.

Proof: Obviously, it can be assumed that $q < p$. By Lemma 3.2 it is enough to show that $L_0 u$ belongs to $L^p(\Omega)$, so that, since

$$L_0 u = Lu - \sum_{i=1}^{2} a_i u_{x_i} - au,$$

we can even reduce to prove that

$$a_i u_{x_i} \in L^p(\Omega), \quad i = 1, 2. \tag{3.9}$$

Observe that there exist $k \in \mathbb{N} \setminus \{1\}$ and $\gamma \in]1, +\infty[$ such that

$$\begin{cases} \gamma^k = \frac{p}{q} > 1, \quad \gamma^{k-1} \neq \frac{2}{q}\,, \\ \frac{1}{\gamma} \geq 1 - q\Big(\frac{1}{2} - \frac{1}{r}\Big). \end{cases} \tag{3.10}$$

In fact, if we choose $k \in \mathbb{N} \setminus \{1\}$ such that

$$\Big(\frac{q}{p}\Big)^{\frac{1}{k}} \geq 1 - q\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad \frac{1}{k} \neq 1 - \log_{\frac{p}{q}} \frac{2}{q}$$

and $\gamma = \Big(\dfrac{p}{q}\Big)^{\frac{1}{k}}$, we have immediately that (3.10) holds. Put now

$$p_h = \gamma^h q, \quad h = 0, 1, \ldots, k. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{cases} p_0 = q,\ p_1 = \gamma q, \ldots, p_{k-1} = \gamma^{k-1} q \neq 2,\ p_k = p, \\ \frac{1}{p_{h+1}} \geq \frac{1}{p_h} + \frac{1}{r} - \frac{1}{2}, \quad h = 0, 1, \ldots, k-1. \end{cases} \tag{3.12}$$

The relation (3.9) can now be obtained by applying k times Theorem 3.1 of [6]. □

Consider the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega). \end{cases} \tag{3.13}$$

We prove here the following existence and uniqueness result dealing with the case $a = 0$.

Lemma 3.4. *Suppose that conditions (h_1) and (h_2') are satisfied, and assume also that $a = 0$. Then the problem (3.13) is uniquely solvable and there exists a positive constant c such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)}, \tag{3.14}$$

where c depends on Ω, p, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$, $\|a_i\|_{L^r(\Omega)}$, $\omega_r[a_i]$, and $p(a_{ij})$ is the extension of a_{ij} to $\mathbb{R}^2$ considered in Lemma 3.1.

PROOF: Assume that u is a solution of (3.13) with $f = 0$, so that u belongs to $W^{2,2}(\Omega)$ by Lemma 3.3. Then u is also a solution of the problem

$$\begin{cases} u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = 0, \end{cases} \tag{3.15}$$

and hence $u = 0$ by the Pucci-Alexandrov maximum principle. The uniqueness of the solution of (3.13) follows.

The above uniqueness result and Lemma 3.1 allow to use the same methods of the proof of Theorem 2.4 in [11], in order to obtain the estimate (3.14) and then the existence of the solution of (3.13). □

It is now possible to obtain the main result of the paper.

Theorem 3.5. *Suppose that conditions* (h_1) *and* (h_2') *are satisfied, and assume also that* $\operatorname{ess\,inf}_\Omega a \geq 0$. *Then the problem* (3.13) *is uniquely solvable and the solution u satisfies the a priori bound*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)}, \tag{3.16}$$

with $c \in \mathbb{R}_+$ depending on Ω, p, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$, $\|a_i\|_{L^r(\Omega)}$, $\|a\|_{L^p(\Omega)}$, $\omega_r[a_i]$, $\omega_p[a]$ and where $p(a_{ij})$ is the extension of a_{ij} to $\mathbb{R}^2$ considered in Lemma 3.1.

PROOF: The proof can be obtained with the same arguments used in [12, Theorem 2.1], replacing Theorems 2.2 and 2.3 of [12] by our Lemmas 3.1, 3.3 and 3.4. □

REFERENCES

[1] Astala K., Iwaniec T., Martin G., *Pucci's conjecture and the Alexandrov inequality for elliptic PDEs in the plane*, to appear.

[2] Caso L., Cavaliere P., Transirico M., *Uniqueness results for elliptic equations with VMO-coefficients*, Int. J. Pure Appl. Math. **13** (2004), 499–512.

[3] Caso L., Cavaliere P., Transirico M., *An existence result for elliptic equations with VMO-coefficients*, J. Math. Anal. Appl., to appear.

[4] Chiarenza F., Frasca M., Longo P., *Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149–168.

[5] Chiarenza F., Frasca M., Longo P., *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.

[6] Glushak A.V., Transirico M., Troisi M., *Teoremi di immersione ed equazioni ellittiche in aperti non limitati*, Rend. Mat. Appl. (7) **9** (1989), 113–130.

[7] Maugeri A., Palagachev D.K., Softova L.G., *Elliptic and Parabolic Equations with Discontinuous Coefficients*, Wiley, Berlin, 2000.

[8] Pucci C., *Equazioni ellittiche con soluzioni in $W^{2,p}$, $p < 2$*, Atti del Convegno sulle Equazioni alle Derivate Parziali (Bologna, 1967), pp. 145–148; MR42-6413.

[9] Talenti G., *Equazioni lineari ellittiche in due variabili*, Matematiche (Catania) **21** (1966), 339–376.

[10] Transirico M., Troisi M., Vitolo A., *BMO spaces on domains of* $\mathbb{R}^n$, Ricerche Mat. **45** (1996), 355–378.

[11] Vitanza C., *$W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients*, Matematiche (Catania) **47** (1992), 177–186.

[12] Vitanza C., *A new contribution to the $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients*, Matematiche (Catania) **48** (1993), 287–296.

Dipartimento di Ingegneria dell'Informazione e Matematica Applicata, Facoltà di Scienze MM.FF.NN., Università di Salerno, via Ponte Don Melillo, I 84084 Fisciano (SA), Italy

E-mail: pcavaliere@unisa.it

Dipartimento di Matematica e Informatica, Facoltà di Scienze MM.FF.NN., Università di Salerno, via Ponte Don Melillo, I 84084 Fisciano (SA), Italy

E-mail: mtransirico@unisa.it

(*Received* November 15, 2004)