Property (a) and dominating families

Samuel Gomes da Silva

Abstract. Generalizations of earlier negative results on Property (a) are proved and two questions on an (a)-version of Jones' Lemma are posed. We discuss these questions in the realm of locally compact spaces. Using dominating families of functions as a tool, we prove that under the assumptions "2^ω is regular" and "$2^\omega < 2^{\omega_1}$" the existence of a T_1 separable locally compact (a)-space with an uncountable closed discrete subset implies the existence of inner models with measurable cardinals. We also use cardinal invariants such as $\mathfrak{d}$ to prove results in the class of locally compact spaces that strengthen, in such class, the negative results mentioned above.

Keywords: property (a), dominating families, small cardinals, inner models of measurability

Classification: Primary 54A25, 54D20; Secondary 54A35

1. Introduction

A topological space satisfies *Property* (a) (or is said to be an (a)-*space*) if for every open cover $\mathcal{U}$ of X and for every dense set $D \subseteq X$ there is a closed and discrete subset $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$ (where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$). Property (a) was introduced by Matveev in [M97] in order to investigate the absoluteness condition in the definition of absolute countable compactness ([M94]). These classes of spaces were motivated by the following characterization of countable compactness: a Hausdorff space X is countably compact if and only if for every open cover $\mathcal{U}$ of X there is a finite subset $F \subseteq X$ such that $X = \mathrm{St}(F,\mathcal{U})$ (3.12.23(d) in [E]). Several questions and results on such spaces may be found in [M94], [M97] and [JMS].

A family of functions is said to be a *dominating family* in an ordered space of functions if it is cofinal in the corresponding order; e.g., in the mod finite order in the functions from ω to ω a family $D \subseteq {}^\omega\omega$ is a dominating family if $(\forall f \in {}^\omega\omega)(\exists g \in D)[f \leq^* g]$ (where $f \leq^* g$ means that $\{n < \omega : g(n) < f(n)\}$ is a finite set). The small cardinal $\mathfrak{d}$ (the $\mathfrak{d}$ominating number) is defined as $\mathfrak{d} = \min\{|D| : D \text{ is a dominating family in } \langle {}^\omega\omega, \leq^*\rangle\} = \mathrm{cf}(\langle {}^\omega\omega, \leq^*\rangle)$. For small cardinals (such as $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{t}$) we refer to [vD]; we will also use the notation in [vD] for orders and quasi-orders, in particular $\subset$ denotes strict inclusion.

The author was supported by FAPESP, Grant 98/03633-2.

For definitions of cardinal functions as density, character, cellularity and extent we refer to [H].

Let us describe the organization of this paper. In Section 2 we prove generalizations of earlier negative results on Property (a), due to Just, Szeptycki and Matveev ([JMS], [M97]). As an application, we give an example of a topological space that compares the presence of Property (a) in paracompact and metacompact spaces. One of the negative results in this section is a version for (a)-spaces of Jones' Lemma; this version is due to Matveev ([M97]).

In Section 3, motivated by a comparison between the proofs of Jones' Lemma for normal spaces and Matveev's (a)-version of the referred lemma, we pose two questions, e.g.: is it consistent that there is a topological space X such that X is an (a)-space which includes a closed and discrete subset of cardinality $d(X)^+$ and $2^{d(X)} < 2^{d(X)^+}$? This is Question 3.1; if we change "density" to "cellularity and character", we get Question 3.3. The search for consistent examples to these questions in spaces constructed from almost disjoint families lead us to deal with small dominating families in the space of functions from ω_1 to ω, and we recall that the existence of such dominating families is related to large cardinals.

In Section 4 we work in the class of locally compact spaces. We relate cofinal families in the family of closed and discrete subsets of a given dense set D to dominating families of functions. Using again the connections between small dominating families and large cardinals, we prove that, under the assumptions "2^ω is regular" and "$2^\omega < 2^{\omega_1}$", the consistency of the existence of a T_1 separable locally compact (a)-space providing a positive answer to Question 3.1 is related to the existence of inner models of measurability. At the end of this section we use cardinal invariants such as $\mathfrak{d}$ to obtain negative results that, restricted to the class of locally compact spaces, strengthen the ones presented in Section 2.

In Section 5 we give some notes and questions.

2. Generalizations of earlier negative results

We prove, in this section, generalizations of results from [JMS] and [M97]. As an application, an example of a topological space (related to the presence of Property (a) in metacompact spaces) is presented in the first subsection.

2.1 A lemma for regular cardinals. The following results generalize a lemma stated for "$\kappa = \omega_1$" in [JMS].

Theorem 2.1. *Let X be a topological space and κ, λ be infinite cardinals with $\lambda = \mathrm{cf}(\kappa)$. Suppose that X includes a dense set D and a closed and discrete subset H such that:*

(1) $|D| = \kappa$;
(2) $|H| \geq \kappa$;
(3) *if* $C \subset D$ *and* $|C| < \kappa$, *then* $\overline{C} \cap H = \emptyset$;
(4) D *does not have closed discrete subsets of size* λ.

Then X does not satisfy Property (a).

PROOF: Let D and H be subsets of X as in the statement. By (2), we may consider $H' \subseteq H$ such that $|H'| = \kappa$. Enumerate $D = \{d_\alpha : \alpha < \kappa\}$ and $H' = \{x_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, let

$$U_\alpha = X \setminus \Big((H' \setminus \{x_\alpha\}) \cup \overline{\{d_\xi : \xi < \alpha\}}\Big).$$

By (3), for every $\alpha < \kappa$ we have $x_\alpha \notin \overline{\{d_\xi : \xi < \alpha\}}$. It follows that, for each $\alpha < \kappa$, U_α is an open neighbourhood of x_α and, moreover, U_α satisfies $U_\alpha \cap H' = \{x_\alpha\}$. Now, consider the open cover of X given by

$$\mathcal{U} = \{X \setminus H'\} \cup \{U_\alpha : \alpha < \kappa\}.$$

Note that, for fixed $\alpha < \kappa$, U_α is the only element of $\mathcal{U}$ that contains x_α.

We claim that $\mathcal{U}$ and D witness that X does not satisfy Property (a). Indeed, let $F \subset D$ be a closed and discrete subset of X. By (4), we have $|F| < \lambda = \mathrm{cf}(\kappa)$, so $\sup(\{\gamma < \kappa : d_\gamma \in F\}) < \kappa$. But then there is $\zeta < \kappa$ such that $F \subseteq \{d_\xi : \xi < \zeta\}$. Thus $F \cap U_\zeta = \emptyset$, which implies $x_\zeta \notin \mathrm{St}(F, \mathcal{U})$. As the closed and discrete $F \subset D$ was arbitrarily chosen, X is not an (a)-space. $\square$

In particular, we generalize for "κ is regular" the lemma for "$\kappa = \omega_1$" in [JMS] mentioned above.

Lemma 2.2. *Let X be a topological space and κ be a regular cardinal. Suppose that X includes a dense set $D \subset X$ and a closed and discrete subset $H \subset X$ such that:*

(1) $|D| = \kappa$;
(2) $|H| \geq \kappa$;
(3) *if* $C \subset D$ *and* $|C| < \kappa$, *then* $\overline{C} \cap H = \emptyset$;
(4) D *does not have closed discrete subsets of size* κ.

Then X does not satisfy Property (a). $\square$

As ω is regular, the preceding result holds for countable dense sets and infinite closed and discrete subsets. In particular, assuming that X is a T_1 space, we have the following corollary:

Corollary 2.3. *Let X be a T_1 separable topological space. Suppose X includes disjoint subsets D and H such that D is a countable dense set and H is an infinite closed and discrete subset. Furthermore, suppose that D does not have infinite closed discrete subsets. Then, X does not satisfy Property (a).* $\square$

As an application of the preceding corollary we will present an example related to metacompact spaces. Recall that a topological space X is said to be *metacompact* if every open cover of X has a point-finite open refinement. It is easy to see (as remarked in [M97]) that paracompact T_1 spaces satisfy Property (a). The following example shows that the same is not true for metacompact spaces.

Example 2.4. A metacompact T_1 first-countable separable locally compact space which does not satisfy Property (a).

Construction: Consider $X = I \cup (\omega \setminus 2)$, where I is the closed unit interval. We topologize X as follows: the basic open neighbourhoods for points in I are the usual Euclidean open neighbourhoods in the interval. If $k \geq 2$, the basic neighbourhoods for the point k are given by the sets of the form

$$B_\epsilon =]1-\epsilon, 1[\cup \{k\}$$

for $\epsilon \in \mathbb{R}$ satisfying $0 < \epsilon < 1$. It is straightforward to check that with this topology the space X is a T_1 non-Hausdorff first-countable separable space and $\omega \setminus 2$ is an infinite closed and discrete subset of X. To see that X is locally compact, just note that, for each $k \geq 2$, $\{[1-\frac{1}{n}, 1[\cup \{k\} : n \geq 1\}$ is a local base of (not closed) compact neighbourhoods of k (compact subsets need not be closed in T_1 spaces).

In order to verify the metacompactness, let $\mathcal{U}$ be an arbitrary open cover of X; we may suppose without loss of generality that $\mathcal{U}$ consists of basic open sets. Consider the family of open sets

$$\mathcal{U}_I = \{U \cap I : U \in \mathcal{U}\}.$$

The family $\mathcal{U}_I$ is an open cover of I consisting of Euclidean open sets, therefore there is $\mathcal{V} \subseteq \mathcal{U}_I$ such that $\mathcal{V}$ is a finite cover of I. For each $k \geq 2$, we fix $U_k \in \mathcal{U}$ such that $k \in U_k$ and define the open set

$$W_k = U_k \cap B_{\frac{1}{k}}.$$

It follows that $\mathcal{V} \cup \{W_k : k \geq 2\}$ is a point-finite refinement of $\mathcal{U}$, as desired. To verify that X is not an (a)-space, we apply Corollary 2.3 for $D = \mathbb{Q} \cap I$ and $H = (\omega \setminus 2)$. □

Question 2.5. *Is there a ZFC example of a metacompact Tychonoff non-(a) space satisfying (some of) the properties of Example 2.4?*

Metacompact normal spaces are countably paracompact, and it is still an open question (due to Matveev [M97]) whether there is an example of a countably paracompact first-countable non-(a) space (at least a consistent one).

2.2 Matveev's (a)-version of Jones' Lemma. The well-known Jones' Lemma for normal spaces has an analogy for Property (a). In fact, there are several results for normal spaces that remain true if the hypothesis "the space is normal" is changed to "the space satisfies Property (a)"; we refer to [JMS] and [M97] for more on this discussion. The "separable version" of Jones' Lemma is frequently given by the statement:

"If X is a separable normal space, then X does not have a closed and discrete subset of size greater than or equal to 2^ω".

The (a)-version of this fact was obtained by Matveev in [M97]:

Theorem 2.6 ([M97]). *If a separable (a)-space X has a closed and discrete subset of size κ, then $\kappa < 2^\omega$.* □

Matveev's result on separable spaces can be extended to the general case $d(X) = \kappa$; this was already remarked by Szeptycki and Vaughan in [SV], but a proof of this was not provided. For the sake of completeness — and in order to proceed some comparisons later — we present here a proof.

Theorem 2.7. *Let X be a topological space and suppose X has a closed and discrete subset of size at least $2^{d(X)}$. Then X does not satisfy Property (a).*

Proof: Let $D \subseteq X$ be a dense set of cardinality $d(X)$. Consider the family of its closed and discrete subsets, say

$$\mathcal{F}_D = \{G \subseteq D : G \text{ is a closed and discrete subset of } X\}.$$

Let H be a closed and discrete subset of X of size at least $2^{d(X)}$. As $|H| \geq 2^{|D|} > |D|$, we may suppose without loss of generality that $H \cap D = \emptyset$.

Enumerating $\mathcal{F}_D = \{G_\alpha : \alpha < \lambda\}$, we have $\lambda \leq 2^{d(X)} \leq |H|$, so we can consider a subset of H of size λ, say $H' = \{x_\alpha : \alpha < \lambda\}$, $H' \subseteq H$. For each $\alpha < \lambda$ we define an open set

$$U_\alpha = X \setminus \Big(G_\alpha \cup (H' \setminus \{x_\alpha\})\Big).$$

Since D and H are disjoint sets and $G_\alpha \subset D$, it is easy to see that, for each $\alpha < \lambda$, U_α is an open neighbourhood of x_α satisfying the conditions

(1) $U_\alpha \cap H' = \{x_\alpha\}$ and
(2) $U_\alpha \cap G_\alpha = \emptyset$.

Thus, if we consider the open cover of X given by

$$\mathcal{U} = \{X \setminus H'\} \cup \{U_\alpha : \alpha < \lambda\},$$

then (1) ensures that U_α is the only element of $\mathcal{U}$ that contains x_α. It follows that $\mathcal{U}$ and D witness that X does not satisfy Property (a). Indeed, if $F \subset D$ is an arbitrary closed and discrete subset of X then there is $\xi < \lambda$ such that $F = G_\xi$ and, by (2), $U_\xi \cap G_\xi = \emptyset$. Therefore

$$\mathrm{St}(F,\mathcal{U}) = \mathrm{St}(G_\xi,\mathcal{U}) \subseteq X \setminus \{x_\xi\}$$

and, as $F \subset D$ was arbitrarily chosen, X is not an (a)-space. □

Looking closely to the arguments of the preceding proof, we also have the following:

Corollary 2.8. *Let X be a topological space and suppose X has subsets D and H such that D is a dense set and H is a closed and discrete subset. If $|\mathcal{F}_D| \leq |H \setminus D|$, then X does not satisfy Property (a).* □

The preceding corollary tries to obtain better upper bounds on the cardinality of a closed and discrete subset of an (a)-space. In fact, several results of this type were obtained by Szeptycki and Vaughan in [SV], using cofinal families (in the sense of inclusion) in the family of the closed and discrete subsets of a given dense set. In Section 4 we will work with such cofinal families too, relating them to dominating families of functions. The importance of presenting here a proof of the (a)-version of Jones' Lemma will be clear in the next section.

Finally, we note that Corollary 2.3 can be obtained from Corollary 2.8: if, in a given T_1 space X, H is an infinite closed and discrete subset disjoint from a countable dense set D without infinite closed discrete subsets, then $\mathcal{F}_D = [D]^{<\omega}$, so $\omega = |\mathcal{F}_D| \leq |H|$, which implies X is not an (a)-space.

3. Two questions on the (a)-version of the Jones' Lemma

Probably, the best way to state Jones' Lemma for normal spaces is the following: "*If X is a normal space, D is a dense subset of X and H is a closed and discrete subset of X, then $2^{|H|} \leq 2^{|D|}$*". This statement "describes the proof", since it is based on the construction of an injective function from $\mathcal{P}(H)$ to $\mathcal{P}(D)$. As $|H| < 2^{|H|} \leq 2^{|D|}$ under the given conditions, from this statement follows the other usual forms of Jones' Lemma, such as "*if X is a normal space then X does not have a closed and discrete subset of cardinality greater than or equal to $2^{d(X)}$*" or even "*if X is a normal space, $d(X) = \kappa$ and $2^\kappa < 2^{\kappa^+}$, then X does not have a closed and discrete subset of cardinality κ^+*". However, comparing these statements with the arguments used by Matveev in his (a)-version of Jones' Lemma, we can see that in the proof of Theorem 2.7 an injective function from $\mathcal{P}(H)$ to $\mathcal{P}(D)$ is not constructed, so the inequality $2^{|H|} \leq 2^{|D|}$ is not established. We present the following question:

Question 3.1. *Is it consistent that there is a topological space X such that X is an (a)-space which includes a closed and discrete subset of cardinality $d(X)^+$ and $2^{d(X)} < 2^{d(X)^+}$?*

Obviously, in a model for a positive answer to the preceding question the inequalities $d(X)^+ < 2^{d(X)} < 2^{d(X)^+}$ must hold. We also point out that a space for a positive answer to the preceding question cannot be paracompact or even metacompact, because of the following result:

Theorem 3.2. *If X is a metacompact space, then $e(X) \leq d(X)$.*

PROOF: Let X be a metacompact space and $D \subseteq X$ a dense subset, with $|D| = \kappa = d(X)$. It is enough to show that if $F \subseteq X$ is a closed and discrete subset of X then $|F| \leq \kappa$. Indeed, let $F \subseteq X$ be a closed discrete subset. For each $x \in F$ we pick an open neighbourhood U_x of x such that $U_x \cap F = \{x\}$. Consider the open cover of X given by

$$\mathcal{U} = \{U_x : x \in F\} \cup \{X \setminus F\}.$$

As X is a metacompact space, there is a point-finite open refinement $\mathcal{V}$ of $\mathcal{U}$, and therefore for each $x \in F$ there is a open neighbourhood V_x of x that satisfies $x \in V_x \subseteq U_x$ with $V_x \in \mathcal{V}$. Obviously, we also have $V_x \cap F = \{x\}$ for every $x \in F$.

For each $d \in D$, we define a subset F_d of F given by

$$F_d = \{x \in F : d \in V_x\}.$$

As D is dense and $\mathcal{V}$ covers X, we have $F = \bigcup_{d \in D} F_d$ and the point-finiteness of $\mathcal{V}$ implies that each one of the sets F_d is a finite set, so $|F| \leq |D| = \kappa$, as desired. □

Regarding the statement of Jones' Lemma for normal spaces using $2^\kappa < 2^{\kappa^+}$, it is well known that an analogous result hold if we change "density" to "cellularity and character", i.e., if X is a normal space, $\kappa = c(X) \cdot \chi(X)$ and $2^\kappa < 2^{\kappa^+}$ then X does not contain a closed and discrete subset of cardinality κ^+. Thus, the following question arises naturally.

Question 3.3. *Is it consistent that there is a topological space X such that X is an (a)-space which includes a closed and discrete subset of cardinality κ^+, for $\kappa = c(X) \cdot \chi(X)$, and $2^\kappa < 2^{\kappa^+}$?*

The search for consistent examples to the preceding questions in spaces constructed from almost disjoint families led us to deal with dominating families.

3.1 Spaces from almost disjoint families. A family $\mathcal{A}$ of infinite subsets of ω is called an *almost disjoint family* (or *a.d. family*) if every pair of distinct elements of $\mathcal{A}$ has finite intersection. A usual construction using an almost disjoint family $\mathcal{A}$ is the corresponding topological space $\Psi(\mathcal{A})$, whose underlying set is $\mathcal{A} \cup \omega$. The points in ω are declared isolated and the basic neighbourhoods of a point $A \in \mathcal{A}$ are given by the sets $\{A\} \cup (A \setminus F)$ for $F \in [\omega]^{<\omega}$. Then, ω is a dense set of isolated points and $\mathcal{A}$ is a closed and discrete subset of $\Psi(\mathcal{A})$. Basic informations on such spaces can be found in [vD].

If $\mathcal{A}$ is a maximal a.d. family, then $\Psi(\mathcal{A})$ is not an (a)-space; the usual argument used to prove this fact is, indeed, an application of Corollary 2.3 for $D = \omega$ and $H = \mathcal{A}$. It is remarkable that $\Psi(\mathcal{A})$ is an (a)-space whenever $|\mathcal{A}| < \mathfrak{p}$ ([SV]).

It is easy to see that a space $\Psi(\mathcal{A})$ has countable density, character and cellularity. Thus, the consistency of the existence of a space $\Psi(\mathcal{A})$ satisfying Property (a) for an a.d. family $\mathcal{A}$ of size ω_1 in a model of "$2^\omega < 2^{\omega_1}$" would provide a positive answer to both Questions 3.1 and 3. This justifies — along with the intrinsic interest on classical combinatoric structures such as almost disjoint families — our interest on such spaces.

The presence of Property (a) in spaces $\Psi(\mathcal{A})$ was characterized in a combinatorial way by Szeptycki and Vaughan in [SV].

Fact 3.4 ([SV]). *If $\mathcal{A} \subseteq [\omega]^\omega$ is an a.d. family, then the corresponding space $\Psi(\mathcal{A})$ satisfies Property (a) if and only if*

$$(\forall\, f : \mathcal{A} \mapsto \omega)(\exists\, P \subset \omega)(\forall\, A \in \mathcal{A})[0 < |P \cap (A \setminus f(A))| < \omega]. \qquad \square$$

Using this characterization, we relate uncountable $\Psi(\mathcal{A})$ spaces satisfying Property (a) to dominating families in $\langle {}^{\omega_1}\omega, \leq \rangle$.

Theorem 3.5. *If there is an a.d. family $\mathcal{A}$ of size ω_1 such that $\Psi(\mathcal{A})$ satisfies Property (a), then there is $\mathcal{F} \subseteq {}^{\omega_1}\omega$ such that $\mathcal{F}$ is a dominating family in $\langle {}^{\omega_1}\omega, \leq \rangle$ and $|\mathcal{F}| = 2^\omega$.*

Proof: Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be as in the statement. For each $P \subseteq \omega$ we define a function $f_P : \omega_1 \mapsto \omega$ such that, for every $\alpha < \omega_1$,

$$f_P(\alpha) = \begin{cases} \max(A_\alpha \cap P) & \text{if } \ 0 < |A_\alpha \cap P| < \omega \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\mathcal{F} = \{f_P : P \subseteq \omega\}$ is a dominating family in $\langle {}^{\omega_1}\omega, \leq \rangle$. Indeed: as $\Psi(\mathcal{A})$ is an (a)-space, Fact 3.4 ensures that for each $h : \mathcal{A} \mapsto \omega$ there is a witness $P_h \subseteq \omega$ satisfying $0 < |P_h \cap (A_\alpha \setminus h(A_\alpha))| < \omega$ for every $\alpha < \omega_1$, and it follows that (naturally identifying ${}^{\mathcal{A}}\omega$ with ${}^{\omega_1}\omega$ under our enumeration of $\mathcal{A}$) the family $\{f_{P_h} : h \in {}^{\mathcal{A}}\omega\} \subseteq \mathcal{F}$ is dominating in $\langle {}^{\omega_1}\omega, \leq \rangle$ as desired. $\square$

Regarding Questions 3.1 and 3.3, we have the following corollary:

Corollary 3.6. *If it is consistent that there is an a.d. family $\mathcal{A}$ of size ω_1 satisfying " $\Psi(\mathcal{A})$ is an (a)-space" + " $2^\omega < 2^{\omega_1}$", then it is consistent that there is a dominating family in $\langle {}^{\omega_1}\omega, \leq \rangle$ of cardinality less than 2^{ω_1}.* $\square$

We recall that the existence of "small dominating families in the space of functions from ω_1 to ω" is related to large cardinals. The existence of dominating families in $\langle {}^{\omega_1}\omega, \leq \rangle$ of size less than 2^{ω_1} was a subject of works due to Jech, Prikry and Steprans, among others. In [JP] it is shown that "$2^\omega < 2^{\omega_1}$" + "2^ω regular" + "there is no inner model with a measurable cardinal" implies that "there is no dominating family in $\langle {}^{\omega_1}\omega, \leq \rangle$ of cardinality 2^ω"; it is also shown in [JP] that there is no dominating family of size less than 2^{ω_1} in $\langle {}^{\omega_1}\omega, \leq \rangle$ if

$\mathfrak{c}$ is a real-measurable cardinal or if "$2^\omega < 2^{\omega_1}$" and "$2^\omega < \aleph_{\omega_1}$". The connection between small dominating families and inner models of measurability in [JP] is established using Jensen's results on the *core model*. Questions related to small dominating families in $\langle {}^{\omega_1}\omega, \leq\rangle$ also appear in Problems 56 and 355 of *Open Problems in Topology* ([vMR]). Note that the referred results (together with Corollary 3.6) show that if we assume "2^ω is regular" and "$2^\omega < 2^{\omega_1}$", then the existence of a model with a topological space $\Psi(\mathcal{A})$ answering in a positive way both Questions 3.1 and 3.3 implies the existence of inner models with measurable cardinals.

Theorem 3.5 will be generalized in the next section for T_1 separable locally compact (a)-spaces. We decide to state Theorem 3.5 because of our special interest in spaces from almost disjoint families (see Question 5.1 in Section 5).

4. Results on locally compact spaces

In this section we keep on working with dominating families, relating them to cofinal families in $\mathcal{F}_D$, for D dense. The results in this section hold for locally compact and even for locally countably compact spaces. We show in the second subsection that the consistency of the existence of a T_1 separable locally compact (a)-space providing a positive answer to Question 3.1 is also related to the existence of inner models of measurability. In the third subsection we use cardinal invariants such as $\mathfrak{d}$ to obtain results that, restricted to the class of locally compact spaces, constitute negative results stronger than the ones presented in Section 2.

4.2 Cofinal families in $\mathcal{F}_D$. Given a dense set D, consider its family of closed discrete subsets, say $\mathcal{F}_D$ (as in Theorem 2.7). If D is countable, then $|\mathcal{F}_D| = \omega$ or $|\mathcal{F}_D| = \mathfrak{c}$, depending on the existence of an infinite closed and discrete subset of D, so for a countable D there is no way to get better upper bounds on $|\mathcal{F}_D|$ other than $\mathfrak{c}$. This explains why it is natural to work with cofinal families in $\mathcal{F}_D$, as in [SV]. For a given dense set D, $\mathcal{F}_D$ will always be ordered by inclusion, so $\mathrm{cf}(\mathcal{F}_D)$ means $\mathrm{cf}(\mathcal{F}_D, \subseteq)$.

We will work in this section with the following cardinal invariants, introduced by Szeptycki and Vaughan:

Definition 4.1 ([SV])**.** The cardinal invariants $\mathrm{ddc}(X)$ and $\mathrm{ddc}_1(X)$ for a topological space X are defined in the following way:

$$\mathrm{ddc}(X) = \min\{\mathrm{cf}(\mathcal{F}_D) : D \text{ is dense in } X\} + \omega,$$
$$\mathrm{ddc}_1(X) = \min\{\mathrm{cf}(\mathcal{F}_D) : D \ \text{ is dense in } X \text{ and } \ |D| = d(X)\} + \omega. \qquad \square$$

It is easy to see that given an arbitrary topological space X we have $\mathrm{ddc}(X) \leq \mathrm{ddc}_1(X) \leq 2^{d(X)}$, but $\mathrm{ddc}(X) < \mathrm{ddc}_1(X)$ is consistent (see [SV]). Also, we may have $\mathrm{ddc}(X) < d(X)$. There is an easy *ZFC* example in [SV] of a metrizable space X such that $\mathrm{ddc}_1(X) = \mathrm{ddc}(X) = \omega < \omega_1 = d(X)$.

4.2 Separable locally compact (a)-spaces with uncountable closed and discrete subsets. In this subsection we work on spaces as in its title. In order to search for a positive answer to Question 3.1 we relate these spaces to dominating families in $\langle {}^{\omega_1}\omega, \leq\rangle$ of size not larger than $\mathfrak{c}$. For another example of the importance of such small dominating families in topology, we refer to [W]: Watson showed that the existence of a countably paracompact separable space with an uncountable closed and discrete subset is equivalent to the existence of a dominating family in $\langle {}^{\omega_1}\omega, \leq\rangle$ of size $\mathfrak{c}$.

Theorem 4.2. *The existence of a T_1 separable locally compact (a)-space X with an uncountable closed and discrete subset implies the existence of a dominating family in $\langle {}^{\omega_1}\omega, \leq\rangle$ of size $\mathrm{ddc}_1(X)$.*

Proof: Let X be a topological space as in the statement, with an uncountable closed and discrete subset H and a countable dense set D such that $\mathrm{cf}(\mathcal{F}_D) = \mathrm{ddc}_1(X) = \kappa$. As $|H| > |D|$, we may suppose without loss of generality that $H = \omega_1 \setminus \omega$ and $D = \omega$; note that, under this assumption, $\omega_1 \setminus \omega$ has no isolated points in X. Let $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ be a cofinal family in $\mathcal{F}_D$ of minimum cardinality. For each $\beta \in \omega_1 \setminus \omega$ we pick an open neighbourhood U_β of β such that

(1) $U_\beta \cap (\omega_1 \setminus \omega) = \{\beta\}$ and
(2) $U_\beta \subseteq K_\beta$, where K_β is a compact subset of X.

For each $\alpha < \kappa$ we define a function $f_\alpha : (\omega_1 \setminus \omega) \mapsto \omega$ such that, for every $\beta \in (\omega_1 \setminus \omega)$,

$$f_\alpha(\beta) = \begin{cases} \max(U_\beta \cap C_\alpha) & \text{if } U_\beta \cap C_\alpha \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the functions in $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ are well defined; (2) ensures that the sets $K_\beta \cap C_\alpha$ (for $\beta \in (\omega_1 \setminus \omega), \alpha < \kappa$) are finite.

We claim that the family $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ is dominating in $\langle {}^{(\omega_1 \setminus \omega)}\omega, \leq\rangle$, and this is sufficient for us. Indeed, consider an arbitrary function $g : (\omega_1 \setminus \omega) \mapsto \omega$. Let $\mathcal{U}$ be the open cover of X given by

$$\mathcal{U} = \{X \setminus (\omega_1 \setminus \omega)\} \cup \{U_\beta \setminus (g(\beta) + 1) : \beta \in (\omega_1 \setminus \omega)\}.$$

Note that, by (1), for each $\beta \in (\omega_1 \setminus \omega)$, the open set $U_\beta \setminus (g(\beta) + 1)$ is the only element of $\mathcal{U}$ that contains β. As X is an (a)-space, there is a closed and discrete subset $F \subset D$ such that $\mathrm{St}(F, \mathcal{U}) = X$. By cofinality, there is $\alpha < \kappa$ such that $F \subseteq C_\alpha$. It is easy to see that f_α dominates g, since for every $\beta \in (\omega_1 \setminus \omega)$ we must have

$$(U_\beta \setminus (g(\beta) + 1)) \cap C_\alpha \neq \emptyset$$

and therefore $f_\alpha(\beta) \geq g(\beta)$, as desired. $\square$

The proof of the preceding theorem give us the following

Corollary 4.3. *Let X be a T_1 separable locally compact (a)-space and assume that X includes disjoint subsets D and H such that D is a countable dense set and H is an infinite closed discrete subset with $|H| = \kappa$. Then for every λ, $\omega \leq \lambda \leq \kappa$, the ordered space of functions $\langle {}^{\lambda}\omega, \leq \rangle$ has a dominating family of size* $\mathrm{cf}(\mathcal{F}_D)$. □

As $\mathrm{ddc}_1(X) \leq \mathfrak{c}$ in Theorem 4.2, we can apply again the facts on small dominating families mentioned in the end of Subsection 3.1 and state the following corollary:

Corollary 4.4. *Assume 2^{ω} is regular and $2^{\omega} < 2^{\omega_1}$. Under these conditions, if there is a T_1 separable locally compact (a)-space that includes an uncountable closed and discrete subset then there is an inner model with a measurable cardinal.* □

Thus, the discussion of Question 3.1, when restricted to the class of separable locally compact spaces, necessarily involves a discussion on the presence of large cardinals.

If we look closely to the proof of Theorem 4.2, we can also point out the following

Remark 4.5. In Theorem 4.2, we can replace the hypothesis "the space is locally compact" by the weaker statement "all the points in the uncountable closed discrete subset have a compact neighbourhood". □

4.3 Results involving cardinals such as $\mathfrak{d}$. In this subsection we use cardinal invariants such as $\mathfrak{d}$ and prove negative results for the class of the locally compact spaces. These results compare with the ones in Section 2 and are stronger in the referred class.

We recall that the small cardinal $\mathfrak{d}$, defined as the minimum cardinality of a dominating family in $\langle {}^{\omega}\omega, \leq^* \rangle$, also satisfies $\mathfrak{d} = \min\{D \subseteq {}^{\omega}\omega : D$ is a dominating family $\langle {}^{\omega}\omega, \leq \rangle\}$(see [vD]).

The following result naturally compares with Corollary 2.3.

Theorem 4.6. *Let X be a T_1 separable locally compact space and assume that X includes disjoint subsets D and H such that D is a countable dense set and H is an infinite closed and discrete subset. Suppose that*

$$\mathrm{cf}(\mathcal{F}_D) < \mathfrak{d}.$$

Then X does not satisfy Property (a).

PROOF: See Corollary 4.3. □

It is easy to see that if H is a closed and discrete subset of X *without isolated points in X*, then $D \setminus H$ is a dense subset of X whenever D is dense in X; this is true without separation axioms, but, if we assume that X is an infinite T_1 space, then $D \setminus H$ is necessarily an infinite dense set. Thus, if X is a T_1 separable space that contains a dense set D with $|D| = d(X) = \omega$ and a closed and discrete subset H without isolated points in X, then $D \setminus H$ is dense and $|D \setminus H| = \omega$; moreover, we can say that $\mathrm{cf}(\mathcal{F}_{D\setminus H}) \leq \mathrm{cf}(\mathcal{F}_D)$ (note that if $\mathcal{G}$ is a cofinal family in $\mathcal{F}_D$ then $\{G \setminus H : G \in \mathcal{G}\}$ is cofinal in $\mathcal{F}_{D\setminus H}$). In particular, if $\mathrm{cf}(\mathcal{F}_D) = \mathrm{ddc}_1(X)$ then $\mathrm{cf}(\mathcal{F}_{D\setminus H}) = \mathrm{ddc}_1(X)$. Therefore, we can present the following corollary of Theorem 4.6:

Corollary 4.7. *Let X be a T_1 separable locally compact space which contains an infinite closed and discrete subset without isolated points in X. Suppose that, for some D countable dense subset of X, the inequality $\mathrm{cf}(\mathcal{F}_D) < \mathfrak{d}$ holds. Then, X is not an (a)-space. In particular, if X is a space under the given conditions such that $\mathrm{ddc}_1(X) < \mathfrak{d}$ then X does not satisfy Property (a).* □

Therefore, for (a)-spaces under the hypothesis of the preceding corollary (T_1, separable, locally compact, etc.), small cofinal families in $\mathcal{F}_D$ cannot be "very small", since they must guarantee that a certain family of functions is a dominating family. On the other hand, for (a)-spaces in general, we have that a small cofinal family in $\mathcal{F}_D$ is already "large enough" to give a strict upper bound on the cardinality of any closed and discrete subset of X disjoint from D.

Proposition 4.8. *Let X be an (a)-space with disjoint subsets D and H such that D is a dense set and H is an infinite closed and discrete subset. Then $|H| < \mathrm{cf}(\mathcal{F}_D)$ and, moreover, $|H| < \mathrm{ddc}(X)$.*

Proof: This is essentially due to Szeptycki and Vaughan (Theorem 8 of [SV]). For the first inequality, let $\kappa = \mathrm{cf}(\mathcal{F}_D)$ and consider an enumeration $\{C_\alpha : \alpha < \kappa\}$ of a cofinal family in $\mathcal{F}_D$ of minimum size. Suppose for a contradiction that $|H| \geq \kappa$; enumerate a subset of size κ, say $H' \subseteq H$, as $H' = \{x_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, we pick an open neighbourhood U_α of x_α such that

(i) $U_\alpha \cap H' = \{x_\alpha\}$ and
(ii) $U_\alpha \cap C_\alpha = \emptyset$.

We may take, e.g., $U_\alpha = X \setminus \Big(C_\alpha \cup (H' \setminus \{x_\alpha\})\Big)$. Consider now the open cover of X given by

$$\mathcal{U} = \{X \setminus H'\} \cup \{U_\alpha : \alpha < \kappa\}$$

and let $G \subseteq D$ be a closed and discrete subset. Taking $\alpha < \kappa$ such that $G \subseteq C_\alpha$, it follows from (i) and (ii) that

$$\mathrm{St}(G,\mathcal{U}) \subseteq \mathrm{St}(C_\alpha,\mathcal{U}) \subseteq X \setminus \{x_\alpha\}$$

and thus $\mathcal{U}$ and D witness that X is not an (a)-space, and this contradicts the hypothesis. For $|H| < \mathrm{ddc}(X)$, as H is disjoint from a dense set we have that H has no isolated points. Thus, if E is a dense subset of X satisfying

$\mathrm{cf}(\mathcal{F}_E) = \mathrm{ddc}(X)$, it suffices to consider the dense set $E \setminus H$ and we have $\mathrm{ddc}(X) \leq \mathrm{cf}(\mathcal{F}_{E\setminus H}) \leq \mathrm{cf}(\mathcal{F}_E) = \mathrm{ddc}(X)$. Now we can just apply the preceding arguments for the dense set $E \setminus H$. $\square$

We present now two topological characterizations of the small cardinal $\mathfrak{d}$.

Theorem 4.9. $\mathfrak{d} = \mathfrak{d}_1 = \mathfrak{d}_2$, *where*

$\mathfrak{d}_1 = \min\{|\mathcal{C}|$: *there is a T_1 separable (a)-space with disjoint subsets D and H such that D is a countable dense set, H is an infinite closed and discrete subset, $\mathcal{C}$ is cofinal in $\mathcal{F}_D$ and each point in H has a compact neighbourhood*$\}$,

$\mathfrak{d}_2 = \min\{|\mathcal{C}|$: *there is a T_1 separable locally compact (a)-space with disjoint subsets D and H such that D is a countable dense set, H is an infinite closed and discrete subset and $\mathcal{C}$ is cofinal in $\mathcal{F}_D$*$\}$.

Proof: The proof of Theorem 4.6 ensures that $\mathfrak{d} \leq \mathfrak{d}_1$ (as in Remark 4.5), and $\mathfrak{d}_1 \leq \mathfrak{d}_2$ is immediate. To establish the equality it suffices now to check $\mathfrak{d}_2 \leq \mathfrak{d}$, and in order to do it we will present an example of a topological space that satisfies the conditions given by the definition of $\mathfrak{d}_2$ and such that there is a cofinal family of size $\mathfrak{d}$ in the family $\mathcal{F}_D$ for the dense set D as in the definition.

Consider a disjoint family $\{X_n : n < \omega\}$ of infinite sets, with $|X_n| = \omega$ for all $n < \omega$. Each set X_n can be enumerated as $X_n = \{x_{n,m} : m < \omega\} \cup \{x_{n,\omega}\}$, and we define $X = \bigcup_{n<\omega} X_n$. We topologize X as follows: the points $x_{n,m}$ for $n, m < \omega$ are declared isolated and each one of the points of the form $x_{n,\omega}$, for $n < \omega$, have their basic neighbourhoods given by the sets

$$B_{n,i} = \{x_{n,\omega}\} \cup \{x_{n,m} : m > i\},$$

for $i < \omega$. In other words, X is (homeomorphic to) a topological sum of ω copies of the ordinal $\omega + 1$ with the order topology. It is also easy to see that X is homeomorphic to a space of the form $\Psi(\mathcal{A})$ in the case of $\mathcal{A}$ being a countable disjoint family that partitions ω in ω infinite sets. X is an (a)-space, since X is metrizable, and it is easy to see that X is a separable normal zero-dimensional locally compact space. Consider now $H = \{x_{n,\omega} : n < \omega\}$ and $D = X \setminus H$; we have that D and H are disjoint sets such that D is a countable dense set and H is an infinite closed and discrete subset. We construct now a dominating family of cardinality $\mathfrak{d}$ in $\mathcal{F}_D$: let $\{f_\alpha : \alpha < \mathfrak{d}\}$ be a dominating family in $\langle {}^\omega\omega, \leq\rangle$. For each $\alpha < \mathfrak{d}$ consider the closed and discrete subset of D given by

$$C_\alpha = \bigcup_{n<\omega} \{x_{n,m} : m \leq f_\alpha(n)\}$$

and let $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{d}\}$. We claim that $\mathcal{C}$ is cofinal in $\mathcal{F}_D$. Indeed, for any closed and discrete $G \subseteq D$ we must have that $G \cap (X_k \setminus \{x_{k,\omega}\})$ is a finite set for each $k < \omega$. Thus we can define a function $f_G : \omega \mapsto \omega$ such that, for every $n < \omega$,

$$f_G(n) = \sup\{m : x_{n,m} \in (X_n \setminus \{x_{n,\omega}\}) \cap G\}$$

and if $\alpha < \mathfrak{d}$ is such that $f_\alpha \geq f_G$, we have clearly $G \subseteq C_\alpha$. $\square$

Let us consider ordered spaces of functions in a more general way. Let θ, λ be infinite cardinals and let $\mathfrak{d}(\theta, \lambda)$ be the cardinal given by

$$\mathfrak{d}(\theta, \lambda) = \min\{\mathcal{F} \subseteq {}^\theta\lambda : \mathcal{F} \text{ is a dominating family in } \langle {}^\theta\lambda, \leq\rangle\}.$$

We can also define

$$\mathfrak{d}^*(\theta, \lambda) = \min\{\mathcal{F} \subseteq {}^\theta\lambda : \mathcal{F} \text{ is a dominating family in } \langle {}^\theta\lambda, \leq^*\rangle\},$$

where "$f <^* g$" means that $(\exists \zeta < \theta)[f(\xi) < g(\xi)$ for every $\zeta \leq \xi < \theta]$. It is remarkable that, for any pair of infinite cardinals $\{\theta, \lambda\}$, we have $\mathfrak{d}(\theta, \lambda) = \mathfrak{d}^*(\theta, \lambda)$; this was established by Comfort in [C] (for families in ${}^\omega\omega$, it was well-known since the 60's). For instance, $\mathrm{cf}(\langle {}^{\omega_1}\omega, \leq\rangle) = \mathrm{cf}(\langle {}^{\omega_1}\omega, \leq^*\rangle)$, where $\langle {}^{\omega_1}\omega, \leq^*\rangle$ is the space of functions from ω_1 to ω with the mod countable order.

The same way Theorem 4.6 compares to Corollary 2.3, the following result naturally compares to Theorem 2.1 and Lemma 2.2.

Theorem 4.10. *Let X be a topological space and θ, λ be infinite cardinals. Suppose that X includes a dense set D and a closed and discrete subset H such that:*

(i) *$|D| = \lambda$;*
(ii) *$|H| = \theta$;*
(iii) *if $C \subset D$ and $|C| < \lambda$, then $\overline{C} \cap H = \emptyset$.*

Furthermore, suppose X is locally compact and assume

$$\mathrm{cf}(\mathcal{F}_D) < \mathfrak{d}(\theta, \lambda).$$

Then X does not satisfy Property (a).

Proof: Enumerate $D = \{d_\alpha : \alpha < \lambda\}$ and $H = \{x_\beta : \beta < \theta\}$. For each $\beta < \theta$ we pick an open neighbourhood U_β of x_β such that

(1) $U_\beta \cap H = \{x_\beta\}$ and
(2) $U_\beta \subseteq K_\beta$, where K_β is a compact subset of X.

Let $\kappa = \mathrm{cf}(\mathcal{F}_D)$ and let $\mathcal{C} = \{C_\xi : \xi < \kappa\}$ be a cofinal family in $\mathcal{F}_D$ of minimum cardinality. For each $\beta < \theta$ and $\xi < \kappa$ we have that $K_\beta \cap C_\xi$ is a finite set, so we can define for each $\xi < \kappa$ a function $f_\xi : H \mapsto D$ such that, for each $x_\beta \in H$,

$$f_\xi(x_\beta) = \begin{cases} d_\zeta & \text{where } \ \zeta = \max\{\delta : d_\delta \in U_\beta \cap C_\xi\}, \text{ if } \ U_\beta \cap C_\xi \neq \emptyset \\ d_0 & \text{otherwise.} \end{cases}$$

By hypothesis, we have $\kappa < \mathfrak{d}(\theta,\lambda)$, so (naturally identifying $\langle {}^H D, \leq\rangle$ with $\langle {}^\theta\lambda, \leq\rangle$ under our enumerations), $\{f_\xi : \xi < \kappa\}$ is not a dominating family in $\langle {}^H D, \leq\rangle$, and therefore

$$(*) \qquad (\exists\, g \in {}^H D)(\forall\, \xi < \kappa)(g \not\leq f_\xi).$$

We use g to construct a open cover of X that, together with D, witnesses that X does not satisfy Property (a). The condition (iii) ensures that for each $\beta < \theta$, the closure of the set $\{d_\delta \in D : \delta \leq \eta$, where $d_\eta = g(x_\beta)\}$ is disjoint from H. Thus, the open set V_β given by

$$V_\beta = U_\beta \setminus \overline{\{d_\delta \in D : \delta \leq \eta, d_\eta = g(x_\beta)\}}$$

is an open neighbourhood of x_β that satisfies $V_\beta \cap H = \{x_\beta\}$. Consider now the open cover of X given by

$$\mathcal{U} = \{X \setminus H\} \cup \{V_\beta : \beta < \theta\}.$$

It is easy to see that V_β is the only element of $\mathcal{U}$ that contains x_β.

Let $F \subset D$ a closed discrete subset. There is $\xi < \kappa$ such that $F \subseteq C_\xi$. By $(*)$, $g \not\leq f_\xi$. Therefore

$$(\exists\, \beta < \theta)[f_\xi(x_\beta) \in \{d_\delta \in D : \delta < \eta, d_\eta = g(x_\beta)\}]$$

and it follows from the definitions of f_ξ and V_β that $C_\xi \cap V_\beta = \emptyset$. Thus

$$\mathrm{St}(F,\mathcal{U}) \subseteq \mathrm{St}(C_\xi,\mathcal{U}) \subseteq X \setminus \{x_\beta\}$$

and, as F was arbitrarily chosen, X is not an (a)-space. $\square$

With respect to consistency results, it seems natural that the applications of Theorem 4.10 will arise in cases where "$\theta \geq \lambda$" and "λ is regular"; indeed, there are consistency results on the cardinals $\mathfrak{d}(\theta,\lambda)$ for the case "$\theta = \lambda''$ with λ regular, as we will remark presently. We also point out that the cases where "$\theta < \lambda$" (for regular λ) do not provide an "elastic structure" with respect to minimum cardinalities of dominating families, since the following results hold:

Lemma 4.11. *Let θ, λ be infinite cardinals, with $\theta < \mathrm{cf}(\lambda)$. Then $\langle {}^\theta\lambda, \leq\rangle$ has a dominating family of cardinality λ.*

Proof: Let θ and λ be as in the lemma. For each $\alpha < \lambda$ we define a function $f_\alpha : \theta \mapsto \lambda$ such that $f_\alpha(\xi) = \alpha$ for every $\xi < \theta$ (i.e. f_α is the constant function of value α). We have now that $\{f_\alpha : \alpha < \lambda\}$ is a dominating family, since, for every $g \in {}^\theta\lambda$, it follows from $\theta < \mathrm{cf}(\lambda)$ that $\beta = \sup\{g(\xi) : \xi < \theta\} < \lambda$. Take $\alpha = \beta + 1$ and the function f_α dominates g. $\square$

Proposition 4.12. *Let θ, λ be infinite cardinals such that λ is regular and $\theta < \lambda$. Then $\mathfrak{d}(\theta, \lambda) = \lambda$.*

PROOF: The preceding lemma ensures that $\mathfrak{d}(\theta, \lambda) \leq \lambda$; to establish the equality, it suffices to check that families of size less than λ cannot be dominating. Indeed, let $\mathcal{F} \subseteq {}^{\theta}\lambda$ with $|\mathcal{F}| < \lambda$. For each $\xi < \theta$, $\sup\{f(\xi) : f \in \mathcal{F}\} < \lambda$, thus the function $g : \theta \mapsto \lambda$ given by $g(\xi) = \sup\{f(\xi) : f \in \mathcal{F}\} + 1$ is not dominated by any function in $\mathcal{F}$. Therefore, $\mathfrak{d}(\theta, \lambda) = \lambda$. $\square$

For the cases "$\theta = \lambda$" (for regular λ) we can say that, writing $\mathfrak{d}(\lambda) = \mathfrak{d}(\lambda, \lambda)$, it is consistent that $\mathfrak{d}(\lambda)$ assumes any "reasonable" value between λ^+ and 2^λ. This was established by Cummings and Shelah in [CS]. Let $\mathfrak{b}(\lambda) = \min\{B \subseteq {}^{\lambda}\lambda : B$ is unbounded in $\langle {}^{\lambda}\lambda, \leq^* \rangle\}$. It was shown in [CS] that, given a regular cardinal λ and a "reasonable" triple of cardinals, it is consistent that this triple assume the values $\mathfrak{b}(\lambda), \mathfrak{d}(\lambda)$ and 2^λ. For what "reasonable" means, we recall the (essentially unique) restrictions on these cardinals in *ZFC*:

Lemma 4.13 ([CS]). *If λ is a regular cardinal, then the following statements hold:*

(i) $\lambda^+ \leq \mathfrak{b}(\lambda)$;
(ii) $\mathfrak{b}(\lambda)$ *is regular;*
(iii) $\mathfrak{b}(\lambda) \leq \mathrm{cf}(\mathfrak{d}(\lambda))$;
(iv) $\mathfrak{d}(\lambda) \leq 2^\lambda$; *and*
(v) $\mathrm{cf}(2^\lambda) > \lambda$. $\square$

In the preceding lemma, (v) is the well-known König's result, (iv) is obvious, (ii) and (iii) are general results on unbounded and dominating families in partial orders and (i) follows from a traditional diagonal argument.

With these restrictions in mind, we can now explain what "the consistency of any reasonable triple" means. Consider a model of *GCH* that contains a "class-function" that for each regular cardinal λ associates a triple of cardinals $(\beta(\lambda), \delta(\lambda), \mu(\lambda))$ satisfying the conditions $\lambda^+ \leq \beta(\lambda) = \mathrm{cf}(\beta(\lambda)) \leq \mathrm{cf}(\delta(\lambda)) \leq \delta(\lambda) \leq \mu(\lambda)$ and $\mathrm{cf}(\mu(\lambda)) > \lambda$ for all regular λ. The main result in [CS] ensures that there is a "class-forcing" that preserves cardinalities and cofinalities and such that the equalities $\mathfrak{b}(\lambda) = \beta(\lambda)$, $\mathfrak{d}(\lambda) = \delta(\lambda)$ and $\mu(\lambda) = 2^\lambda$ (for every regular λ) hold in its generical extensions.

We present now a corollary of the proof of Theorem 4.10.

Corollary 4.14. *Let θ, λ be infinite cardinals, with $\theta \geq \lambda$, and suppose X is a locally compact topological space which includes a dense set D and a closed and discrete subset H such that D and H satisfy the conditions of Theorem* 4.10. *Furthermore, assume* $\mathrm{cf}(\mathcal{F}_D) < \mathfrak{d}(\lambda)$. *Then, X does not satisfy Property* (a). $\square$

We end this section with a metatheorem which describes the "flavour" of the consistency results that may arise from our results.

Theorem 4.15. *Suppose X is a T_1 separable locally compact space which includes an infinite closed and discrete subset without isolated points in X. Assume that, for some countable dense set D,*

$$ZFC \vdash \text{“}\mathrm{cf}(\mathcal{F}_D) \in \{\omega_1, \mathfrak{p}, \mathfrak{t}, \mathfrak{b}, \mathfrak{s}, \mathfrak{a}\}\text{”}.$$

Then, it is consistent that X does not satisfy Property (a). In particular, if X is a space under the given conditions such that $ZFC \vdash$ “$\mathrm{ddc}_1(X) \in \{\omega_1, \mathfrak{p}, \mathfrak{t}, \mathfrak{b}, \mathfrak{s}, \mathfrak{a}\}$”, then it is consistent that X is not an (a)-space.

Proof: Let $\kappa = \mathrm{cf}(\mathcal{F}_D)$. If $\kappa \in \{\omega_1, \mathfrak{p}, \mathfrak{t}, \mathfrak{b}, \mathfrak{s}, \mathfrak{a}\}$, then the strict inequality “$\kappa < \mathfrak{d}$” is consistent (see [vD]). Let M be a model of *ZFC* such that

$$M \models \text{“}\kappa < \mathfrak{d}\text{”}.$$

By Corollary 4.7,

$$M \models \text{“}X \text{ is not an } (a)\text{-space”}$$

and thus it is consistent that X does not satisfy Property (a). □

5. Notes and questions

We first ask if some kind of “reciprocal statements” of two results in this paper are true.

Question 5.1. *Does “ $2^\omega < 2^{\omega_1}$ ” + “there is a dominating family of size not larger than $\mathfrak{c}$ in ${}^{\omega_1}\omega$” imply that there is an almost disjoint family $\mathcal{A}$ whose corresponding space $\Psi(\mathcal{A})$ answers positively both Questions* 3.1 *and* 3.3*?*

In a more general way:

Question 5.2. *Does “ $2^\omega < 2^{\omega_1}$ ” + “there is a dominating family of size not larger than $\mathfrak{c}$ in ${}^{\omega_1}\omega$” imply that there is a T_1 separable space that answers positively Question* 3.1*? The same question can be posed adding local compactness.*

We note that the results in Section 4 give us informations on classes of spaces that satisfy Property (a), e.g. metric spaces. We can say that if X is a separable locally compact metric space that contains an infinite closed and discrete subset without isolated points in X, then any cofinal family in the family of the closed and discrete subsets of an arbitrary countable dense set must have size at least $\mathfrak{d}$.

Theorem 3.2, Lemma 4.11 and Proposition 4.12 are probably well-known, but we did not find any reference for them.

Acknowledgments. The author is grateful to the referee for his careful reading of the paper and for providing several useful comments and suggestions.

Most of the results in this paper are from the author's Ph.D thesis, written under the supervision of Lucia Renato Junqueira; I would like to thank her for the guidance. I also would like to thank Ofelia Teresa Alas and Piotr Koszmider for their helpful questions and suggestions.

References

[C] Comfort W.W., *Cofinal families in certain function spaces*, Comment. Math. Univ. Carolinae **29** (1988), no. 4, 665–675.

[CS] Cummings J., Shelah S., *Cardinal invariants above the continuum*, Ann. Pure Appl. Logic **75** (1995), no. 3, 251–268.

[E] Engelking R., *General Topology*, Heldermann Verlag, Sigma Series in Pure Mathematics 6, Berlin, 1989.

[H] Hodel R., *Cardinal functions I*, in: Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pp. 1–61.

[JMS] Just W., Matveev M.V., Szeptycki P.J., *Some results on property* (a), Topology Appl. **100** (2000), no. 1, 103–111.

[JP] Jech T., Prikry K., *Cofinality of the partial ordering of functions from* ω_1 *into* ω *under eventual domination*, Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 1, 25–32.

[M94] Matveev M.V., *Absolutely countably compact spaces*, Topology Appl. **58** (1994), no. 1, 81–92.

[M97] Matveev M.V., *Some questions on property* (a), Questions Answers Gen. Topology **15** (1997), no. 2, 103–111.

[SV] Szeptycki P.J., Vaughan J.E., *Almost disjoint families and property* (a), Fund. Math. **158** (1998), no. 3, 229–240.

[vD] van Douwen E.K., *The integers and topology*, in: Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pp. 111–167.

[vMR] van Mill J., Reed G.M. (editors), *Open Problems in Topology*, North-Holland, Amsterdam, 1990.

[W] Watson W.S., *Separation in countably paracompact spaces*, Trans. Amer. Math. Soc. **290** (1985), no. 2, 831–842.

Instituto de Matematica e Estatistica, University of São Paulo,
Caixa Postal 66281, 05311-970 São Paulo, Brazil

E-mail: samuel@ime.usp.br

(*Received* August 16, 2004, *revised* July 12, 2005)