Some results on the recognizability of the linear groups over the binary field

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Abstract. In this paper, we first find the set of orders of all elements in some special linear groups over the binary field. Then, we will prove the characterizability of the special linear group $\mathrm{PSL}(13,2)$ using only the set of its element orders.

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1. Introduction

Let G be a finite group and $\pi_e(G)$ be the set of orders of all elements in G. Clearly, $\pi_e(G)$ is a subset of the set of natural numbers, also $\pi_e(G)$ is closed and partially ordered by the divisibility relation. Hence, $\pi_e(G)$ is uniquely determined by $\mu(G)$, the set of elements that are maximal under the divisibility relation. If Ω is a subset of natural numbers, then $h(\Omega)$ denotes the number of non-isomorphic finite groups G such that $\pi_e(G) = \Omega$. It is clear that $h(\pi_e(G)) \geq 1$, for any group G. Following W.J. Shi, we say that a finite group G is *non-distinguishable* if $h(\pi_e(G)) = \infty$; and *distinguishable* if $h(\pi_e(G)) < \infty$. Moreover, a distinguishable group G is called *k-distinguishable* if $h(\pi_e(G)) = k$ $(k < \infty)$. Usually, a 1-distinguishable group G is called a *recognizable* (or *characterizable*) group.

To every finite group G we associate a graph known as its *prime graph* denoted by $\Gamma(G) = (V(G), E(G))$. For this graph we have $V(G) = \pi(G)$, the set of all prime divisors of the order of G, and for two vertices $p, q \in V(G)$ we have $\{p, q\} \in E(G)$ if and only if $pq \in \pi_e(G)$. Denote the connected components of $\Gamma(G)$ by $\pi_i(G) = \pi_i$, $i = 1, 2, \dots, t(G)$, where $t(G)$ is the number of connected components. If $2 \in \pi(G)$ we set $2 \in \pi_1$.

By [8] we have

$$t(\mathrm{PSL}(n,2)) = \begin{cases} 1 & \text{if } \ n \neq p, \ p+1; \\ 2 & \text{if } \ n = p \ \text{ or } \ p+1, \end{cases}$$

where p is an odd prime number. When $n = p$ or $p + 1$, $\mathrm{PSL}(n, 2)$ has two components, one of them is

$$\pi_1 = \pi(2\textstyle\prod_{i=1}^{p-1}(2^i - 1)), \quad (\text{resp.} \quad \pi_1 = \pi(2(2^{p+1} - 1)\prod_{i=1}^{p-1}(2^i - 1)))$$

and the other in any case is

$$\pi_2 = \pi(2^p - 1).$$

Characterization of finite groups through their element orders is one of the most interesting problems in finite group theory. This problem was first introduced by W.J. Shi in [15]. There are some results in the literature showing that certain groups are characterizable (see references and Table 1 in [12]). In particular, it was proved that the following simple groups are characterizable: $\mathrm{PSL}(n, 2)$ for $n = 3, 4, 5, 6, 7, 8$ (see [17], [13], [3], [16], [4], [5]). About simple groups $\mathrm{PSL}(9, 2)$ and $\mathrm{PSL}(10, 2)$, the problem is still open. In fact, since the prime graphs of these groups are connected, the problem is more difficult. Moreover, in [3], Darafsheh and Moghaddamfar put forward the following conjecture:

Conjecture. *For all positive integers $n \geq 3$, the simple groups* $\mathrm{PSL}(n, 2)$ *are characterizable.*

In this paper, we will prove that the conjecture is correct for the special linear group $\mathrm{PSL}(13, 2)$. Finally we will prove that:

Main Theorem. *Let G be a finite group. Then $G \cong \mathrm{PSL}(13, 2)$ if and only if $\pi_e(G) = \pi_e(\mathrm{PSL}(13, 2))$.*

In the following, groups considered are finite and simple groups are non-abelian. We also use the notation $\mathrm{PSL}(n, q)$ or $L_n(q)$ for the projective special linear group. It is clear that $\mathrm{PSL}(n, 2) = \mathrm{GL}(n, 2)$.

2. On set of orders of elements in $\mathrm{PSL}(n, 2)$

In this section, we first introduce some notation which are taken from [7]. Then we continue to find some properties of the set $\pi_e(\mathrm{PSL}(n, 2))$, whose proof is based on examining the structure of $\mathrm{PSL}(n, 2)$ and on several arithmetic arguments. Finally, we calculate the set of orders of elements of the projective special linear groups $\mathrm{PSL}(n, 2)$ where $n = 9, 10, 11, 12, 13$.

Notation. Let $f(x) = x^m - a_{m-1}x^{m-1} - \ldots - a_0$ be a polynomial over $\mathrm{GF}(q)$ of degree m. Using Green's notation ([7]), let

$$U(f) = U_1(f) := \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ a_0 & a_1 & a_2 & a_3 & \ldots & a_{m-1} \end{bmatrix}$$

denote its companion matrix. We also set

$$U_l(f) := \begin{bmatrix} U(f) & 1_m & 0 & 0 & \dots & 0 \\ 0 & U(f) & 1_m & 0 & \dots & 0 \\ 0 & 0 & U(f) & 1_m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & U(f) \end{bmatrix},$$

with l diagonal blocks $U(f)$, where 1_m is the identity matrix.

If $\lambda = \{l_1, l_2, \dots, l_p\}$ is a partition of a positive integer k whose p parts are written in descending order *i.e.*

$$l_1 \geq l_2 \geq \dots \geq l_p > 0,$$

then we set

$$U_\lambda(f) := \mathrm{diag}\{U_{l_1}(f), U_{l_2}(f), \dots, U_{l_p}(f)\}.$$

We denote by $c(n, q)$ the number of conjugacy classes of $\mathrm{GL}(n, q)$. In general, there is a generating function for $c(n, q)$ as follows:

$$\sum_{n=0}^{\infty} c(n, q) x^n = \prod_{m=1}^{\infty} p(x^m)^{w(m,q)}, \tag{1}$$

where

$$w(m, q) = \frac{1}{m} \sum_{k|m} \mu(k) q^{m/k} \tag{2}$$

is the number of irreducible polynomials $f(x)$ of degree m over $\mathrm{GF}(q)$. We recall that in equations (1) and (2)

$$p(x) = \frac{1}{(1-x)(1-x^2)\dots} = \sum_{d=0}^{\infty} p_d x^d \tag{3}$$

is the partition function (in this power series the coefficient p_d is the number of partitions of d), and μ is the Möbius function.

By definition the *order* of $f(x) \in \mathrm{GF}(q)[x]$, $f(0) \neq 0$, is the smallest natural number e such that $f(x) \big| x^e - 1$, denoted by $\mathrm{ord}(f)$. If A is an element of $\mathrm{GL}(n, q)$ with minimal polynomial $f(x)$, then it is known that the order of A in $\mathrm{GL}(n, q)$ is equal to the order of $f(x)$. Therefore finding the orders of polynomials over

GF(q) is important for finding the orders of elements in the general linear groups. Since we are interested in GL($n,2$), in what follows we turn to the field GF(2).

Suppose that $A \in \mathrm{GL}(n,2)$ has characteristic polynomial $f_{m_1}^{k_1} f_{m_2}^{k_2} \dots f_{m_s}^{k_s}$, where f_{m_i}, $1 \leq i \leq s$, is an irreducible polynomial over GF(2) of degree m_i, and $k_1, k_2, \dots, k_s$ are positive integers. Evidently $\sum_{i=1}^{n} m_i k_i = n$, and moreover A is conjugate to one of the matrices of the form

$$\mathrm{diag}\{U_{\nu_1}(f_{m_1}), U_{\nu_2}(f_{m_2}), \dots, U_{\nu_s}(f_{m_s})\}$$

in GL($n,2$), where $\nu_1, \nu_2, \dots, \nu_s$ are certain partitions of $k_1, k_2, \dots, k_s$ respectively. In this case, we denote the conjugacy class c of A by the symbol

$$c = (f_{m_1}^{\nu_1} f_{m_2}^{\nu_2} \dots f_{m_s}^{\nu_s}).$$

Furthermore, if B is conjugate to $\mathrm{diag}\{U_{k_1}(f_{m_1}), U_{k_2}(f_{m_2}), \dots, U_{k_s}(f_{m_s})\}$, then we have

$$o(B) = \mathrm{l.c.m.}\,\{o(U_{k_1}(f_{m_1})), o(U_{k_2}(f_{m_2})), \dots, o(U_{k_s}(f_{m_s}))\}. \tag{4}$$

Hence, among all elements of GL($n,2$) having the same characteristic polynomial $f_{m_1}^{k_1} f_{m_2}^{k_2} \dots f_{m_s}^{k_s}$, $o(B)$ is maximal.

If $w(d,2) = k$, then it means that there are k irreducible polynomials of degree d over GF(2), say, $g_1, g_2, \dots, g_k$. Certainly $o(U_1(g_i)) = \mathrm{ord}(g_i)$ divides $2^d - 1$, and also there exist g_j, $1 \leq j \leq k$, such that $o(U_1(g_j)) = \mathrm{ord}(g_j) = 2^d - 1$.

We are now ready for the first result.

Lemma 1. *Let $n = k_1m_1 + k_2m_2 + \cdots + k_sm_s$, where $k_1, k_2, \dots, k_s$, $m_1, m_2, \dots, m_s$ are positive integers and $n \geq 3$. Let $e = \mathrm{l.c.m.}\,(2^{m_1} - 1, 2^{m_2} - 1, \dots, 2^{m_s} - 1)$ and t be the smallest integer with $2^t \geq \max(k_1, k_2, \dots, k_s)$. Then $2^t e \in \pi_e(\mathrm{PSL}(n,2))$.*

Proof: Let $A \in \mathrm{PSL}(n,2) \cong \mathrm{GL}(n,2)$ have characteristic polynomial $f = f_{m_1}^{k_1} f_{m_2}^{k_2} \dots f_{m_s}^{k_s}$, where f_{m_i} is an irreducible polynomial over GF(2) of degree m_i and $o(U_1(f_{m_i})) = \mathrm{ord}(f_{m_i}) = 2^{m_i} - 1$. Now, if A is conjugate to

$$\mathrm{diag}\{U_{k_1}(f_{m_1}), U_{k_2}(f_{m_2}), \dots, U_{k_s}(f_{m_s})\},$$

then by (4) and Theorem 3.8 in [10], we obtain that

$$\begin{aligned}
o(A) &= \mathrm{l.c.m.}\,\{o(U_{k_1}(f_{m_1})), o(U_{k_2}(f_{m_2})), \dots, o(U_{k_s}(f_{m_s}))\} \\
&= \mathrm{l.c.m.}\,\{2^{t_1}\,\mathrm{ord}(f_{m_1}), 2^{t_2}\,\mathrm{ord}(f_{m_2}), \dots, 2^{t_s}\,\mathrm{ord}(f_{m_s})\} \\
&= 2^t \times \mathrm{l.c.m.}\,\{\mathrm{ord}(f_{m_1}), \mathrm{ord}(f_{m_2}), \dots, \mathrm{ord}(f_{m_s})\} \\
&= 2^t \times \mathrm{l.c.m.}\,\{2^{m_1} - 1, 2^{m_2} - 1, \dots, 2^{m_s} - 1\},
\end{aligned}$$

where t_i is the smallest integer with $2^{t_i} \geq k_i$, and $t = \max(t_1, t_2, \dots, t_s)$. □

Corollary 1. *The following statements hold.*

(a) *For $i \geq 1$ we have $\pi_e(\mathrm{PSL}(i,2)) \subset \pi_e(\mathrm{PSL}(i+1,2))$. In particular, for every $1 \leq i \leq n$, $2^i - 1$ belong to $\pi_e(\mathrm{PSL}(n,2))$.*

(b) *$(q^n-1)/d(q-1)$ and $(q^{n-1}-1)/d$ belong to $\mu(L_n(q))$, where $d = (n, q-1)$. In particular, $2^{n-1}-1$ and 2^n-1 belong to $\mu(L_n(2))$.*

(c) *$2^s \in \mu(\mathrm{PSL}(n,2))$, where s is the smallest integer with $2^s \geq n$.*

(d) *$2^n - 1$ is the maximal number in $\mu(\mathrm{PSL}(n,2))$.*

(e) *$k(2^{n-2}-1) \in \pi_e(\mathrm{PSL}(n,2))$ if and only if $k = 1, 2$ or 3. Moreover $2(2^{n-2}-1)$ is the maximal even number in $\mu(\mathrm{PSL}(n,2))$.*

(f) *Let $n = \sum_{i=1}^{s} m_i$, where $m_1, m_2, \dots, m_s$ are positive integers and for every $i, j = 1, 2, \dots, s$, $(m_i, m_j) = 1$. Then $\prod_{i=1}^{s}(2^{m_i}-1) \in \mu(\mathrm{PSL}(n,2))$.*

Proof: (a) Evidently, $\mathrm{PSL}(i,2) \hookrightarrow \mathrm{PSL}(i+1,2)$ for every $i \geq 1$, and so $\pi_e(\mathrm{PSL}(i,2)) \subseteq \pi_e(\mathrm{PSL}(i+1,2))$. Moreover, $\mathrm{PSL}(i,2)$ contains a Singer cycle of order $2^i - 1$. This proves part (a).

(b) These facts are quite well-known, see for instance [9].

(c) Take $A \in c = (f_1^n)$, where f_1 is irreducible polynomial over GF(2) of degree 1. Then the result follows from Lemma 1.

(d) Let $A \in \mathrm{PSL}(n,2)$ have characteristic polynomial $f = f_{m_1}^{k_1} f_{m_2}^{k_2} \dots f_{m_s}^{k_s}$, where f_{m_i} is an irreducible polynomial over GF(2) of degree m_i. Assume that t and e are as in Lemma 1. Clearly $o(A)$ divides $2^t e$, and by noticing that $\sum_{i=1}^{n} m_i k_i = n$, we conclude that

$$o(A) \leq 2^t e \leq 2^t(2^{m_1}-1)(2^{m_2}-1)\dots(2^{m_s}-1) \leq 2^{(t+\sum_{i=1}^{s} m_i)} - 1 \leq 2^n - 1,$$

completing the part (d).

(e) Assume that $A \in \mathrm{PSL}(n,2)$. Evidently $A \in c = (f_{n-2} f_i^k)$ if and only if $(i,k) = (1,2)$ or $(2,1)$. Using (4), in the first case we have $o(A) = 2(2^{n-2}-1)$, and in the latter case we have

$$o(A) = \begin{cases} 2^{n-2} - 1 & \text{if } n \text{ is even,} \\ 3(2^{n-2}-1) & \text{if } n \text{ is odd.} \end{cases}$$

Hence $k(2^{n-2}-1) \in \pi_e(\mathrm{PSL}(n,2))$ if and only if $k = 1, 2$ or 3. Now, with the same argument as in (d) we may prove that $2(2^{n-2}-1)$ is the maximal even number in $\pi_e(\mathrm{PSL}(n,2))$, which implies that $2(2^{n-2}-1) \in \mu(\mathrm{PSL}(n,2))$.

(f) Assume that $A \in c = (f_{m_1}, f_{m_2}, \dots, f_{m_s})$, where f_{m_i} is an irreducible polynomial over GF(2) of degree m_i, and $o(U_1(f_{m_i})) = 2^{m_i}-1$. Now by Lemma 1, it is easy to see that

$$o(A) = \mathrm{l.c.m.}\,(2^{m_1}-1, 2^{m_2}-1, \dots, 2^{m_s}-1) = \prod_{i=1}^{s}(2^{m_i}-1),$$

since $(2^{m_i}-1, 2^{m_j}-1) = 2^{(m_i,m_j)}-1 = 1$.

Now, we show that $o(A) \in \mu(\mathrm{PSL}(n,2))$. Assume that $\mathrm{PSL}(n,2)$ contains an element, say, B, such that $o(A)$ divides $o(B)$. Suppose that $B \in (f_{d_1}^{k_1} f_{d_2}^{k_2} \dots f_{d_z}^{k_z})$, where f_{d_j}, $1 \le j \le z$, is an irreducible polynomial over GF(2) of degree d_j, $\sum_{j=1}^{z} k_j d_j = n$ and $o(U_1(f_{d_j})) = 2^{d_j}-1$. From Lemma 1, we have $o(B) = 2^t e$, where t is the least positive integer such that $2^t \ge \max(k_1, k_2, \dots, k_z)$ and $e =$ l.c.m. $(2^{d_1}-1, 2^{d_2}-1, \dots, 2^{d_z}-1)$. Since $o(A)$ divides $o(B)$, for each i there is a $j_{(i)}$ such that $m_i \mid d_{j_{(i)}}$, $1 \le i \le s$. Hence $m_i \le d_{j_{(i)}}$ implying $\sum_{i=1}^{s} m_1 \le \sum_{i=1}^{s} d_{j_{(i)}}$. On the other hand, since $k_i \ge 1$, we obtain

$$\sum_{j=1}^{z} d_j \le \sum_{i=1}^{z} k_j d_j = \sum_{i=1}^{s} m_1 \le \sum_{i=1}^{s} d_{j_{(i)}}.$$

Therefore equality holds, which implies $s = z$, $k_i = 1$ and $d_{j_{(i)}} = m_i$, for all i, $1 \le i \le s$. Hence $A = B$, and the result follows. $\square$

Lemma 2. *Let $G = \mathrm{PSL}(n,2)$, where $n = 9, 10, 11, 12, 13$. Then $\mu(G)$ is given as in Table 4.*

Proof: First of all, using (2) and (3) we calculate the values of $w(n,2)$ and p_n where $1 \le n \le 13$, and list them in Table 1.

Table 1. The number of irreducible polynomials of degree n over $\mathbb{Z}_2$, and the number of partitions of n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$w(n,2)$	2	1	2	3	6	9	18	30	56	99	186	335	630
p_n	1	2	3	5	7	11	15	22	30	42	56	77	101

Next, we calculate the values of $c(n,2)$, where $1 \le n \le 13$. In fact, using (1) we obtain

$$(5)\quad \sum_{n=0}^{\infty} c(n,2)x^n = \prod_{m=1}^{\infty} p(x^m)^{w(m,2)} = 1 + x + 3x^2 + 6x^3 + 14x^4 + 27x^5 + 60x^6$$
$$+117x^7 + 246x^8 + 490x^9 + 1002x^{10} + 1998x^{11} + 4031x^{12} + 8066x^{13} + \cdots .$$

Finally, we calculate $o(U_k(f_i))$ with $k \ge 2$ and $ki \le 13$, which is given in Table 2.

TABLE 2. The order of $A \in \mathrm{GL}(ki, 2)$ having characteristic polynomial f_i^k, where $k > 1$ and $ki \leq 13$.

k	f_1	f_2	f_3	f_4	f_5	f_6
2	2	6	14	30	62	126
3	4	12	28	60		
4	4	12	28			
$5 \leq k \leq 6$	8	24				
$7 \leq k \leq 8$	8					
$9 \leq k \leq 13$	16					

First suppose $G = \mathrm{PSL}(9, 2)$. By (5), G contains 490 conjugacy classes. Now, by the previous explanations and using Lemma 1 and Table 2, we can easily list these conjugacy classes and find the maximum order for all elements in the conjugacy classes having the same characteristic polynomial, which is denoted by m in the last column of Table 3. Therefore, we derive the set $\mu(G)$ from this column, as required. Note that, in Table 3, $\mathrm{Par}(k)$ denotes the set of partitions of k. We can find $\mu(\mathrm{PSL}(n, 2))$ for $n = 10, 11, 12$ and 13, in a similar manner. The final result is tabulated in Table 4. □

TABLE 3. The order of elements of the simple group $\mathrm{PSL}(9, 2)$.

Type of c	Conditions	Number	m
(f_1^r)	$r \in \mathrm{Par}(9)$	30	$16 = 2^4$
$(f_2^r f_1)$	$r \in \mathrm{Par}(4)$	5	$12 = 2^2.3$
$(f_2^r f_1^s)$	$r, s \in \mathrm{Par}(3)$	9	$12 = 2^2.3$
$(f_2^r f_1^s)$	$r \in \mathrm{Par}(2), s \in \mathrm{Par}(5)$	14	$24 = 2^3.3$
$(f_2 f_1^r)$	$r \in \mathrm{Par}(7)$	15	$24 = 2^3.3$
(f_3^r)	$r \in \mathrm{Par}(3)$	10	$28 = 2^2.7$
$(f_3^r f_2 f_1)$	$r \in \mathrm{Par}(2)$	5	$42 = 2.3.7$
$(f_3^r f_1^s)$	$r \in \mathrm{Par}(2),\ s \in \mathrm{Par}(3)$	15	$28 = 2^2.7$
$(f_3 f_2^r)$	$r \in \mathrm{Par}(3)$	6	$84 = 2^2.3.7$
$(f_3 f_2^r f_1^s)$	$r, s \in \mathrm{Par}(2)$	8	$42 = 2.3.7$
$(f_3 f_2 f_1^r)$	$r \in \mathrm{Par}(4)$	10	$84 = 2^2.3.7$
$(f_3 f_1^r)$	$r \in \mathrm{Par}(6)$	22	$56 = 2^3.7$
$(f_4^r f_1)$	$r \in \mathrm{Par}(2)$	9	$30 = 2.3.5$
$(f_4 f_3 f_2)$		6	$105 = 3.5.7$
$(f_4 f_3 f_1^r)$	$r \in \mathrm{Par}(2)$	12	$210 = 2.3.5.7$
$(f_4 f_2^r f_1)$	$r \in \mathrm{Par}(2)$	6	$30 = 2.3.5$
$(f_4 f_2 f_1^r)$	$r \in \mathrm{Par}(3)$	9	$60 = 2^2.3.5$
$(f_4 f_1^r)$	$r \in \mathrm{Par}(5)$	21	$120 = 2^3.3.5$

Type of c	Conditions	Number	m
(f_5f_4)		18	$465 = 3.5.31$
$(f_5f_3f_1)$		12	217=7.31
$(f_5f_2^r)$	$r \in \mathrm{Par}(2)$	12	$186 = 2.3.31$
$(f_5f_2f_1^r)$	$r \in \mathrm{Par}(2)$	12	$186 = 2.3.31$
$(f_5f_1^r)$	$r \in \mathrm{Par}(4)$	30	$124 = 2^2.31$
(f_6f_3)		18	$63 = 3^2.7$
$(f_6f_2f_1)$		9	$63 = 3^2.7$
$(f_6f_1^r)$	$r \in \mathrm{Par}(3)$	27	$252 = 2^2.3^2.7$
(f_7f_2)		18	$381 = 3.127$
$(f_7f_1^r)$	$r \in \mathrm{Par}(2)$	36	$254 = 2.127$
(f_8f_1)		30	$255 = 3.5.17$
(f_9)		56	$511 = 7.73$
Total		490	

TABLE 4. *The set of orders of elements of some projective special linear groups.*

G	$\mu(G)$
$\mathrm{PSL}(9,2)$	16, 56, 120, 124, 186, 210, 217, 252, 254, 255, 381, 465, 511
$\mathrm{PSL}(10,2)$	16, 120, 168, 248, 252, 315, 372, 381, 420, 434, 465, 508, 510, 511, 651, 889, 930, 1023
$\mathrm{PSL}(11,2)$	48, 120, 248, 315, 372, 420, 504, 508, 762, 868, 889, 930, 1020, 1022, 1023, 1533, 1785, 1905, 1953, 2047
$\mathrm{PSL}(12,2)$	48, 112, 504, 630, 744, 840, 868, 1016, 1020, 1302, 1524, 1533, 1785, 1778, 1860, 1905, 1953, 2044, 2046, 2047, 2667, 3255, 3937, 4095
$\mathrm{PSL}(13,2)$	112, 240, 504, 744, 840, 1016, 1260, 1524, 1736, 1860, 2040, 2044, 2604, 3066, 3255, 3556, 3570, 3810, 3906, 3937, 4092, 4094, 4095, 6141, 7161, 7665, 7905, 8001, 8191

3. Recognizing $\mathrm{PSL}(13,2)$ by its order elements

Our main result of this section is the characterization of $\mathrm{PSL}(13,2)$ by its order elements, in fact we prove the statement of the Main Theorem.

We begin with a well-known theorem due to Gruenberg and Kegel.

Gruenberg-Kegel Theorem (see [18, Theorem A]). *If G is a finite group with disconnected prime graph $\Gamma(G)$ then one of the following holds.*

(1) *$t(G) = 2$ and G is either a Frobenius or a 2-Frobenius group.*
(2) *G is an extension of a $\pi_1(G)$-group N by a group G_1, where $P \trianglelefteq G_1 \trianglelefteq \mathrm{Aut}(P)$, P is a non-abelian simple group and G_1/P is a $\pi_1(G)$-group. Moreover $t(P) \geq t(G)$ and for every i, $2 \leq i \leq t(G)$, there exists j, $2 \leq j \leq t(P)$, such that $\pi_j(P) = \pi_i(G)$.*

By Lemma 2, we know that $26 \notin \mu(\mathrm{PSL}(13,2))$. Now, in the following lemma we prove that there is an outer automorphism of $\mathrm{PSL}(13,2)$ of order 26, which certainly proves that $\pi_e(\mathrm{PSL}(13,2)) \subsetneqq \pi_e(\mathrm{Aut}(\mathrm{PSL}(13,2)))$.

Lemma 3. *The group* $\mathrm{Aut}(\mathrm{PSL}(13,2))$ *contains an element of order* 26.

Proof: Let θ be an involuntary graph automorphism of $G = \mathrm{PSL}(13,2)$. Using the notation in [2] we have $G^+ = \mathrm{Aut}(G) = G \cdot \langle\theta\rangle = G \cup \theta G$. The conjugacy classes of G^+ which lie in θG are called negative classes and by Theorem 1 in [2], G^+ has only one negative conjugacy class of involutions with representative θI and we have

$$|C_{G^+}(\theta I)| = 2^{37}(2^2-1)(2^4-1)\cdots(2^{12}-1) = 2^{37}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 31,$$

where I is the identity matrix. Now, we deduce that $2\cdot 13 \in \pi_e(G^+)$. □

Lemma 4. *If G is a simple group of Lie type such that*

$$8191 \in \pi(G) \subseteq \pi(\mathrm{PSL}(13,2)),$$

then G is isomorphic to $\mathrm{PSL}(13,2)$ *or* $\mathrm{PSL}(2,8191)$.

Proof: Suppose G is a finite simple group of Lie type over a finite field of order $q = p^n$, where p is a prime and n is a natural number. Evidently $p \in \pi(G)$, hence p may be equal to $2, 3, 5, 7, 11, 13, 17, 23, 31, 73, 89, 127$ or 8191. If $p = 2$, then it is clear that the order of 2 modulo 8191 is 13, and there is no natural number m such that $2^m + 1 \equiv 0 (\mathrm{mod}\ 8191)$. Thus if $2^k - 1$ divides $|G|$ and $8191 \in \pi(2^k - 1)$, for some k, then k must be a multiple of 13. Therefore, from Table 6 in [1], the only candidate for G under our assumptions is $A_{12}(2) \cong \mathrm{PSL}(13,2)$. Suppose $p = 3$. In this case the calculations show that the order of 3 modulo 8191 is greater than 100. Now, if G is a simple group of Lie type in characteristic 3, the order of which is divisible by 8191, from Table 6 in [1], no candidates for G will arise. Similarly for $p = 5, 7, 11, 13, 17, 23, 31, 73, 89, 127$, there is no group with the above property. If $p = 8191$, then q must be a power of 8191 and the only possible group is $G = A_1(8191) \cong \mathrm{PSL}(2,8191)$. The lemma is proved. □

Lemma 5. *The special linear group* $\mathrm{PSL}(13,2)$ *has a Frobenius subgroup of shape* $3^3 : 13$.

Proof: Let F be the finite field with 3^3 elements. Let H be the cyclic subgroup of $F^\times$ with order 13. Then it is easy to verify that

$$G = \{f_{a,b} : F \to F \mid f_{a,b}(x) = ax + b,\ a \in H,\ b \in F,\ \forall\, x \in F\}$$

is a Frobenius group with complement H and kernel the additive group of F which is an elementary abelian group of order 3^3. Therefore the existence of the Frobenius group G of the shape $3^3 : 13$ is established.

But from [6, p. 68] the complex character table of the group G can be constructed. In particular, G has 13 irreducible complex characters of degree 1 and two with degree 13, apparently the characters of degree 13 are faithful. But it is well-known that the degrees of the irreducible characters of any finite group G over a field whose characteristic does not divide the order of G are the same as the degrees of ordinary irreducible characters of G. Therefore the Frobenius group $G = 3^3 : 13$ has an irreducible character of degree 13 over a field with characteristic 2 which is denoted by K. Therefore we have the faithful representation $G \to \mathrm{GL}(13, K)$ affording the character of degree 13. But the above representation can be realized over a field with two elements. Therefore we have a monomorphism $G \to \mathrm{GL}(13, 2)$ proving the lemma. $\square$

Now we are able to prove the Main Theorem of this section.

Proof of Main Theorem: We only need to prove the sufficiency part. Let G be a finite group for which $\pi_e(G) = \pi_e(\mathrm{PSL}(13, 2))$. Then $t(G) = 2$ and the connected components of the prime graph of $\Gamma(G)$ are:

$$\pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 23, 31, 73, 89, 127\} \quad \text{and} \quad \pi_2(G) = \{8191\}$$

(see [8]). We will prove that G is isomorphic to $\mathrm{PSL}(13, 2)$. First of all, from [11] G is non-soluble and so G is not a 2-Frobenius group. On the other hand, since $2 \cdot 13 \notin \pi_e(G)$, by Lemma 2.7 in [3] it follows that G cannot be a Frobenius group.

Now, we adhere to the notation of item (2) of the Gruenberg-Kegel theorem. We invoke the Classification of Finite Simple Groups to eliminate all possibilities for P (see [1]). Note that $\pi_2(G) = \pi_j(P) = \{8191\}$ for some $j \geq 2$. We claim that $P \cong \mathrm{PSL}(13, 2)$. If P is an alternating group A_n, $n \geq 5$, then since $8191 \in \pi(P)$, we deduce $n \geq 8191$. But then $19 \in \pi(G)$, which is a contradiction. Also, P cannot be a sporadic simple group, because otherwise the maximum prime in P would be 71, whereas $8191 \in \pi(P)$, which is a contradiction. Finally, we assume that P is a simple group of Lie type. In this case, by Lemma 4, we obtain $P \cong \mathrm{PSL}(13, 2)$ or $\mathrm{PSL}(2, 8191)$. If $P \cong \mathrm{PSL}(2, 8191)$, then we get $4096 \in \pi_e(P) \setminus \pi_e(G)$, which is a contradiction. Therefore, $P \cong \mathrm{PSL}(13, 2)$, as claimed.

Now, we show that $N = 1$. Assume the contrary. Without loss of generality we may assume that $N = O_r(G)$ for some prime $r \in \pi_1(G)$. Moreover, we may assume that N is an elementary abelian subgroup and $C_{G_1}(N) = N$. Let $K = \langle A \rangle$ where $A \in \mathrm{GL}(12, 2)$ and $o(A) = 2^{12} - 1$. Then

$$L = \left\{ \left[\begin{array}{c|c} 1 & a_1 \;\; a_2 \;\; \ldots \;\; a_{12} \\ \hline 0 & X \end{array} \right] \,\Big|\, X \in K, a_i \in GF(2), 1 \leq i \leq 12 \right\} \leq \mathrm{PSL}(13, 2).$$

We put

$$S = \left\{ \left[\begin{array}{c|c} 1 & a_1 \;\; a_2 \;\; \ldots \;\; a_{12} \\ \hline 0 & I \end{array} \right] \,\Big|\, a_i \in GF(2), 1 \leq i \leq 12 \right\},$$

and

$$T = \left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array} \right] \,\middle|\, X \in K \right\}.$$

Then we form the semi-direct product $L = T \rtimes S = 2^{12} : (2^{12}-1) \leq \mathrm{PSL}(13,2) \leq G_1 \leq G/N$. Now, if $r \neq 2$, then by Lemma 6 in [14], we get $r \cdot (2^{12}-1) \in \pi_e(G)$, which contradicts Lemma 2. Thus, N is a non-trivial 2-subgroup. In this case, we have $3^3 : 13 < \mathrm{PSL}(13,2)$ by Lemma 5, and again by Lemma 6 in [14], we obtain $26 \in \pi_e(G)$, which contradicts Lemma 2. Therefore $N = 1$.

Finally, we claim that $G \cong \mathrm{PSL}(13,2)$. Because $N = 1$, we obtain

$$\mathrm{PSL}(13,2) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(13,2)).$$

Moreover, since $|\,\mathrm{Out}(\mathrm{PSL}(13,2))| = 2$, it follows that $G \cong \mathrm{PSL}(13,2)$ or $G \cong \mathrm{Aut}(\mathrm{PSL}(13,2))$. However, by Lemma 3 we have $\pi_e(G) \subsetneqq \pi_e(\mathrm{Aut}(\mathrm{PSL}(13,2)))$, which implies that $G \cong \mathrm{PSL}(13,2)$, as claimed. $\square$

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