Cardinal invariants of universals

Gareth Fairey, Paul Gartside, Andrew Marsh

Abstract. We examine when a space X has a zero set universal parametrised by a metrisable space of minimal weight and show that this depends on the σ-weight of X when X is perfectly normal. We also show that if Y parametrises a zero set universal for X then $hL(X^n) \leq hd(Y)$ for all $n \in \mathbb{N}$. We construct zero set universals that have nice properties (such as separability or ccc) in the case where the space has a K-coarser topology. Examples are given including an S space with zero set universal parametrised by an L space (and vice versa).

Keywords: zero set universals, continuous function universals, S and L spaces, admissible topology, cardinal invariants, function spaces

Classification: 54C30, 54C50, 54D65, 54D80, 54E35

1. Introduction

In this paper we deal with continuous function universals and zero set universals. A universal will in some appropriate sense parametrise all objects in a certain class. More specifically we can define a *continuous function universal* as follows. Given a space X we say that a space Y parametrises a continuous function universal for X via the function F if $F : X \times Y \to \mathbb{R}$ is continuous and for any continuous $f : X \to \mathbb{R}$ there exists some $y \in Y$ such that $F(x, y) = f(x)$ for all $x \in X$. We will use F^y to denote the corresponding function on X. Note that if the y above is *unique* then, in effect, Y is the set of all continuous real valued functions on X with an **admissible** topology (the evaluation map is continuous, see [1]).

Given a space X we say that a space Y parametrises a zero set universal for X if there exists U, a zero set in $X \times Y$ such that for all $A \subset X$ with A a zero set there exists $y \in Y$ such that $U^y = \{x \in X : (x, y) \in U\} = A$. This zero set U must be witnessed by some continuous function $F : X \times Y \to \mathbb{R}$ and we will refer to such a function as the parametrising function. Note that the particular case when X is perfectly normal, and Y is the set of closed subsets has been studied under the name **continuous perfect normality** ([15], [8]).

The third author would like to acknowledge the support of a National University of Ireland Travelling Studentship.

In [4] the authors investigate spaces that have universals parametrised by compact or Lindelöf spaces. In this paper we tackle a number of other questions regarding the parametrisation of continuous function universals and zero set universals.

First of all we investigate the case where X has a zero set universal parametrised by a metrisable space of *minimal weight*. Parametrisation by metrisable spaces is trivial as we can take a sufficiently large set with the discrete topology. We show that we can parametrise a zero set universal for a space X by a metrisable space Y where $w(Y) = w(X)$ when the σ-Z-weight of X equals the weight of X. The σ-Z-weight is defined to be the minimal size of a collection $\mathcal{B}$ of open subsets of X such that every cozero subset is the countable union of elements from $\mathcal{B}$.

Examining how the hereditary properties of the parametrising space Y bound the hereditary properties of X leads us to the inequality $hL(X^n) \leq hd(Y)$ for all $n \in \omega$. This is more than we get when Y parametrises an open universal for X (see [7]). We construct a number of examples including a hereditarily separable, hereditarily Lindelöf space X with $nw(X) = \aleph_1$ such that X has a continuous function universal parametrised by a space Y with $hL(Y^\omega) = hd(Y^\omega) = \omega$. This has applications to admissible topologies.

In the final section we look at when the parametrising space is separable, ccc or has calibre ω_1. First we describe how to construct a continuous function universal for a space X when X has a K-coarser topology (a coarser topology τ such that each point of X has a neighbourhood base consisting of τ-compact sets). This general construction will allow us to construct continuous function universals with 'nice' properties once we know that the space in question has a K-coarser topology with the appropriate properties. As an example we show that the Sorgenfrey line has a continuous function universal parametrised by a separable space.

We will use the following theorem frequently. For a proof see [4].

Theorem 1. *Let X be Tychonoff. If Y parametrises a zero set universal for X then some subspace of Y^ω parametrises a continuous function universal for X.*

2. Parametrisation by metrisable spaces of minimal weight

In this section we examine the situation where a space X has a zero set universal or continuous function universal parametrised by a metrisable space of minimal weight. The weight of the parametrising space is an upper bound for the weight of the space X so the minimal weight of the parametrising space is the weight of X. The key notion is that of the σ-weight of X: the minimal size of a base for X such that every open set in X is the countable union of elements from the base.

We begin by dealing with the case when X is perfectly normal. We use $D(\kappa)$ to denote the discrete space of size κ.

Theorem 2. *Let X be a perfectly normal space with $w(X) = \kappa$. Then the following are equivalent:*

(i) *X has a zero set universal parametrised by $D(\kappa)^\omega$,*
(ii) *X has a zero set universal parametrised by a first countable space of weight κ,*
(iii) *X has σ-weight κ,*
(iv) *X has a continuous function universal parametrised by a metrisable space of weight κ.*

Proof: Let X be a perfectly normal space with $w(X) = \kappa$. Clearly (i) implies (ii) and (iv) implies (ii).

(ii) implies (iii): Let X have a zero set universal parametrised by Y and let $F : X \times Y \to \mathbb{R}$ be the relevant parametrising function. Let $\mathcal{U} = F^{-1}(\mathbb{R} \setminus 0)$. Assume that Y is first countable and that $\{C_\alpha : \alpha \in \kappa\}$ is a basis for Y. For each $\alpha \in \kappa$ we define $B_\alpha = \bigcup\{U : U \text{ is open}, U \times C_\alpha \subset \mathcal{U}\}$. Let $\mathcal{B} = \{B_\alpha : \alpha \in \kappa\}$. We show that $\mathcal{B}$ is a σ-basis. Fix $U \subset X$ such that U is open. Since X is perfectly normal then U is in fact a cozero set, so there is some $y \in Y$ such that $U = \mathcal{U}^y$. Let $\{C_{\alpha_n} : n \in \omega\}$ be a countable basis at y. Then $U = \bigcup\{B_{\alpha_n} : n \in \omega\}$.

(iii) implies (i): Let $\mathcal{B} = \{B_\lambda : \lambda \in \kappa\}$ be a σ-base for X. As X is perfectly normal we can assume that each $B_\lambda \in \mathcal{B}$ is a cozero set as witnessed by $f_\lambda : X \to I$. For all $\alpha \in D(\kappa)^\omega$ let α_n denote the n'th element of the sequence α. We define $F : X \times D(\kappa)^\omega \to \mathbb{R}$ by

$$F(x, \alpha) = \sum_{n=0}^{\infty} \frac{f_{\alpha_n}(x)}{2^{n+1}}$$

for all $\alpha \in D(\kappa)^\omega$ and $x \in X$. We must show that (a) F is continuous and (b) that for all U open in X there is some $\alpha \in D(\kappa)^\omega$ such that $(F^\alpha)^{-1}(\mathbb{R} \setminus \{0\}) = U$.

To show (b) we fix U open in X. Now since $\mathcal{B}$ is a σ-base we can find $\alpha \in D(\kappa)^\omega$ such that $\bigcup\{B_{\alpha_n} : n \in \omega\} = U$. Then for all $x \in X$ we know that $F(x, \alpha) = 0$ if and only if $x \notin \bigcup\{B_{\alpha_n} : n \in \omega\} = U$.

To show (a) we fix $x \in X$, $\alpha \in D(\kappa)^\omega$ and U open such that $F(x, \alpha) \in U$. Find $N \in \omega$ such that $(F(x, \alpha) - 2^{-N}, F(x, \alpha) + 2^{-N}) \subset U$. For each $j \in \omega$ such that $j \leq N$ define $U_j = f_{\alpha_j}^{-1}(f_{\alpha_j}(x) - 2^{-N-1}, f_{\alpha_j}(x) + 2^{-N-1})$. Let $V = \bigcap_{j=0}^{N} U_j$ and $W = \prod_{j=0}^{N}\{\alpha_j\} \times \prod_{j=N+1}^{\infty} D(\kappa)$. If $(x', \beta) \in V \times W$ then

$$\begin{aligned}
|F(x, \alpha) - F(x', \beta)| &\leq \sum_{j=0}^{\infty} \frac{|f_{\alpha_j}(x) - f_{\beta_j}(x)|}{2^{j+1}} \\
&= \sum_{j=0}^{N} \frac{|f_{\alpha_j}(x) - f_{\beta_j}(x)|}{2^{j+1}} + \sum_{j=N+1}^{\infty} \frac{|f_{\alpha_j}(x) - f_{\beta_j}(x)|}{2^{j+1}}
\end{aligned}$$

$$\leq \sum_{j=0}^{N} 2^{-N-1} 2^{-j-1} + \sum_{j=N+1}^{\infty} 2^{-j-1} \leq \frac{1}{2^N}.$$

This shows that $F(x', \beta) \in U$ and so we are done.

(i) implies (iv): We know that if a space Y parametrises a zero set universal for X then by Theorem 1 some subspace of Y^ω parametrises a continuous function universal for X. Since metrisability and weight are countably productive and hereditary we are done. □

In the class of Tychonoff spaces the crucial property is σ-Z-weight: the minimal size of a collection $\mathcal{B}$ of open subsets of X such that every cozero subset is the countable union of elements from $\mathcal{B}$. The following version of Theorem 2 can be proved in much the same way.

Theorem 3. *Let X be Tychonoff with $w(X) = \kappa$. The following are equivalent:*

(i) *X has a continuous function universal parametrised by a metrisable space of weight κ,*

(ii) *X has a zero set universal parametrised by a first countable space of weight κ,*

(iii) *X has σ-Z-weight κ.*

Corollary 4. *Every Lindelöf T_3 space has a continuous function universal parametrised by a metrisable space of minimal possible weight.*

In [5] it is shown that if X is a T_3 Lindelöf space containing an uncountable discrete space then no metrisable space of minimal weight parametrises an open universal for X. So we get the following.

Corollary 5. *There exists a Tychonoff space for which σ-Z-weight is strictly less than σ-weight.*

We now examine the special case where the space in question is the discrete space of size κ. It is clear from Theorem 2 that $D(\kappa)$ has a continuous function universal parametrised by a metrisable space of weight κ if and only if there is a subcollection $\mathcal{C}$ of $\mathcal{P}(\kappa)$ of size κ such that every subset of κ is the union of countably many elements of $\mathcal{C}$. We will call such a cardinal a *σ-cardinal*.

We know the following about σ-cardinals (see [5]).

Theorem 6. *Let κ be a cardinal.*

(i) *If κ is a σ-cardinal then κ has countable cofinality.*

(ii) *If κ is a strong limit and $cf(\kappa) = \omega$ then κ is a σ-cardinal.*

(iii) *If (GCH) holds and κ is a σ-cardinal then κ is a strong limit.*

(iv) *It is consistent that $\aleph_\omega$ is a σ-cardinal and that $2^{\aleph_0} = \aleph_{\omega+1}$.*

Also connected with the question of parametrisation of $D(\kappa)$ is the notion of supermetrisability: a space X of weight κ is said to be supermetrisable if and only there is a finer metrisable topology on X of weight $\leq \kappa$.

Theorem 7. *The following are equivalent:*

(i) *κ is a σ-cardinal,*
(ii) *all spaces of weight $\leq \kappa$ have σ-weight $\leq \kappa$,*
(iii) *all Tychonoff spaces of weight $\leq \kappa$ have a continuous function universal parametrised by a metrisable space of weight $\leq \kappa$,*
(iv) *$\mathbb{R}^\kappa$ is supermetrisable,*
(v) *all Tychonoff spaces of weight κ are supermetrisable,*
(vi) *$D(\kappa)$ has a zero set universal and a continuous function universal parametrised by a metrisable space of weight κ.*

Proof: (i) implies (ii): Let κ be a σ-cardinal and let $\mathcal{C} \subset \mathcal{P}(\kappa)$ be such that every subset of κ is a countable union of elements from $\mathcal{C}$. Let $\mathcal{B} = \{B_\alpha : \alpha \in \kappa\}$ be a basis for a space X. Then $\mathcal{B}' = \{\bigcup\{B_\alpha : \alpha \in C\} : C \in \mathcal{C}\}$ is a σ-basis for X.

(ii) implies (iii): This is shown in Theorem 3.

(iii) implies (iv): Note that the set of continuous functions on the discrete space of size κ is simply $\mathbb{R}^\kappa$. Now there is some metrisable space Y of weight κ that parametrises a continuous function universal for $D(\kappa)$ (via the function F). For each function $f \in \mathbb{R}^\kappa$ choose $y_f \in Y$ such that $F(x, y_f) = f(x)$ for all $x \in \kappa$. The space $\{y_f : f \in \mathbb{R}^\kappa\}$ witnesses that $\mathbb{R}^\kappa$ is supermetrisable.

(iv) implies (v): This follows from the fact that every Tychonoff space of weight $\leq \kappa$ embeds in $\mathbb{R}^\kappa$.

(v) implies (vi): Let τ be a finer metric topology on $\mathbb{R}^\kappa$ of weight κ. Then $(\mathbb{R}^\kappa, \tau)$ parametrises a continuous function universal for $D(\kappa)$ via the evaluation map.

(vi) implies (i): This follows from Theorem 2. □

3. Hereditary properties

Assume that Y parametrises a zero set universal for X. We examine how cardinal invariants of Y bound cardinal invariants of X. In particular we are interested in hereditary properties such as the hereditary density. The case is very much the same as for open universals studied in [7]. However in the case where Y is hereditarily separable we will show that in fact X^n is hereditarily Lindelöf for all $n \in \omega$. When dealing with open universal we only get $hL(X) \leq hd(Y)$.

Theorem 8. *Let X be Tychonoff and assume that X has a zero set universal parametrised by Y. Then the following are true:*

$$w(X) \leq nw(Y), \qquad hd(X) \leq hL(Y),$$
$$hc(X) \leq hc(Y), \qquad hL(X^n) \leq hd(Y) \qquad \textit{for all } n \in \mathbb{N}.$$

Proof: The first three statements can be proved by minor modifications of the proofs in [7]. We will prove that $hL(X^n) \leq hd(Y)$. We first prove the following. If

X is Tychonoff and has zero set universal parametrised by Y such that $d(Y) \leq \kappa$, then $\exists Z \subseteq \mathbb{R}^\kappa$ and a 1–1 continuous surjection $G : X \to Z$.

Proof of claim: We must show that if D is a dense subset of Y such that $|D| = \kappa$, then $\{F^d : d \in D\}$ separates points in X. From this, we can define $G : X \to \mathbb{R}^D$ by $G(x)(d) = F(x, d)$, then take $Z = \mathrm{ran}(G)$.

Let x_1 and x_2 be two distinct points in X. Therefore there is $y \in Y$ such that $F(x_1, y) = 0$ and $F(x_2, y) \neq 0$, and hence disjoint open sets U_1 and U_2 such that $F(x_1, y) \in U_1$ and $F(x_2, y) \in U_2$. By continuity of F, there are open sets V_1 and V_2 in Y, both containing y such that $\{x_i\} \times V_i \subseteq F^{-1}(U_i)$ for $i = 1, 2$. Hence for $y' \in V_1 \cap V_2$, we see that $F(x_1, y') \neq F(x_2, y')$; also, since $V_1 \cap V_2$ is open in Y, it must meet every dense set, so there is such a y' within any dense set.

Proof of main result: Let $\kappa = hd(Y)$. Then from the above condition, we know firstly that $hL(X) \leq \kappa$. Now, for a contradiction, we suppose that, for some $n \in \mathbb{N}$ (we pick the least such), $\{\mathbf{x}_\alpha : \alpha < \kappa^+\}$ is a right-separated subset of X^n, witnessed by the basic open sets $\{\prod_{i=1}^n V_\alpha^i : \alpha < \kappa\}$. By Proposition 21 of [7], this can be done in a more symmetric way, so that for $\alpha, \beta < \kappa^+$, whenever $\mathbf{x}_\beta \in \sigma \prod_{i=1}^n V_\alpha^i$ for some $\sigma \in S_n$ (where the group action permutes co-ordinates), then $\beta \leq \alpha$. Also, we can assume that all points are off the diagonal, since $hL(X) \leq \kappa$ (and the diagonal is homeomorphic to X). In fact, we can use this last fact to guarantee that each point $\mathbf{x}_\alpha$ has no two co-ordinates equal.

Now the diagonal of X, denoted Δ_X, is precisely the set $\{(x, y) \in X \times X : G(x) = G(y)\}$, where G is as defined previously. Having defined $S = \{(i, j) : 1 \leq i < j \leq n\}$, the subset corresponding to the diagonal in $X_i \times X_j$ is $\bigcap_{\alpha<\kappa} \bigcap_{m<\omega} \{(y, z) \in X_i \times X_j : |G(y)_\alpha - G(z)_\alpha| < 1/m\}$. Hence, defining $W_{(\alpha.\omega+m)} = \{(y, z) \in X \times X : |G(y)_\alpha - G(z)_\alpha| < 1/m\}$, we see that

$$\begin{aligned}
\{\mathbf{x} \in X^n : \forall (i, j) \in S, \mathbf{x}(i) = \mathbf{x}(j)\} &= \bigcap_{(i,j)\in S} \{\mathbf{x} \in X^n : (\mathbf{x}(i), \mathbf{x}(j)) \in \Delta_X\} \\
&= \bigcap_{(i,j)\in S} \bigcap_{\alpha<\kappa} \{\mathbf{x} \in X^n : (\mathbf{x}(i), \mathbf{x}(j)) \in W_\alpha\} \\
&= \bigcap_{\alpha<\kappa} \bigcap_{(i,j)\in S} \{\mathbf{x} \in X^n : (\mathbf{x}(i), \mathbf{x}(j)) \in W_\alpha\} \\
&= \bigcap_{\alpha<\kappa} W'_\alpha
\end{aligned}$$

where W'_α is the basic open set $\{\mathbf{x} \in X^n : \forall (i, j) \in S\ (\mathbf{x}(i), \mathbf{x}(j)) \in W_\alpha\}$. By the pigeon-hole principle, there must be one such W'_α which contains the diagonal, but which misses every $\mathbf{x}_\beta$ (taking a sub-family of $\mathbf{x}_\alpha$s if necessary).

Also, we can refine each of the open sets V_α^i so that $\mathbf{x}_\alpha \in \prod_{i=1}^n V_\alpha^i$ still, but now, $(V_\alpha^i)^n \subseteq W'_\alpha$. Since X is Tychonoff, we can even make $\{V_\alpha^i : 1 \leq i \leq n\}$

pairwise disjoint cozero sets. Therefore, for all $\alpha < \kappa^+$, there is some y_α in Y such that $\bigcup_{i=1}^n V_\alpha^i = \{x \in X : F(x, y_\alpha) \neq 0\}$. Hence, there are open subsets of Y, $\{U_\alpha : \alpha < \kappa^+\}$, so that, for each $\alpha < \kappa^+$, $y_\alpha \in U_\alpha$ and $\{\mathbf{x}_\alpha(i) : 1 \leq i \leq n\} \times U_\alpha \subseteq F^{-1}(\mathbb{R} \setminus \{0\})$. As a result, whenever $y_\beta \in U_\alpha$, then for $i = 1, \ldots, n$, $\mathbf{x}_\alpha(i) \in \bigcup_{i=1}^n V_\alpha^i$. In other words, there is some $\sigma \in S_n$ such that $\mathbf{x}_\alpha \in \sigma \prod_{i=1}^n V_\beta^i$. But this means that $\alpha \leq \beta$, so $\{y_\alpha : \alpha < \kappa^+\}$ is left-separated, hence $hd(Y) \geq \kappa^+$ which contradicts our original hypothesis. □

4. S spaces and L spaces

In this section we construct three related examples of 'bad' spaces with continuous function universals parametrised by a 'good' space. For instance, there is a non metrisable space with a continuous function universal parametrised by a space whose countable power is both hereditarily Lindelöf and hereditarily separable.

The spaces are constructed from the interaction of the Baire metric topology on a fixed subset, A, of ω^ω with various orders on A. More particularly, given a partial order $\preceq$ on A, then (as in Todorčević [14]) we define the intersection topology $A[\preceq]$ from a countable local base at each $a \in A$ made up of the sets $B_\preceq(n, a)$ for $n < \omega$ where $B_\preceq(n, a) = \{b \in A : (b \preceq a) \wedge (a\restriction_n = b\restriction_n)\}$. The two orders that we consider here are: $\leq$ which is defined component-wise (i.e. $a \leq b$ precisely if, for all n, $a_n \leq b_n$), and the lexicographic order $\leq_L$ (where $a \leq_L b$ precisely if, at the smallest n such that $a_n \neq b_n$, then $a_n < b_n$).

Assuming that $\mathfrak{b} = \omega_1$, we can find a subset A of ω^ω with order type ω_1 under $\leq^*$ (where $\langle a_n \rangle_{n<\omega} \leq^* \langle b_n \rangle_{n<\omega}$ precisely if there is some $N < \omega$ such that, for all $n \geq N$, $a_n \leq b_n$). Todorčević has shown that, in this case, various intersection topologies on A behave as follows:

- ([14, Theorem 0.6]) $A[\leq]$ is a strong S space (i.e. the countable power of $A[\leq]$ is hereditarily separable, but $A[\leq]$ itself fails to be Lindelöf).
- ([14, Theorem 0.6]) $A[\geq]$ is a strong L space (i.e. its countable power is hereditarily Lindelöf, but is itself non-separable).
- ([14, Theorem 3.0]) $A[\leq_L]$ and $A[\geq_L]$ are each homeomorphic to a subspace of the Sorgenfrey line such that the countable power of each is hereditarily separable and hereditarily Lindelöf. Neither space, however, is metrisable as each fails to have a countable base.

Once we know these basic properties, we can use them, along with the definition of the local base at a point to produce the following three examples:

Example 9 ($\mathfrak{b} = \aleph_1$). *There is an L space with a zero set universal parametrised by a strong S space.*

Hence there is a non-hereditarily separable space with a continuous function universal parametrised by a space whose countable power is hereditarily separable.

Proof: The 'hence' part follows from the first assertion and Theorem 1.

Let $X = A[\geq]$ and $Y = (A[\leq] \times \omega)^\omega$. We shall show that X has a zero set universal parametrised by Y. Since X is an L space and hence hereditarily Lindelöf and regular, then it must indeed be perfectly normal. Therefore, every open set is a cozero set, and from a cover by basic open sets we can find a countable sub-cover. In other words, the cozero sets in X are precisely those of the form $\bigcup\{B_\geq(m_n, a_n) : n < \omega\}$ where each $m_n < \omega$ and $a_n \in A$.

Thus, we have a way of coding up the zero sets of X by points in Y, which we use to define a function $F : X \times Y \to 2^\omega$ by $F(x, \langle a_n, m_n\rangle_{n<\omega}) = \langle b_n\rangle_{n<\omega}$ where $b_n = 1$ if $x \in B_\geq(m_n, a_n)$ and $b_n = 0$ otherwise. Now, $F(x, \langle a_n, m_n\rangle_{n<\omega}) = \langle 0\rangle$ precisely if, for each n, $x \notin B_\geq(m_n, a_n)$, i.e. $x \notin \bigcup\{B_\geq(m_n, a_n) : n < \omega\}$.

Since, with the canonical embedding of the Cantor set within $[0, 1]$ (where $\langle b_n\rangle_{n<\omega}$ maps to $\Sigma_{n<\omega} 2 \cdot b_n/3^{(n+1)}$), only $\langle 0\rangle$ maps to 0, it will suffice to show that this function F is continuous. This simplifies to checking continuity with respect to each of the two types of sub-basic open sets in 2^ω:

- For $U(m, 0) = \{\langle b_n\rangle_{n<\omega} : b_m = 0\}$, we let $(x, \langle a_n, m_n\rangle_{n<\omega}) \in F^{-1}(U(m, 0))$. Hence, equivalently, $x \notin B_\geq(m_m, a_m) = \{b \in A : b \geq a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$. Therefore, one of two cases will apply. Firstly, if $x\restriction_{m_m} \neq a_m\restriction_{m_m}$, we let $n = m_m$. Otherwise, $x\restriction_{m_m} = a_m\restriction_{m_m}$ and $x \ngeq a_m$, so there is some $n > m_m$ such that $x(n-1) < a_m(n-1)$, in which case $x\restriction_n \neq a_m\restriction_n$. Thus, if $y \in B_\geq(n, x)$ and $\langle c_i, r_i\rangle_{i<\omega}$ is such that $c_m \in B_\leq(n, a_m)$ and $r_m = m_m$, then $y\restriction_n = x\restriction_n \neq a_m\restriction_n = c_m\restriction_n$. In the second case, it must also be the case that $y(n) < c_m(n)$ and $y\restriction_{m_m} = c_m\restriction_{m_m}$, so in both cases, $(y, \langle c_i, r_i\rangle_{i<\omega}) \in F^{-1}(U(m, 0))$ which is therefore open.
- For $U(m, 1) = \{\langle b_n\rangle_{n<\omega} : b_m = 1\}$, we let $(x, \langle a_n, m_n\rangle_{n<\omega}) \in F^{-1}(U(m, 1))$. Hence, equivalently, $x \in B_\geq(m_m, a_m) = \{b \in A : b \geq a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$, so, in other words, $x \geq a_m$ and $x\restriction_{m_m} = a_m\restriction_{m_m}$. Now, if $y \in B_\geq(m_m, x)$ and $c_m \in B_\leq(m_m, a_m)$, then $y \geq x \geq a_m \geq c_m$ and $y\restriction_{m_m} = x\restriction_{m_m} = a_m\restriction_{m_m} = c_m\restriction_{m_m}$, so $y \in B_\geq(m_m, c_m)$ which is enough to guarantee that $F^{-1}(U(m, 1))$ is open too.

Since, to prove continuity of a function, it suffices to show that the inverse image of each open set in a sub-basis is open, we have established that F is a continuous function, so Y parametrises a zero set universal for X, as required. □

Example 10 ($\mathfrak{b} = \aleph_1$)**.** *There is an S space with a zero set universal parametrised by a strong L space.*

Hence there is a non-hereditarily Lindelöf space with continuous function universal parametrised by a space with hereditarily Lindelöf power.

Proof: The 'hence' part follows from the first assertion and Theorem 1.

Let $X = A[\leq]$ and $Y = 2^\omega \times (A[\geq] \times \omega)^\omega$. We shall show that X has a zero set universal parametrised by Y. First, we show that X is perfectly normal, by proving that any open subset is a cozero set. Using that information, we create a closed-set universal, as in Gartside and Lo [7], and prove that it is, in fact, also a zero set universal.

Let (Z, τ) be such that $Z \subseteq \mathbb{R}$ and τ is finer than the Euclidean topology on Z. We say that (Z, τ) is a "Kunen line"-type space if, for every subset S of X, $|\overline{S}^d \setminus \overline{S}^\tau| \leq \aleph_0$, where d represents the Euclidean metric topology. Todorčević [14, Chapter 2] constructs a topology $A[H]$ on A of "Kunen line"-type which is a locally compact strong S space. This topology is based on, and has finer topology than, $A[<^*]$, but he observes (on page 24 of [14]): "Note that … we could have added the condition $a \leq b$ in this definition of H instead of $a <^* b$. But since we don't have a use for this, we keep the definition as it is." If the definition for $A[H]$ is changed in this way, then the topology of $A[\leq]$ is sandwiched between those of $A[H]$ and the Euclidean topology on A (considered as a subspace of the irrationals), which shows that $A[\leq]$ is a "Kunen line"-type space. Hence, every open subset of X is the union of a Euclidean-type open set with a countable union of $A[\leq]$-type basic open sets. Now, Euclidean-type open sets are cozero sets, as the topology is finer than the Euclidean topology, and basic open sets are clopen sets, so countable unions of basic open sets are cozero sets also, so therefore *any* open set in $A[\leq]$ is a cozero set.

As shown in [4], there is a zero set universal for the Euclidean zero sets, given via the non-negative continuous function $F_1 : X \times 2^\omega \to \mathbb{R}$. We use the same method as in Example 9 to produce a zero set universal for the countable unions of basic open sets. We define a function $F_2 : X \times (A[\geq] \times \omega)^\omega$ by $F_2(x, \langle a_n, m_n \rangle_{n<\omega}) = \sum_{n<\omega} 2 \cdot b_n / 3^{n+1}$ where $b_n = 1$ if $x \in B_\leq(m_n, a_n)$ and $b_n = 0$ otherwise. Now, $F_2(x, \langle a_n, m_n \rangle_{n<\omega}) = 0$ precisely if, for each n, $x \notin B_\leq(m_n, a_n)$, i.e. $x \notin \bigcup\{B_\leq(m_n, a_n) : n < \omega\}$, so that each zero set which is the complement of a countable union of basic open sets is parametrised by this function. The proof that this function F_2 is continuous follows the same pattern as the proof that the function F in Example 9 is continuous:

- For $U(m, 0) = \{\langle b_n \rangle_{n<\omega} : b_m = 0\}$, we let $(x, \langle a_n, m_n \rangle_{n<\omega}) \in F_2^{-1}(U(m, 0))$. Hence, equivalently, $x \notin B_\leq(m_m, a_m) = \{b \in A : b \leq a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$. Therefore, one of two cases will apply. Firstly, if $x\restriction_{m_m} \neq a_m\restriction_{m_m}$, we let $n = m_m$. Otherwise, $x\restriction_{m_m} = a_m\restriction_{m_m}$ and $x \not\leq a_m$, so there is some $n > m_m$ such that $x(n-1) > a_m(n-1)$, in which case $x\restriction_n \neq a_m\restriction_n$. Thus, if $y \in B_\leq(n, x)$ and $\langle c_i, r_i \rangle_{i<\omega}$ is such that $c_m \in B_\geq(n, a_m)$ and $r_m = m_m$, then $y\restriction_n = x\restriction_n \neq a_m\restriction_n = c_m\restriction_n$. In the second case, it must also be the case that $y(n) > c_m(n)$ and $y\restriction_{m_m} = c_m\restriction_{m_m}$, so in both cases, $(y, \langle c_i, r_i \rangle_{i<\omega}) \in F^{-1}(U(m, 0))$ which is seen to be an open set.

- For $U(m, 1) = \{\langle b_n \rangle_{n<\omega} : b_m = 1\}$, we let $(x, \langle a_n, m_n \rangle_{n<\omega}) \in F^{-1}(U(m, 1))$. Hence, equivalently, $x \in B_\leq(m_m, a_m) = \{b \in A : b \leq a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$, so, in other words, $x \leq a_m$ and $x\restriction_{m_m} = a_m\restriction_{m_m}$. Now, if $y \in B_\leq(m_m, x)$ and $c_m \in B_\geq(m_m, a_m)$, then $y \leq x \leq a_m \leq c_m$ and $y\restriction_{m_m} = x\restriction_{m_m} = a_m\restriction_{m_m} =$

$c_m\restriction_{m_m}$, so $y \in B_{\leq}(m_m, c_m)$ which is enough to guarantee that $F^{-1}(U(m,1))$ is open too.

Thus, both F_1 and F_2 are continuous functions, so defining $F : X \times Y \to \mathbb{R}$ by $F(x, (\langle i_n\rangle_{n<\omega}, \langle a_n, m_n\rangle_{n<\omega})) = F_1(x, \langle i_n\rangle_{n<\omega}) + F_2(x, \langle a_n, m_n\rangle_{n<\omega})$ will also give a continuous function. As defined, $F(x, (\langle i_n\rangle_{n<\omega}, \langle a_n, m_n\rangle_{n<\omega})) = 0$ precisely when $F_1(x, \langle i_n\rangle_{n<\omega}) = 0$ and $F_2(x, \langle a_n, m_n\rangle_{n<\omega}) = 0$, so every open set is the cozero set corresponding to F_y for some $y \in Y$, as the union of a Euclidean cozero set with a countable union of basic open sets. □

Example 11 ($\mathfrak{b} = \aleph_1$). *There is a hereditarily separable, hereditarily Lindelöf non-metrisable space X with $nw(X) = \aleph_1$, which has a zero set universal parametrised by a space which is both hereditarily separable and hereditarily Lindelöf.*

Hence there is a space with uncountable netweight despite having a continuous function universal parametrised by a space whose countable power is hereditarily separable and hereditarily Lindelöf.

Proof: The ‘hence’ part follows from the first assertion and Theorem 1.

Let $X = A[\geq_L]$ and $Y = (A[\leq_L] \times \omega)^\omega$. We shall show that X has a zero set universal parametrised by Y. Now, since X is hereditarily Lindelöf, we know, again, that the cozero sets in X are precisely the open sets, and that, moreover, each of these sets is a countable union of basic open sets. Hence, we can define F in a similar fashion to our previous example, by $F(x, \langle a_n, m_n\rangle_{n<\omega}) = \langle b_n\rangle_{n<\omega}$, where $b_n = 1$ if $x \in B_{\geq_L}(m_n, a_n)$ and $b_n = 0$ otherwise. With this definition, it is clear that F has all properties (save continuity) that are needed. It just remains to check continuity. This simplifies to checking continuity with respect to each of the two types of sub-basic open sets in 2^ω:

- For $U(m,0) = \{\langle b_n\rangle_{n<\omega} : b_m = 0\}$, we let $(x, \langle a_n, m_n\rangle_{n<\omega}) \in F^{-1}(U(m,0))$. Hence, equivalently, $x \notin B_{\geq_L}(m_m, a_m) = \{b \in A : b \geq_L a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$. Therefore, one of two cases will apply. Firstly, if $x\restriction_{m_m} \neq a_m\restriction_{m_m}$, we let $n = m_m$. Otherwise, $x\restriction_{m_m} = a_m\restriction_{m_m}$ and $x \not\geq_L a_m$, so $x <_L a_m$, in which case, pick the least $n > m_m$ for which $x\restriction_n \neq a_m\restriction_n$. Thus, if $y \in B_{\geq_L}(n, x)$ and $\langle c_i, r_i\rangle_{i<\omega}$ is such that $c_m \in B_{\leq_L}(n, a_m)$ and $r_m = m_m$, then $y\restriction_n = x\restriction_n \neq a_m\restriction_n = c_m\restriction_n$. In the second case, it must also be the case that $y <_L c_m$ and $y\restriction_{m_m} = c_m\restriction_{m_m}$, so in both cases, $(y, \langle c_i, r_i\rangle_{i<\omega}) \in F^{-1}(U(m,0))$ which is seen to be an open set.
- For $U(m,1) = \{\langle b_n\rangle_{n<\omega} : b_m = 1\}$, we let $(x, \langle a_n, m_n\rangle_{n<\omega}) \in F^{-1}(U(m,1))$. Hence, equivalently, $x \in B_{\geq_L}(m_m, a_m) = \{b \in A : b \geq_L a_m \wedge a_m\restriction_{m_m} = b\restriction_{m_m}\}$, so, in other words, $x \geq_L a_m$ and $x\restriction_{m_m} = a_m\restriction_{m_m}$. Now, if $y \in B_{\geq_L}(m_m, x)$ and $c_m \in B_{\geq_L}(m_m, a_m)$, then $y \geq_L x \geq_L a_m \geq_L c_m$ and $y\restriction_{m_m} = x\restriction_{m_m} = a_m\restriction_{m_m} = c_m\restriction_{m_m}$, so $y \in B_{\geq_L}(m_m, c_m)$ which is enough to guarantee that $F^{-1}(U(m,1))$ is open too.

Hence, by the standard argument from a sub-basis of the range space, F is continuous, so shows that Y parametrises a zero set universal for X as required. $\square$

Observe that this last example is an uncountable subspace of the Sorgenfrey line. Also note that these examples also give rise to S and L admissible topologies.

5. Separability and chain conditions

5.1 Sufficient conditions. We will find some sufficient conditions for a space to have a continuous function universal parametrised by a separable space, a ccc space or a space with calibre ω_1. These rely on the idea of a K-coarser topology on a space.

Definition 12. Let τ, σ be two topologies on a set X with $\tau \subset \sigma$. We say that τ is a K-coarser topology if (X, σ) has a neighbourhood basis consisting of τ-compact neighbourhoods.

The existence of a K-coarser topology τ on a space (X, σ) will allow us to construct a continuous function universal for (X, σ) by refining the topology on $C_k(X)$ without adding "too many" open sets.

Fix a space (X, σ). Let $\mathfrak{U} = \{(r, q) : r, q \in \mathbb{Q}, r < q\}$ and $\mathfrak{U}_{\mathbb{Q}} = \mathfrak{U} \cup \{\{q\} : q \in \mathbb{Q}\}$. Fix $\mathcal{C} = \langle C_0, \ldots, C_n \rangle$ where each $C_i \subset X$ and $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$ where each $U_i \subset \mathbb{R}$. Define $W'(\mathcal{C}, \mathcal{U}) = \{f \in \mathbb{R}^X : \forall i \leq n (f[C_i] \subset U_i)\}$ and $W(\mathcal{C}, \mathcal{U}) = \{f \in C(X) : \forall i \leq n (f[C_i] \subset U_i)\}$.

If $\mathfrak{B} \subset \mathcal{P}(\mathbb{R})$ and τ, σ are topologies on X we define the space $C_{k_\tau}((X, \sigma), \mathfrak{B})$ to have as its underlying set $C(X, \sigma)$ and a basis

$$\mathcal{S} = \{W(\mathcal{C}, \mathcal{U}) : \mathcal{C} \in \mathcal{P}(X)^{<\omega}, \mathcal{U} \in \mathfrak{B}^{<\omega}, |\mathcal{C}| = |\mathcal{U}|, \forall C \in \mathcal{C} (C \text{ is } \tau\text{–compact})\}.$$

For any set A the set $A^{<\omega}$ is the collection of all finite partial functions from ω into X whose domain consists of some initial segment of ω. Note that $C_{k_\sigma}((X, \sigma), \mathfrak{U})$ is simply the space $C_k(X, \sigma)$.

Let τ be a K-coarser topology on (X, σ). The space $C_{k_\tau}((X, \sigma), \mathfrak{U}_{\mathbb{Q}})$ parametrises a continuous function universal for (X, σ) via the evaluation map. In addition this space is T_2 and 0-dimensional, and so the space is Tychonoff. Although it may be easier to work with the space $C_{k_\tau}((X, \sigma), \mathfrak{U})$ it is difficult to see how one would show that this space is Tychonoff. We summarise with the following theorem.

Theorem 13. *Let τ be a K-coarser topology on (X, σ).*

(i) *The space $C_{k_\tau}((X, \sigma), \mathfrak{U}_{\mathbb{Q}})$ parametrises a continuous function universal for (X, σ) via the evaluation map.*
(ii) *$C_{k_\tau}((X, \sigma), \mathfrak{U}_{\mathbb{Q}})$ is T_2 and 0-dimensional, and hence is Tychonoff.*

We will now show that $C_{k_\tau}((X, \tau), \mathfrak{U}_{\mathbb{Q}})$ is a dense subspace of $C_{k_\sigma}((X, \sigma), \mathfrak{U}_{\mathbb{Q}})$. Towards this end we have the following theorem.

Theorem 14. *Let X be a Tychonoff space. Let $\mathcal{C} = \langle C_0, \dots, C_n \rangle$ consist of subsets of X and let $\mathcal{U} = \langle U_0, \dots, U_n \rangle$ where each $U_i \in \mathfrak{U}$. Assume $\mathcal{D} = \langle D_0, \dots, D_m \rangle$ consists of subsets of X and that $\mathcal{V} = \langle \{q_0\}, \dots, \{q_m\} \rangle$ where each $q_j \in \mathbb{Q}$.*

If either (i) *each C_i and D_i is compact or* (ii) *each C_i and D_i is a zero set and if there exists $f \in W'(\mathcal{C},\mathcal{U}) \cap W'(\mathcal{D},\mathcal{V})$ then there exists $g \in W(\mathcal{C},\mathcal{U}) \cap W(\mathcal{D},\mathcal{V})$.*

The following lemma will simplify this proof.

Lemma 15. *Let $\mathcal{C} = \langle C_i : i \leq n \rangle$ consist of subsets of X. Define $E_A = \bigcap_{j \in A} C_j \setminus \bigcup_{j \notin A} C_j$ and for each $A \subset n+1$ define $o(A) = |(n+1) \setminus A|$. Let $I \subset \mathcal{P}(n+1)$ satisfy: there exists $k \leq n$ such that $o(A) \leq k+1$ for all $A \in I$ and if $o(A) \leq k$ then $A \in I$ (we say that such an I is* downward closed). *Then $\bigcup\{E_A : A \in I\} = \bigcup\{\bigcap_{j \in A} C_j : A \in I\}$.*

Now we are ready to prove Theorem 14.

Proof: We only give the proof for case (i) as case (ii) can be proved with minor modifications of the same argument. Let $\mathcal{C}, \mathcal{U}, \mathcal{D}$ and $\mathcal{V}$ be as in the statement of the lemma, case (i).

Assume that there exists some $f \in W'(\mathcal{C},\mathcal{U}) \cap W'(\mathcal{D},\mathcal{V})$. We divide our proof into two parts. First we show that (a) there exists $h \in W(\mathcal{C},\mathcal{U})$ and then we show that (b) there exists $g \in W(\mathcal{C},\mathcal{U}) \cap W(\mathcal{D},\mathcal{V})$.

Part (a): For each $i \leq n$ we will recursively define a continuous function h_i satisfying: for all $A \subset n+1$ such that $o(A) \leq i$ and for all $x \in E_A$ we have $h_i(x) \in \bigcap\{U_j : j \in A\}$. This will suffice as defining $h = h_n$ we must have that $h \in W(\mathcal{C},\mathcal{U})$.

To construct h_0: there is only one $A \subset n+1$ such that $o(A) = 0$, that is $A = n+1$. If $E_A \neq \emptyset$ then we can choose $r_0 \in \bigcap \mathcal{U}$. We define a function h_0 by setting for each $x \in X$ that $h_0(x) = r_0$. If $E_A = \emptyset$ then any choice of h_0 will do.

Assume that for some $k < n$ and for all $i \leq k$ we have the required function h_i. To construct h_{k+1}: Let $\langle A_0, \dots, A_l \rangle$ be an ordering of the set $\{A : o(a) = |k+1|\}$. We claim that for each $s \leq l$ we can recursively define a continuous function h^s_{k+1} satisfying: (1) for all $A \subset n+1$ such that $o(A) \leq k$ and for all $x \in E_A$ we have $h^s_{k+1}(x) \in \bigcap\{U_j : j \in A\}$ and (2) for all $i \leq s$ and for all $x \in E_{A_i}$ we have $h^s_{k+1}(x) \in \bigcap\{U_j : j \in A_i\}$. Then defining $h_{k+1} = h^l_{k+1}$ we will have constructed the required h_{k+1}.

All that remains to be shown is that the claim is true. Let $h^{-1}_{k+1} = h_k$. Assume that for some $s < l$ and all $i \leq s$ we have defined the required h^i_{k+1}. Let $Z^{s+1}_{k+1} = \{x : \exists j \leq n (x \in C_j \wedge h^s_{k+1}(x) \notin U_j)\}$ and note that Z^{s+1}_{k+1} is a compact set. To see this we can rewrite Z^{s+1}_{k+1} as

$$Z^{s+1}_{k+1} = \bigcup_{j \leq n} (C_j \cap (f^s_{k+1})^{-1}(\mathbb{R} \setminus U_j)).$$

To define h^{s+1}_{k+1}: if $E_{A_{s+1}} \cap Z^{s+1}_{k+1} = \emptyset$ then let $h^{s+1}_{k+1} = h^s_{k+1}$ and note that this function satisfies (1) and (2) as described in the previous paragraph. If not then find $r^{s+1}_{k+1} \in \bigcap\{U_j : j \in A_{s+1}\}$. By Lemma 15 the set $\bigcup\{E_A : o(A) \leq k\} \cup \bigcup\{E_{A_i} : i \leq s\}$ is compact and from the definitions is disjoint from Z^{s+1}_{k+1}.

We can now find a continuous function p^{s+1}_{k+1} such that $p^{s+1}_{k+1} \restriction Z^{s+1}_{k+1} = 1$ and $p^{s+1}_{k+1} \restriction (\bigcup\{E_A : o(A) \leq k\} \cup \bigcup\{E_{A_i} : i \leq s\}) = 0$ and $p^{s+1}_{k+1}[X] \subset [0,1]$.

We define the function h^{s+1}_{k+1} by setting for each $x \in X$ that $h^{s+1}_{k+1}(x) = h^s_{k+1}(x) - h^s_{k+1}(x)p^{s+1}_{k+1}(x) + r^{s+1}_{k+1}p^{s+1}_{k+1}(x)$. This function is continuous and it satisfies (1) and (2) as described earlier.

Part (b): We will now recursively define for each $k \leq m$ a continuous function g_k such that $g_k \in W(\mathcal{C},\mathcal{U}) \cap W(\langle D_0, \dots, D_k\rangle, \langle\{q_0\}, \dots, \{q_k\}\rangle)$. Let $g_{-1} = h$. Assume that there is $k < m$ such that for each $i \leq k$ we have defined the required g_i. Find a continuous function p_{k+1} that satisfies: for all $x \in D_{k+1}$ we have $p_{k+1}(x) = 1$, for all $i \leq k$ and $x \in D_i$ we have $p_{k+1}(x) = 0$ and for all $j \leq n$ such that $D_{k+1} \cap C_j = \emptyset$ and $x \in C_j$ we have $p_{k+1}(x) = 0$. Now we define the function g_{k+1} by setting for each $x \in X$ that $g_{k+1}(x) = g_k(x) - p_{k+1}(x)g_k(x) + p_{k+1}(x)q_{k+1}$. It is easily verified that $g_{k+1} \in W(\mathcal{C},\mathcal{U}) \cap W(\langle D_0, \dots, D_{k+1}\rangle, \langle\{q_0\}, \dots, \{q_{k+1}\}\rangle)$. Now defining $g = g_m$ we have constructed the required function. $\square$

The next result follows almost immediately from Theorem 14.

Corollary 16. *Fix a Tychonoff space (X,σ) and let τ be a K-coarser topology. Then $C_{k_\tau}((X,\tau),\mathfrak{U}_{\mathbb{Q}})$ is a dense subspace of $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$.*

Now we have reduced the problem of showing that $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ is separable, ccc or has calibre ω_1 to that of demonstrating that $C_{k_\tau}((X,\tau),\mathfrak{U}_{\mathbb{Q}})$ has these properties. From now on we will write $C_k(X,\mathfrak{U}_{\mathbb{Q}})$ and $C_k(X)$ as there will be only one topology considered on X.

Fix an arbitrary space X. We will investigate when $C_k(X,\mathfrak{U}_{\mathbb{Q}})$ is ccc, separable or has calibre ω_1.

Lemma 17. *Fix a Tychonoff space X. Then*

(i) *$C_k(X,\mathfrak{U}_{\mathbb{Q}})$ is separable if and only if $C_k(X)$ is separable,*
(ii) *$C_k(X,\mathfrak{U}_{\mathbb{Q}})$ is ccc if and only if $C_k(X)$ is ccc,*
(iii) *$C_k(X,\mathfrak{U}_{\mathbb{Q}})$ has calibre ω_1 if and only if $C_k(X)$ has calibre ω_1.*

Proof: The identity map is a continuous function from $C_k(X,\mathfrak{U}_{\mathbb{Q}})$ onto $C_k(X)$ and so one implication is immediate in (i), (ii) and (iii).

Part (i): Assume that $C_k(X)$ is separable and so X has a coarser separable metric topology τ. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a basis for τ. Assume that $\mathcal{A}$ is closed under finite unions. For each $n, m \in \omega$ let $f_{n,m} : X \to \mathbb{R}$ be a τ-continuous function satisfying $f_{n,m}(x) = 1$ when $x \in \overline{A_m}$ and $f_{n,m}(x) = 0$ when $x \in X \setminus A_n$.

Of course this is only well-defined when $\overline{A_m} \subset A_n$ and if this does not hold then we let $f_{n,m}(x) = 0$ for all $x \in X$. The linear span of $\{f_{n,m} : n, m \in \omega\}$ over $\mathbb{Q}$ is a countable set that is dense in $C_k(X, \mathfrak{U}_{\mathbb{Q}})$.

Part (iii): Assume that $C_k(X)$ has calibre ω_1. Fix an uncountable collection $\mathcal{W} = \{W_\alpha : \alpha \in \omega_1\}$ of basic open non-empty subsets of $C_k(X, \mathfrak{U}_{\mathbb{Q}})$. So we can assume that each W_α is of the form $W(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap W(\mathcal{D}_\alpha, \mathcal{V}_\alpha)$ where for all $\alpha \in \omega_1$: $\mathcal{C}_\alpha = \{C_i^\alpha : i \leq n_\alpha\}$ consists of zero sets of X, $\mathcal{U}_\alpha = \{U_i^\alpha : i \leq n_\alpha\}$ where each $U_i \in \mathfrak{U}$, $\mathcal{D}_\alpha = \{D_j^\alpha : j \leq m_\alpha\}$ consists of pairwise disjoint zero subsets of X and $\mathcal{V}_\alpha = \{\{q_j^\alpha\} : j \leq m_\alpha\}$ where each $q_j^\alpha \in \mathbb{Q}$.

By passing to an uncountable subcollection we can assume that for all $\alpha, \beta \in \omega_1$ we have $\mathcal{U}_\alpha = \mathcal{U}_\beta$ and $\mathcal{V}_\alpha = \mathcal{V}_\beta$. We will drop the subscripts and use $\mathcal{U}$ and $\mathcal{V}$ to denote these sets. Assume that $|\mathcal{U}| = n$ and $|\mathcal{V}| = m$. We can write $\mathcal{V}$ as $\{\{q_j\} : j \leq m\}$ where each $q_j \in \mathbb{Q}$. Choose $\delta > 0$ such that $4\delta < \min\{|q_i - q_j| : i, j \leq m\}$. We define a new collection $\mathcal{V}'$ by defining for each $j \leq m$ the set $V_j' = (q_j - \delta, q_j + \delta)$ and letting $\mathcal{V}' = \{V_j' : j \leq m\}$. Note that for each $\alpha \in \omega_1$ we know that $W(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap W(\mathcal{D}_\alpha, \mathcal{V}_\alpha')$ is a non-empty subset of $C_k(X)$. So there is some $f \in C(X)$ and $A \subset \omega_1$ such that $|A| = \omega_1$ and $f \in \bigcap\{W(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap W(\mathcal{D}_\alpha, \mathcal{V}_\alpha') : \alpha \in A\}$.

We define two collections of zero sets $\mathcal{C}$ and $\mathcal{D}$ by defining for each $i \leq n$, $C_i = f^{-1}(\overline{U_i})$ and for each $j \leq m$ we define $D_j = f^{-1}(\overline{V_i'})$. Note that if $j, j' \leq m$ then $D_j \cap D_{j'} = \emptyset$ when $j \neq j'$. We can define a new function f' by setting $f'(x) = f(x)$ when $x \notin \bigcup \mathcal{D}$ and $f'(x) = q_j$ when $x \in D_j$. Now we have that $f' \in W'(\mathcal{C}, \mathcal{U}) \cap W'(\mathcal{D}, \mathcal{V})$ and so applying Lemma 14 we know that there exists $g \in W(\mathcal{C}, \mathcal{U}) \cap W(\mathcal{D}, \mathcal{V})$. But

$$W(\mathcal{C}, \mathcal{U}) \cap W(\mathcal{D}, \mathcal{V}) \subset \bigcap_{\alpha \in A} W_\alpha$$

and so we are done.

Part (ii) can be proved in much the same way as part (iii). □

Corollary 18. *Let (X, σ) be a Tychonoff space and let τ be a K-coarser topology.*

(i) *If (X, τ) is second countable then (X, σ) has a continuous function universal parametrised by a separable space.*
(ii) *If $C_k(X, \tau)$ is ccc then (X, σ) has a continuous function universal parametrised by a ccc space.*
(iii) *If $C_k(X, \tau)$ has calibre ω_1 then (X, σ) has a continuous function universal parametrised by a space with calibre ω_1.*

We can also derive sufficient conditions in the ccc or calibre ω_1 cases that do not depend on the properties of an external object (such as $C_k(X, \tau)$). In [13] necessary and sufficient conditions on X for $C_k(X)$ to have calibre ω_1 are described and in [12] the same is done for the ccc case. We can summarise these results in the following lemma. Note that $\mathcal{C}$ is an n-chain of sets if $\mathcal{C}$ is an ordered collection of $n+1$ many sets $\langle C_0, \dots, C_n \rangle$ and $C_i \cap C_j = \emptyset$ if $|i-j| > 1$.

Lemma 19. *Let X be a Tychonoff space. $C_k(X)$ is ccc if and only for all $n > 1$ and for every collection of n-chains of compact sets $\{\langle F^0_\alpha, \dots, F^n_\alpha \rangle : \alpha \in \omega_1\}$ there are $\alpha_1, \alpha_2 \in \omega_1$ and an n-chain of zero sets $\{C_i : i \leq n\}$ satisfying:* (a) *for $j = 1, 2$ and $i \leq n$ we have $F^i_{\alpha_j} \subset C_i$ and* (b) *for $j = 1, 2$ and $i < n$ we have $C_i \cap C_{i+1} = \emptyset$ if and only if $F^i_{\alpha_j} \cap F^{i+1}_{\alpha_j} = \emptyset$.*

$C_k(X)$ has calibre ω_1 if and only if for all $n > 1$ and every collection of n-chains of compact sets $\{\langle F^0_\alpha, \dots, F^n_\alpha \rangle : \alpha \in \omega_1\}$ there is some $A \subset \omega_1$ with $|A| = \omega_1$ and an n-chain of zero sets $\{C_i : i \leq n\}$ satisfying: (a) *for $\alpha \in A$ and $i \leq n$ we have $F^i_\alpha \subset C_i$ and* (b) *for $\alpha \in A$ and $i < n$ we have $C_i \cap C_{i+1} = \emptyset$ if and only if $F^i_\alpha \cap F^{i+1}_\alpha = \emptyset$.*

These results are useful when dealing with spaces that are not locally compact, as in the locally compact case $C_k(X)$ itself will parametrise a continuous function universal for X. For example we can now construct a separable space Y that parametrises a continuous function universal for the Sorgenfrey line. If we let X be the disjoint sum of $\mathfrak{c}^+$ many copies of the Sorgenfrey line then we know that X has no continuous function universal parametrised by a separable space as $C_p(X)$ is not even separable. But since X will have a K-coarser metric topology we can construct a continuous function universal parametrised by a ccc space.

5.2 Necessary conditions. We will deal first with the case where a space X has a continuous function universal parametrised by a separable space. We say a space (X, σ) is co-SM if and only if there is a separable metric topology $\tau \subset \sigma$ such that (X, σ) has a neighbourhood basis of τ-closed sets.

Lemma 20. *Let X be a Tychonoff space. If X has a continuous function universal parametrised by a separable space then X is co-SM.*

Proof: Let Y be a separable metric space that parametrises a continuous function universal for X via the function $F : X \times Y \to \mathbb{R}$. Let D be a countable dense subset of Y. Each $d \in D$ represents the continuous function F^d. Let τ be the coarsest topology that makes each F^d continuous and note that τ is separable metric.

Fix x in open U. Pick $y \in Y$ so that $F(x, y) = 1$ and $F[(X \setminus U) \times \{y\}] = \{0\}$. By continuity of F at (x, y) there are open V and W with $x \in V$, $y \in W$ and $F[V \times W] \subseteq (\frac{2}{3}, \frac{4}{3})$.

Claim: If $x' \notin \overline{U}$ then there is a τ-open T containing x' disjoint from V.

From the claim it follows that $\overline{V}^\tau \subseteq \overline{U}$, and by regularity of X, the τ-closed neighbourhoods of x form a local base — as required for co-SM.

If we assume that $x' \notin \overline{U}$ then we must have that $F(x', y) = 0$. So by continuity of F at (x', y) there are open V' and W' with $x' \in V'$ and $y \in W'$ so that $F[V' \times W'] \subseteq (\frac{-1}{3}, \frac{1}{3})$.

Pick $d \in D \cap (W \cap W')$. Then $d \in W'$ so $F(x', d) \in (\frac{-1}{3}, \frac{1}{3})$. Hence by τ-continuity of F^d at x', there is a τ-open $T \ni x'$ such that $F[T \times \{d\}] \subseteq (\frac{-1}{3}, \frac{1}{3})$. Since $d \in W$, $F[V \times \{d\}] \subseteq (\frac{2}{3}, \frac{4}{3})$. Hence V and T are disjoint — as required.

□

Note that this falls short of the sufficient condition given previously leading to the following question.

Problem 21. *Is there a Tychonoff space X such that X is co-SM but X can have no continuous function universal parametrised by a separable space?*

Turning our attention to parametrising spaces which are ccc we introduce the following two properties.

Definition 22. A space X has the property P_1 if and only if for every pair of disjoint compact subsets (K, L) there exists a pair of open sets $U(K, L), V(K, L)$ with $K \subset U(K, L)$, $L \subset V(K, L)$ and $\overline{U(K, L)} \cap \overline{V(K, L)} = \emptyset$ satisfying the following:

for any collection $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact sets there exists α_1, α_1 such that

$$\overline{\bigcup_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})} \cap \overline{\bigcup_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i})} = \emptyset.$$

Definition 23. A space X has the property P_2 if and only if for every pair of disjoint compact subsets (K, L) there exists a pair of open sets $U(K, L), V(K, L)$ with $K \subset U(K, L)$, $L \subset V(K, L)$ and $\overline{U(K, L)} \cap \overline{V(K, L)} = \emptyset$ satisfying the following:

for any collection $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact sets there exists α_1, α_1 such that

$$\bigcup_{i=1,2} K_{\alpha_i} \subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

and

$$\bigcup_{i=1,2} L_{\alpha_i} \subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i}).$$

Lemma 24. *Let X be a Tychonoff space. If X has a zero set universal parametrised by a ccc space then X has property P_1 and every compact subspace has property P_2.*

Proof: Let Y be ccc and assume that Y parametrises a zero set universal for X via the continuous function $F : X \times Y \to \mathbb{R}$. Let Z be the disjoint sum of ω many copies of Y and let Y_n denote the nth copy of Y that is a subspace of Z. Define a function $F' : X \times Z \to \mathbb{R}$ by letting $F'(x,z) = nF(x,z)$ when $z \in Y_n$. Finally let $G = |F'|$. Note that Z parametrises a zero set universal for X via G, that Z is ccc and that for any pair of disjoint compact sets $K, L \subset X$ there exists $z \in Z$ such that $G^z[K] = 0$ and $G^z[L] \subset [1,\infty)$. We say that such a z separates K and L.

We will first show that X has property P_2 on its compact subspaces. Fix a compact subspace C. Let K, L be disjoint compact subsets of C. We show how to construct the required $U(K,L)$ and $V(K,L)$. Since K and L are compact we can find $z(K,L) \in Z$ that separates K and L. Let $U(K,L) = \{x \in C : G(x, z(K,L)) < \frac{1}{4}\}$ and $V(K,L) = \{x \in C : G(x, z(K,L)) > \frac{3}{4}\}$. Find open $W(K,L)$ such that $z(K,L) \in W(K,L)$ and for all $(x,z_1),(x,z_2) \in C \times W(K,L)$ we have $|G(x,z_1) - G(x,z_2)| < \frac{1}{8}$.

Now take a collection $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact subsets of C. Look at the corresponding collection $\{W(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$. Since Z is ccc there must be $z \in Z$ and $\alpha_1, \alpha_2 \in \omega_1$ such that $z \in W(K_{\alpha_1}, L_{\alpha_1}) \cap W(K_{\alpha_2}, L_{\alpha_2})$. We claim that

$$\bigcup_{i=1,2} K_{\alpha_i} \subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

and

$$\bigcup_{i=1,2} L_{\alpha_i} \subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i})$$

as required. We will only show that $K_{\alpha_1} \subset U(K_{\alpha_2}, L_{\alpha_2})$ as the other cases can be dealt with similarly. Fix $x \in K_{\alpha_1}$. Note that $G(x,z) < \frac{1}{8}$ since $G(x, z(K_{\alpha_1}, L_{\alpha_1})) = 0$ and $z \in W(K_{\alpha_1}, L_{\alpha_1})$. But then $G(x, z(K_{\alpha_2}, L_{\alpha_2})) < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ and so $x \in U(K_{\alpha_2}, L_{\alpha_2})$.

Now we will show that X has property P_1. The proof is similar to the P_1 case and so we will only show how to construct $U(K,L)$ and $V(K,L)$. Let K, L be disjoint compact subsets of X. Find $z \in Z$ that separates K and L. Using the compactness of K and L and the continuity of G find open $U(K,L), V(K,L), W(K,L)$ such that $K \subset U(K,L)$, $L \subset V(K,L)$ and $z \in W(K,L)$ satisfying: for all $(x,z') \in U(K,L) \times W(K,L)$, $G(x,z') < \frac{1}{4}$ and for all $(x,z') \in V(K,L) \times W(K,L)$, $G(x,z') > \frac{3}{4}$. □

Lemma 25. *Let X be a compact Hausdorff space. If X has property P_2 then X is metrisable.*

Proof: It suffices to find a countable T_1-separating collection of open subsets of X (see for example [9]). Let $\mathcal{C} = \{(K_\alpha, L_\alpha) : \alpha \in I\}$ be a collection of disjoint pairs of compact subsets of X that satisfies $(*)$: for all $\alpha_1, \alpha_2 \in I$ either

$$\bigcup_{i=1,2} K_{\alpha_i} \not\subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

or

$$\bigcup_{i=1,2} L_{\alpha_i} \not\subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i}).$$

Assume that $\mathcal{S} = \{U(K_\alpha, L_\alpha) : \alpha \in I\} \cup \{V(K_\alpha, L_\alpha) : \alpha \in I\}$ is not a T_1-separating collection. We will show that we can find (K, L) such that $\mathcal{C} \cup \{(K, L)\}$ satisfies the same property $(*)$ as $\mathcal{C}$. Since $\mathcal{S}$ is not a T_1-separating collection there exist $x_1, x_2 \in X$ such that for all $C \in \mathcal{S}$ we have $x_1 \in C$ implies $x_2 \in C$. Let $K = \{x_1\}$ and let $L = \{x_2\}$. Fix $\alpha \in I$. If $x_1 \in U(K_\alpha, L_\alpha)$ and $x_2 \in V(K_\alpha, L_\alpha)$ then $x_2 \notin U(K_\alpha, L_\alpha)$, contradicting the choice of x_1, x_2. So condition $(*)$ holds for $\mathcal{C} \cup \{(K, L)\}$.

Now let $\mathcal{C}$ be a collection of disjoint pairs of compact subsets of X that is maximal with respect to $(*)$ (i.e. $\mathcal{C}$ satisfies $(*)$, but for any collection $\mathcal{D}$, if $\mathcal{C} \subsetneq \mathcal{D}$ then $\mathcal{D}$ does not have property $(*)$). Since X has P_2 we must have that $\mathcal{C}$ is countable. But $\mathcal{S}$ as described above must be a T_1-separating collection, and so we are done. □

Problem 26. *Does the property P_1 imply the property P_2? If not is the property P_1 equivalent to metrisability in compact spaces?*

References

[1] Arens R., Dugundji J., *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5–31.
[2] Arhangel'skii A.V., *Topological Function Spaces*, Kluwer Academic Publishers, 1992.
[3] Engelking R., *General Topology*, Heldermann, Berlin, 1989.
[4] Gartside P., Marsh A., *Compact universals*, Topology Appl. **143** (2004), no. 1–3, 1–13.
[5] Gartside P.M., Knight R.W., Lo J.T.H., *Parametrizing open universals*, Topology Appl. **119** (2002), no. 2, 131–145.
[6] Gartside P.M., Lo J.T.H., *The hierarchy of Borel universal sets*, Topology Appl. **119** (2002), 117–129.
[7] Gartside P.M., Lo J.T.H., *Open universal sets*, Topology Appl. **129** (2003), no. 1, 89–101.
[8] Gruenhage G., *Continuously perfect normal spaces and some generalizations*, Trans. Amer. Math. Soc. **224** (1976), 323–338.
[9] Gruenhage G., *Generalized metric spaces*, in Handbook of Set-theoretic Topology, North Holland, Amsterdam, 1984, pp, 423–501.
[10] Gul'ko S.P., *On properties of subsets of Σ-products*, Soviet Math. Dokl. **18** (1977), 1438–1442.

[11] Hodel R., *Cardinal functions I*, in Handbook of Set-theoretic Topology, North Holland, Amsterdam, 1984, pp. 1–61.
[12] Marsh A., *Topology of function spaces*, PhD. Thesis, Univ. Pittsburgh, 2004.
[13] Nakhmanson L.B., *The Suslin number and calibres of the ring of continuous functions*, Izv. Vyssh. Uchebn. Zaved. Mat. (1984), no. 3, 49–55.
[14] Todorčević S., *Partition Problems in Topology*, Contemporary Mathematics 84, Amer. Math. Soc., Providence, RI, 1989.
[15] Zenor P., *Some continuous separation axioms*, Fund. Math. **90** (1975/1976), no. 2, 143–158.

School of Physics and Astronomy, The University of Manchester, Manchester M13 9PL, United Kingdom

Department of Mathematics, 301 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260, USA

E-mail: gartside@math.pitt.edu

Department of Mathematics, NUI, Galway, Newcastle Road, Galway, Ireland

(*Received* November 5, 2004, *revised* August 29, 2005)