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Remarks on an article of J.P. King

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Abstract. The present note discusses an interesting positive linear operator which was recently introduced by J.P. King. New estimates in terms of the first and second modulus of continuity are given, and iterates of the operators are considered as well. For general King operators the second moments are minimized.

Keywords: positive linear operators, degree of approximation, contraction principle, second order modulus, second moments

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1. Introduction

In [4] J.P. King defined the following interesting (and somewhat exotic) sequence of linear and positive operators $V_n : C[0,1] \to C[0,1]$ which generalize the classical Bernstein operators B_n:

$$V_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right) \tag{1}$$

for all $f \in C[0,1]$, $0 \le x \le 1$, where $r_n : [0,1] \to [0,1]$ are continuous functions.

We list some of their properties.

Property 1.1. *If $\{V_n\}_{n\in\mathbb{N}}$ are the operators defined in (1) we have*

$$\begin{aligned} V_n(e_0;x) &= e_0(x) \\ V_n(e_1;x) &= r_n(x) \quad \text{and} \\ V_n(e_2;x) &= \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2 \end{aligned} \tag{2}$$

where $e_i(x) = x^i$, $i = 0,1,2$, are the classical test functions for positive linear operator approximation.

The equation $V_n(e_1;x) = r_n(x)$ shows that the classical Bernstein operator B_n, which is obtained for $r_n(x) = x$, is the unique mapping of the form (1) which reproduces linear functions.

Theorem 1.2. *One has* $\lim_{n\to\infty} V_n f(x) = f(x)$ *for each* $f \in C[0,1]$, $x \in [0,1]$, *if and only if* $\lim_{n\to\infty} r_n(x) = x$.

Choosing the "right" r_n function, J.P. King proved the following:

Theorem 1.3. *Let* $\{V_n^*\}_{n\in\mathbb{N}}$ *be the sequence of operators defined in (1) with*

$$(3) \qquad r_n^*(x) := \begin{cases} r_1^*(x) = x^2, & n = 1, \\ r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & n = 2, 3, \ldots \end{cases}$$

Then:

(i) $V_n^*(e_2; x) = e_2(x)$, $n \in \mathbb{N}$; $x \in [0,1]$,
(ii) $V_n^*(e_1; x) \neq e_1(x)$,
(iii) $\lim_{n\to\infty} V_n^*(f; x) = f(x)$ *for each* $f \in C[0,1]$.

Remark 1.4. Since $V_n^* e_1 = r_n^*$, it is clear that V_n^* is not a polynomial operator.

J.P. King also gave quantitative estimates for V_n^* in terms of the classical first order modulus $\omega_1(f; \cdot)$ using a result of O. Shisha and B. Mond [8].

Theorem 1.5. *For* $\{V_n^*\}_{n\in\mathbb{N}}$ *defined in* (1) *we have*

$$(4) \quad |V_n^*(f; x) - f(x)| \leq 2\omega_1\left(f; \sqrt{2x(x - V_n^*(e_1; x))}\right), \quad f \in C[0,1];\ x \in [0,1].$$

Remark 1.6. From the fact that $V_n^*(e_1; x) = r_n^*(x)$ and $x \geq r_n^*(x)$ the square root in (4) indeed represents a real number.

From Theorem 1.5 one can easily obtain that V_n^* interpolates f at the endpoints:

Proposition 1.7. *With* $\{V_n^*\}_{n\in\mathbb{N}}$ *from* (1) *we have* $V_n^*(f; 0) = f(0)$ *and* $V_n^*(f; 1) = f(1)$, *i.e.,* V_n^* *interpolates at the endpoints* 0 *and* 1.

Proof: We put $\alpha_n(x) := \sqrt{2x(x - V_n^*(e_1; x))}$. For $x = 0$ we have $\alpha_n(0) = 0$, so $\omega_1(f; \alpha_n(0)) = 0$. That means $V_n^*(f; 0) = f(0)$. For $x = 1$ we have $V_n^*(e_1; 1) = r_n^*(1)$, and if we insert in (3) the value 1, we obtain $r_n^*(1) = 1$. That leads us again to $\omega_1(f; \alpha_n(1)) = 0$ and $V_n^*(f; 1) = f(1)$. □

Remark 1.8. For a linear and positive operator $L : C[0,1] \to C[0,1]$ with $Le_i = e_i$, $i = 0, 1$, it is known that L interpolates f in 0 and 1. This follows easily, if we insert $x = 0$ and $x = 1$ in

$$|L(f; x) - f(x)| \leq 2 \cdot \omega_1(f; L(|t - x|; x)).$$

The latter inequality can be found in Mamedov's article [5]. We observe now, with the help of the operators introduced by J.P. King, that the above property is only necessary and not sufficient. Indeed, the V_n^*, $n \in \mathbb{N}$, interpolate f in 0 and 1, they are linear and positive, but $V_n^* e_1 \neq e_1$.

2. Quantitative estimates with ω_2

From Păltănea's Theorem in [6, p. 28], the following is known:

Theorem 2.1. *Let $L : C[0,1] \to C[0,1]$ be a positive and linear operator. Then we have*

$$\begin{aligned}|L(f;x) - f(x)| \leq{}& |L(e_0;x) - e_0(x)| \cdot |f(x)| \\ &+ |L(e_1 - x;x)| \cdot \frac{1}{h}\omega_1(f;h) \\ &+ \left(L(e_0;x) + \frac{1}{2} \cdot \frac{1}{h^2} \cdot L((e_1 - x)^2;x)\right)\omega_2(f;h);\end{aligned}$$

where $h > 0$, $f \in C[0,1]$, $x \in [0,1]$, and ω_2 is the classical second order modulus defined by

$$\omega_2(f;h) := \sup_{|t|\leq h}\{|f(x+t) - 2f(x) + f(x-t)| \mid x,\ x \pm t \in [0,1]\}.$$

For V_n^* this means:

$$\begin{aligned}|V_n^*(f;x) - f(x)| \leq{}& (x - r_n^*(x)) \cdot \frac{1}{h}\omega_1(f;h) \\ &+ \left(1 + \frac{1}{h^2}x(x - r_n^*(x))\right)\omega_2(f;h),\end{aligned}$$

and for $h := \sqrt{x - r_n^*(x)}$ we arrive at

(5)
$$|V_n^*(f;x) - f(x)| \leq \sqrt{x - r_n^*(x)} \cdot \omega_1(f; \sqrt{x - r_n^*(x)}) + (1 + x)\omega_2(f; \sqrt{x - r_n^*(x)}).$$

If $f \in C^1[0,1]$ then due to the fact that $\omega_1(f;h) = O(h)$ and also $\omega_2(f;h) = O(h)$ we have the approximation order $O(\sqrt{x - r_n^*(x)})$, when $n \to \infty$. For $f \in C^2[0,1]$ having similar properties for the moduli $\omega_1(f;h) = O(h)$ and $\omega_2(f;h) = O(h^2)$ we obtain $O(x - r_n^*(x))$, $n \to \infty$.

3. Iterates of V_n^*

This section is motivated by recent papers of O. Agratini and I.A. Rus ([1], [7]) in which the contraction principle was used to show the following result of Kelisky and Rivlin [3].

Theorem 3.1. *If $n \in \mathbb{N}$ is fixed, then for all $f \in C[0,1]$, $x \in [0,1]$*

$$\lim_{m\to\infty} B_n^m(f;x) = f(0) + [f(1) - f(0)] \cdot x = B_1(f;x).$$

For "over-iterated" King operators V_n^* we have a similar result, but with a different limiting operator.

Theorem 3.2. *If $n \in \mathbb{N}$ is fixed, then for all $f \in C[0,1]$, $x \in [0,1]$*

$$\lim_{m\to\infty} (V_n^*)^m(f;x) = f(0) + [f(1) - f(0)] \cdot x^2 = V_1^*(f;x).$$

Proof: Following Rus we consider the Banach space $(C[0,1], \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the Chebyshev norm. Let

$$X_{\alpha,\beta} = \{f \in C[0,1] : f(0) = \alpha, f(1) = \beta\},\ \alpha, \beta \in \mathbb{R}.$$

We note that

a) $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$;
b) $X_{\alpha,\beta}$ is an invariant subset of V_n^* for all $\alpha, \beta \in \mathbb{R}$, $n \in \mathbb{N}$ (see Proposition 1.7);
c) $C[0,1] = \bigcup_{\alpha,\beta\in\mathbb{R}} X_{\alpha,\beta}$ is a partition of $C[0,1]$.

Now we show that

$$V_n^*|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$$

is a contraction for all $\alpha, \beta \in \mathbb{R}$.

Let $f, g \in X_{\alpha,\beta}$. From (1) we have

$$\begin{aligned}
|V_n^*(f;x) - V_n^*(g;x)| &= |V_n^*(f-g;x)| \\
&= \left| \sum_{k=1}^{n-1} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} \cdot (f-g)\left(\frac{k}{n}\right) \right| \\
&\le |1 - (r_n^*(x))^n - (1 - r_n^*(x))^n| \cdot \|f - g\|_\infty \\
&\le \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|_\infty,
\end{aligned}$$

recalling that $r_n^* : [0,1] \to [0,1]$.

Hence $\|V_n^* f - V_n^* g\|_\infty \le \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|_\infty$, and thus $V_n^*|_{X_{\alpha,\beta}}$ is contractive.

On the other hand $\alpha + (\beta - \alpha)e_2 \in X_{\alpha,\beta}$ is a fixed point for V_n^*.

If $f \in C[0,1]$ is arbitrarily given, then $f \in X_{f(0),f(1)}$ and from the contraction principle [2] we know that

$$\lim_{m\to\infty} (V_n^*)^m f = f(0) + (f(1) - f(0))e_2,$$

which concludes the proof. □

4. Polynomial operators of King's type

Can we find *polynomial* operators of the form (1) that reproduce e_2? The answer is negative!

Indeed, by the last two equations of (2) and the condition $V_n(e_2;x) = x^2$, r_n must be a polynomial of first degree. We put $r_n(x) = ax + b$ and we get:

$$x^2 = \frac{n-1}{n}a^2x^2 + \left(\frac{a}{n} + \frac{2(n-1)ab}{n}\right)x + \left(\frac{b}{n} + \frac{n-1}{n}b^2\right).$$

This leads to the equations:

$$\begin{cases} 1 = & \frac{n-1}{n}a^2, \\ 0 = & \frac{a}{n} + \frac{2(n-1)ab}{n}, \\ 0 = & \frac{b}{n} + \frac{n-1}{n}b^2. \end{cases}$$

So $a = \pm\sqrt{\frac{n}{n-1}}$ and $b = 0$ or $b = \frac{1}{1-n}$. But for these values the second equation is not satisfied. One open question remains: Can we find another type of linear and positive polynomial operators L for which $Le_2 = e_2$?

5. General case

In this section we want to "optimize" the second moments $V_n((e_1 - x)^2; x)$, $x \in [0,1]$, of the general V_n and study in this case which properties remain.

The second moments are in the general case

$$\begin{aligned} \alpha_n^2(x) = V_n((e_1-x)^2;x) &= \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2 - 2xr_n(x) + x^2 \\ &= \frac{1}{n}r_n(x)(1-r_n(x)) + (r_n(x)-x)^2, \end{aligned} \tag{6}$$

where $0 \le r_n(x) \le 1$ are continuous functions. We want to find r_n so that α_n^2 is minimal.

We define $g_x : [0,1] \to [0,1]$, $x \in [0,1]$ a fixed parameter, by $g_x(y) := \frac{1}{n}y(1-y) + (y-x)^2$. We can write $g_x(y) = \left(1 - \frac{1}{n}\right)y^2 + \left(\frac{1}{n} - 2x\right)y + x^2$. Because $1 - \frac{1}{n} > 0$, $n = 2, 3, \ldots$, the function g_x admits a minimum point:

$$y_{\min} = -\frac{\frac{1}{n} - 2x}{2 - \frac{2}{n}} = \frac{2nx - 1}{2n - 2}.$$

We need $0 \le y_{\min} \le 1$, which means $\frac{1}{2n} \le x \le 1 - \frac{1}{2n}$, $n = 2, 3, \ldots$

We define $r_n^{\min} : [0,1] \to [0,1]$ by

$$(7)\qquad r_n^{\min}(x) := \begin{cases} 0, & x \in \left[0, \frac{1}{2n}\right), \\ \frac{2nx-1}{2n-2}, & x \in \left[\frac{1}{2n}, 1-\frac{1}{2n}\right], \\ 1, & x \in \left(1-\frac{1}{2n}, 1\right]. \end{cases}$$

Theorem 5.1. *The function $r_n^{\min}$ defined in* (7) *yields the minimum value for α_n^2.*

Proof: For $x \in \left[\frac{1}{2n}, 1-\frac{1}{2n}\right]$ this was proven before. It remains to show the above affirmation for $x \in \left[0, \frac{1}{2n}\right)$ and $x \in \left(1-\frac{1}{2n}, 1\right]$.

First case: $x \in \left[0, \frac{1}{2n}\right) \Rightarrow r_n^{\min}(x) = 0$ and we have to prove that $g_x(y) \geq g_x(0)$ for each $y \in [0,1]$ or $\frac{1}{n}y(1-y) + (y-x)^2 \geq x^2$ for each $x \in [0,1]$. But the latter is equivalent to $\frac{1}{2n} + y\left(\frac{1}{2} - \frac{1}{2n}\right) \geq x$, which is true due to our choice of x.

Second case: $x \in \left(1-\frac{1}{2n}, 1\right] \Rightarrow r_n^{\min}(x) = 1$ and we have to prove that $g_x(y) \geq g_x(1)$ for each $y \in [0,1]$ or $\frac{1}{n}y(1-y) + (y-x)^2 \geq (1-x)^2$. This means $\left(1-\frac{1}{2n}\right) - (1-y)\left(\frac{1}{2} - \frac{1}{2n}\right) \leq x$, which is again true due to our choice of x. $\square$

The operators V_n defined via $r_n^{\min}$ we denote by $V_n^{\min}$.

Property 5.2. *For the* (*minimal*) *second moments α_n^2 of $V_n^{\min}$ we have the representation*

$$\alpha_n^2(x) = \begin{cases} x^2, & x \in \left[0, \frac{1}{2n}\right), \\ \frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right), & x \in \left[\frac{1}{2n}, 1-\frac{1}{2n}\right], \\ (1-x)^2, & x \in \left(1-\frac{1}{2n}, 1\right]. \end{cases}$$

Proof: This follows immediately from the general form

$$\frac{1}{n} r_n(x)(1-r_n(x)) + (r_n(x)-x)^2$$

and the above representation of $r_n^{\min}(x)$. $\square$

Using Păltănea's theorem again we arrive at

$$\begin{aligned} |V_n^{\min}(f;x) - f(x)| &\leq |x - r_n^{\min}(x)| \cdot \frac{1}{h} \cdot \omega_1(f;h) \\ &\quad + \left(1 + \frac{1}{2} \cdot \frac{1}{h^2} \cdot \alpha_n^2(x)\right) \cdot \omega_2(f;h), \quad h > 0. \end{aligned}$$

For $h = |\alpha_n(x)|$ we obtain

$$|V_n^{\min}(f;x) - f(x)| \leq \frac{|x - r_n^{\min}(x)|}{|\alpha_n(x)|} \cdot \omega_1(f;|\alpha_n(x)|) + \frac{3}{2} \cdot \omega_2(f;|\alpha_n(x)|).$$

Note that $|x - r_n^{\min}(x)| = |V_n^{\min}(e_1 - x;x)| \leq V_n^{\min}(|e_1 - x|;x) \leq \sqrt{V_n^{\min}((e_1-x)^2;x)} = |\alpha_n(x)|$, and thus $\frac{|x - r_n^{\min}(x)|}{|\alpha_n(x)|} \leq 1$, $x \in [0,1]$.

Remark 5.3. (i) From the definition of $r_n^{\min}$ we have $\lim_{n\to\infty} r_n^{\min}(x) = x$ and from Theorem 1.2 $\lim_{n\to\infty} V_n(f;x) = f(x)$.
The latter fact is also a consequence of our second application of Theorem 2.1 for $V_n^{\min}$.

(ii) $V_n^{\min}$ does not reproduce e_2. Starting from (2) we see that $V_n^{\min}(e_2;x) = 0 \neq x^2$, $x \in \left(0, \frac{1}{2n}\right)$.

(iii) The interpolation properties at the endpoints remain. Indeed, $V_n^{\min}(f;0) = \binom{n}{0}(1-r_n(0))^n f(0) = f(0)$, and $V_n^{\min}(f;1) = \binom{n}{n} f(\frac{n}{n}) = f(1)$.

(iv) For $f \in C^1[0,1]$ we have, with a constant c independent of x,

$$|V_n^{\min}(f;x) - f(x)| \leq c \cdot (|x - r_n^{\min}(x)| + |\alpha_n(x)|) =$$

$$= c\cdot \begin{cases} 2x, & x \in \left[0, \frac{1}{2n}\right), \text{ hence } O\left(\frac{1}{n}\right), \\ \frac{|\frac{1}{2}-x|}{n-1} + \sqrt{\frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right)}, & x \in \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right], \text{ hence } O\left(\frac{1}{\sqrt{n}}\right), \\ 2(1-x), & x \in \left(1 - \frac{1}{2n}, 1\right], \text{ hence } O\left(\frac{1}{n}\right). \end{cases}$$

So the degree of approximation is better close to the endpoints, a fact shared by the Bernstein operators where $r_n(x) = x$.

(v) If $f \in C^2[0,1]$, then

$$|V_n^{\min}(f;x) - f(x)| \leq c \cdot (|x - r_n^{\min}(x)| + \alpha_n^2(x)) =$$

$$= c \cdot \begin{cases} x + x^2, & x \in \left[0, \frac{1}{2n}\right), \\ \frac{|\frac{1}{2}-x|}{n-1} + \frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right), & x \in \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right], \\ (1-x) + (1-x)^2, & x \in \left(1 - \frac{1}{2n}, 1\right]. \end{cases}$$

So for C^2-functions we get a global degree of approximation of order $O\left(\frac{1}{n}\right)$ which is also the case for the classical Bernstein operators.

References

[1] Agratini O., Rus I.A., *Iterates of a class of discrete linear operators*, Comment. Math. Univ. Carolinae **44** (2003), 555–563.

[2] Beresin I.S., Zhidkov N.P., *Numerische Methoden II*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1971.

[3] Kelisky R.P., Rivlin T.J., *Iterates of Bernstein polynomials*, Pacific J. Math. **21** (1967), 511–520.

[4] King P.J., *Positive linear operators which preserve x^2*, Acta Math. Hungar. **99** (2003), 203–208.

[5] Mamedov R.G., *On the order of approximation of functions by sequences of linear positive operators* (Russian), Dokl. Akad. Nauk SSSR **128** (1959), 674–676.

[6] Păltănea R., *Approximation by linear positive operators: Estimates with second order moduli*, Ed. Univ. Transilvania, Braşov, 2003.
[7] Rus I.A., *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl. **292** (2004), 259–261.
[8] Shisha O., Mond B., *The degree of convergence of linear positive operators*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 1196–1200.

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