$C(X)$ can sometimes determine X without X being realcompact

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Abstract. As usual $C(X)$ will denote the ring of real-valued continuous functions on a Tychonoff space X. It is well-known that if X and Y are realcompact spaces such that $C(X)$ and $C(Y)$ are isomorphic, then X and Y are homeomorphic; that is $C(X)$ *determines* X. The restriction to realcompact spaces stems from the fact that $C(X)$ and $C(\upsilon X)$ are isomorphic, where υX is the (Hewitt) realcompactification of X. In this note, a class of locally compact spaces X that includes properly the class of locally compact realcompact spaces is exhibited such that $C(X)$ determines X. The problem of getting similar results for other restricted classes of generalized realcompact spaces is posed.

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1. Introduction

All topological spaces considered are assumed to be Tychonoff spaces (i.e., subspaces of compact Hausdorff spaces) and although some definitions will be recalled, familiarity with the notation and terminology in [GJ76] is assumed. As usual $C(X)$ will denote the ring of real-valued continuous functions on a (Tychonoff) space X under the usual pointwise operations, and $C^*(X)$ denoted its subring of bounded functions. If, whenever a space X is dense in a space Y and each $f \in C(X)$ has a continuous extension $\upsilon f \in C(Y)$, it follows that $Y = X$, then X is said to be *realcompact*. The essentially unique compact space βX such that every $f \in C^*(X)$ has a continuous extension $\beta f \in C(\beta X)$ is called the *Stone-Čech compactification* of X. It is known that $\upsilon X = \{p \in \beta X :$ every $f \in C(X)$ has a continuous extension to $X \cup \{p\}\}$ is the largest realcompact subspace of βX that contains X. The space υX is called the (*Hewitt*) *realcompacification* of X, and it follows that $C(X)$ and $C(\upsilon X)$ are isomorphic. (See Chapters 6–8 of [GJ76].)

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In their paper [BD92], R.L. Blair and E.K. van Douwen generalized the concept of realcompactness by defining a space X to be *nearly realcompact* if $\beta X \backslash \upsilon X$ is dense in $\beta X \backslash X$. Clearly every realcompact space has this property and they give many examples of nearly realcompact spaces that are not realcompact and they show that the product of any nowhere locally compact nearly realcompact space with an arbitrary space is nearly realcompact. They also gave some topological characterizations of this class of spaces but did not consider whether $C(X)$ determines X in this case.

Their study of such spaces was continued by J. Schommer in [S94] who noted that these spaces were considered previously by D. Johnson and M. Mandelker in [JM73] who called them η-compact spaces. They are defined in Section 6 of the paper [JM73] by the property that every $f \in C(X)$ with pseudocompact support belongs to all free maximal ideals of $C(X)$ and this is shown to characterize nearly realcompactness. This will be used below to show that $C(X)$ determines X if X is locally compact and nearly realcompact.

2. Fast sets and strongly real maximal ideals

The main tool used in [S94] is the notion of a fast set. To define it, we need to remind the reader of some notation and terminology as a well-known theorem due to Gelfand and Kolmogoroff. (See Chapter 7 of [GJ76].) As usual, $Z(f) = f^{\leftarrow}(0)$ and $\operatorname{coz}(f) = X \backslash Z(f)$. A subspace S of X is called *relatively pseudocompact* if $f|_S$ is bounded for all $f \in C(X)$, and it is not difficult to show that a subset A of X is *relatively pseudocompact* if and only if $\operatorname{cl}_{\beta X} A \subseteq \upsilon X$.

2.1 Theorem (Gelfand and Kolmogoroff)**.** *There is a one-one correspondence between the points of βX and the maximal ideals of $C(X)$ given by $p \to M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$.*

2.2 Definition. A subset F of X is said to be *fast* if it is closed in $X \cup J(X)$, where $J = J(X) = \operatorname{cl}_{\beta X}(\beta X \backslash \upsilon X)$, and $\mathcal{F}(X)$ denotes the family of fast subsets of X.

The following properties of fast sets are established in [S94].

2.3 Theorem. *For any space X*

(1) *Every fast subset of X is necessarily closed, and the converse is true if and only if X is pseudocompact.*
(2) *Every compact set is a fast set in X and the converse is true if and only if X is nearly realcompact.*
(3) *Finite unions, arbitrary intersections, and closed subsets of a fast set are fast.*
(4) *Every fast set in X is relatively pseudocompact, but the converse need not be true* (e.g. $[0, \omega_0)$ *is a relatively pseudocompact subset of* $[0, \omega_1)$, *but is not fast*).

(5) *Let* $\epsilon X = \beta X \backslash (J \backslash X)$. *Then a closed subset* F *of* X *is fast in* X *if and only if there exists a compact set* K *in* ϵX *with* $K \cap X = F$ *and* X *is nearly realcompact if and only if* $X = \epsilon X$.

(6) *A closed subset* F *of* X *is fast in* X *if and only if there exists a compact set* K *such that for any open neighborhood* V *of* K *there exists a pseudocompact subset* P *of* X *so that* $F \backslash V$ *and* $X \backslash P$ *can be completely separated.*

If $I \subset C(X)$, then $\mathcal{Z}(\mathcal{I})$ denotes $\{Z(f) : f \in I\}$. A maximal ideal M of $C(X)$ is called *real* or *hyper-real* according as $C(X)/M$ is the real field $\mathbb{R}$ or contains it properly.

Recall that for any $p \in \beta X$, we let O^p denote $\{f \in C(X) : \mathrm{cl}_{\beta X} Z(f)$ is a neighborhood of $p\}$ and note that it is the intersections of all of the prime ideals of $C(X)$ that are contained in M^p. Thus we can associate to each maximal ideal M of $C(X)$, a unique Z-ideal O_M in the sense that O_M is contained in the unique maximal ideal M. Henceforth in this terminology, when we express M as M^p, then $O_M = O^p$.

2.4 Definition. A maximal ideal M is said to be a *strongly real maximal ideal* (*or SRM ideal*) if there is a $Z \in \mathcal{Z}(O_M)$ which is fast in X.

It follows from 2.3(4) that every SRM ideal is real. As we show next, the converse need not hold.

2.5 Example. Not every real maximal ideal need be an SRM ideal. Indeed, if p is a nonisolated point of a P-space X, then $M^p = M_p$ is not an SRM ideal. For, every P-space is nearly realcompact by Proposition 4.5 of [S94], and hence every fast subset of X is compact by (2) of Theorem 2.3 above. Because every compact P-space is finite, the conclusion follows.

2.6 Theorem. *The family of all SRM ideals in* $C(X)$ *is* $\{M^p : p \in \beta X \backslash J\}$.

Proof: Let $p \in \beta X \backslash J$. Since J is compact, there exists a zero set neighborhood Z of p in βX which does not meet J. Clearly Z is a compact subset of ϵX, so $Z \cap X$ is a fast zero set of X. Since Z is a neighborhood of p, $p \in \mathrm{cl}_{\beta X}(Z \cap X)$ and this latter set is a neighborhood of p. Thus $Z \cap X$ is in $Z(O^p)$. So M^p is an SRM ideal.

Suppose conversely that M^p is an SRM ideal for some p in βX. Then there exists a fast zero set $Z \in \mathcal{Z}(M^p)$ such that $\mathrm{cl}_{\beta X} Z$ is a neighborhood of p. By the definition of fast sets, it follows that $p \notin J$. So $p \in \beta X \backslash J$. $\square$

Another characterization of SRM ideals will follow. First, more of the contents of Schommer's paper [S94] will be reviewed and more terminology will be introduced.

X is called *locally fast* if each of its points has a neighborhood whose closure is fast, and is called *locally pseudocompact* if each of its points of has a pseudocompact neighborhood. $N_p(X)$ denotes the set of points at which X fails to be locally pseudocompact and $N_F(X)$ denotes the set of points at which X fails to be locally fast.

2.7 Lemma. (a) $N_p(X) = N_F(X) = X \cap J$.

(b) *X is locally pseudocompact if and only if X is locally fast if and only if $X \cap J = \emptyset$.*

(c) *If X is nearly realcompact, then $X \cap J = \emptyset$ if and only if X is locally compact.*

Proof: By Propositions 3.3, 3.8, and 3.9 of [S94], (a) holds, and (b) follows immediately from (a). If X is locally compact, then it is locally pseudocompact. So $X \cap J = \emptyset$ by (b). If $X \cap J = \emptyset$, then it is locally fast by (b), and is locally compact by Theorem 2.3(2). So (c) holds. □

Example 2.5 shows that not every fixed maximal ideal need be an SRM ideal. Thus the following theorem characterizes locally pseudocompact spaces in terms of SRM ideals and gives a necessary and sufficient condition for every fixed maximal ideal of $C(X)$ to be an SRM ideal.

2.8 Theorem. *A space X is locally pseudocompact if and only if every fixed maximal ideal of $C(X)$ is an SRM ideal.*

Proof: If X is locally pseudocompact, then $X \cap J = \emptyset$ by Lemma 2.7 and it follows that $X \cap J = \emptyset$. So $X \subset (\beta X \backslash J)$. It follows from Theorem 2.6 that every fixed maximal ideal of $C(X)$ is an SRM ideal.

Conversely, suppose that every fixed maximal ideal of $C(X)$ is an SRM ideal. By Theorem 2.6, it follows that $X \subset (\beta X \backslash J)$. Thus $J \cap X = \emptyset$, and by Lemma 2.7, this implies that X is locally pseudocompact. □

2.9 Theorem. *A space X is nearly realcompact if and only if every SRM ideal is fixed.*

Proof: Suppose X is nearly realcompact and M is an SRM ideal of $C(X)$. Then $\mathcal{Z}(M)$ contains a fast zero set $\mathcal{Z}$. Since X is nearly realcompact, every fast set is compact by (2) of Theorem 2.3 and hence $Z(M)$ containing the compact zero set $\mathcal{Z}$, is fixed by 4.10 of [GJ76]. So M is a fixed maximal ideal.

Suppose conversely that X is not nearly realcompact. Then by (5) of Theorem 2.3, $\epsilon X \backslash X \neq \emptyset$, and M^p is an SRM ideal which not fixed, for each $p \in \epsilon X \backslash X$. □

While Theorem 2.8 may appear superficially to be an algebraic characterization of nearly realcompact space, this is not the case because the definition of SRM ideals involves topological concepts. To get such an algebraic characterization, we need to recall from [JM73] that $C_\psi(X)$ denotes the ideal of functions in $C(X)$ with pseudocompact support, and prove two preliminary results.

2.10 Lemma. *A function $f \in C(X)$ is in $C_\psi(X)$ if and only if $fg \in C^*(X)$ whenever $g \in C(X)$.*

Proof: Any $g \in C(X)$ is bounded on $\mathrm{cl}_X \operatorname{coz} f$ whenever $f \in C_\psi(X)$. So $fg \in C^*(X)$ if $f \in C_\psi(X)$ and $g \in C(X)$.

Suppose that $fg \in C^*(X)$ whenever $g \in C(X)$. Then $f = f.1 \in C^*(X)$. If also, $f \notin C_\psi(X)$, then $\mathrm{cl}_X \operatorname{coz} f$ is not pseudocompact and $\operatorname{coz} f$ is not relatively pseudocompact as is noted in Theorem 2.1 of [Ma71]. Then, by 1.20 of [GJ76], there is an $h \in C(X)$ and a C-embedded copy S of N contained in $\operatorname{coz} f$ such that $h \,|\, S$ diverges to ∞. Because S is C-embedded in X, there is a $g \in C(X)$ with the same values as $\frac{h}{f}$ on S. Then $fg \,|\, S = h \,|\, S$ is unbounded, so $fg \notin C^*(X)$; contrary to the assumption. $\square$

2.11 Lemma. *For any space X, the followings are equivalent.*

(1) *M is an SRM ideal.*
(2) *$C_\psi(X)$ is not contained in M.*

Proof: If (1) holds, then there exists $Z \in Z(O_M)$ which is fast in X. Let $Z = Z(f)$ for some $f \in O_M$. Then by [GJ76, 7.12(b)], there exists $g \notin M$ such that $fg = 0$. This implies that $Z(f) \cup Z(g) = X$. Now by Theorem 2.3(4), $Z(f)$ is indeed a relatively pseudocompact subset of X. Thus for each $h \in C(X)$, gh is bounded on $Z(f)$ and 0 on $Z(g)$ and hence $gh \in C^*(X)$. By Lemma 2.10, it follows that $g \in C_\psi(X)$.

Suppose conversely that there is an $f \in C_\psi(X) \backslash M$. Then there is a $Z \in \mathcal{Z}(O_M)$ disjoint from $Z(f)$. For otherwise, $Z \cap Z(f) \neq \emptyset$ for all $Z \in \mathcal{Z}(O_M)$. Thus $\{Z(f) : f \in O_M\} \cup \{Z(f)\}$ is a family of zero sets with the finite intersection property and so is contained in the unique $\mathcal{Z}$-ultrafilter $\mathcal{Z}(M)$ because O_M is contained in the unique maximal ideal M. This implies that $Z(f) \in \mathcal{Z}(M)$ and so $f \in M$, a contradiction. Because $f \in C_\psi(X)$, its support P is pseudocompact. Moreover, since $Z \cap Z(f) = \emptyset$, it follows that $\mathrm{cl}_{\beta X} Z \cap \mathrm{cl}_{\beta X} Z(f) = \emptyset$. Since P is pseudocompact and $Z \subset P$, $\mathrm{cl}_{\beta X} Z \subset \upsilon X$ and $\mathrm{cl}_{\beta X} Z(f) \supset \beta X \backslash \upsilon X$. This implies that $J(X) \subset \mathrm{cl}_{\beta X} Z(f)$ and $\mathrm{cl}_{\beta X} Z \cap J(X) = \emptyset$. Thus no limit point of Z in βX lies in J; that is Z is closed in $X \cup J$. Hence by definition Z is a fast set in X. This shows that M is an SRM ideal. $\square$

We use the last two lemmas to provide an algebraic characterization of SRM ideals.

2.12 Theorem. *A maximal ideal M of $C(X)$ is an SRM ideal if and only if there is an $f \in (C(X) \backslash M)$ such that $fg \in C^*(X)$ for all $g \in C(X)$.*

Proof: Let M be an SRM ideal. By Lemma 2.11, there is an $f \in C_\psi(X) \backslash M$. Then by Lemma 2.10 $fg \in C^*(X)$ for all $g \in C(X)$. So the conclusion holds.

Conversely, suppose there is an $f \in C(X) \backslash M$ with the property that $fg \in C^*(X)$, for all $g \in C(X)$. Then by Lemma 2.8, $f \in C_\psi(X)$ which is not in M.

This shows that $C_\psi(X)$ is not contained in M. By Lemma 2.11, M is an SRM ideal. □

2.13 Theorem. *If X is nowhere locally pseudocompact, then it is nearly real compact.*

PROOF: Since X is nowhere locally pseudocompact, $C_\psi(X) = \{0\}$, so no real maximal ideal is SRM and hence X is nearly realcompact by Theorem 2.9. □

$I(X)$ and $C_K(X)$ will denote respectively the intersection of all of the free maximal ideals and the functions in $C(X)$ with compact support. As was introduced by Johnson and Mandelker in [JM73], a space X is said to be *μ-compact*, *η-compact*, or *ψ-compact* according as $C_K(X) = I(X)$, $C\psi(X) = I(X)$ or $C_K(X) = C\psi(X)$. While proofs of all of the following items may be found in [JM73], we include them here to give an alternate proof to each of the items using the notion of SRM ideal.

2.14 Theorem (Johnson and Mandelker)**.**

(a) $C\psi(X) = \bigcap_{p\in\beta X\setminus\upsilon X} M^p$.
(b) $\bigcap_{p\in J} M^p = \bigcap_{p\in\beta X\setminus\epsilon X} M^p = \bigcap_{p\in\beta X\setminus\upsilon X} M^p = C_\psi(X)$.
(c) *A space X is nearly realcompact if and only if it is η-compact.*
(d) *Every ψ-compact space is nearly realcompact.*

PROOF: (a) Since $\beta X\setminus\upsilon X \subset J$, by Theorem 2.6 and Lemma 2.9, $C_\psi(X) \subset \bigcap_{p\in\beta X\setminus\upsilon X} M^p$. For the converse let $f \in \bigcap_{p\in\beta X\setminus\upsilon X} M^p$. Then $f \in C^*(X)$. Now $fg \in \bigcap_{p\in\beta X\setminus\upsilon X} M^p$ for all $g \in C(X)$, which implies that $fg \in C^*(X)$. So $f \in C_\psi X$ by Lemma 2.10.

(b) By Theorem 2.6 and Lemma 2.11, we have $C_\psi(X) \subset \bigcap_{p\in J} M^p \subset \bigcap_{p\in\beta X\setminus\ \epsilon X} M^p \subset \bigcap_{p\in\beta X\setminus\upsilon X} M^p = C_\psi(X)$. Hence the result follows.

(c) If X is nearly realcompact, then $\beta X\setminus\epsilon X = \beta X\setminus X$ by Theorem 2.1. So from the Theorem 2.14(b) we have $\bigcap_{p\in\beta X\setminus\epsilon X} M^p = C_\psi(X)$ or equivalently, $\bigcap_{p\in\beta X\setminus X} M^p = C_\psi(X)$ and from the definition of η-compactness, it follows that X is η-compact.

(d) By assumption $C_K(X) = C_\psi(X)$. Suppose X is not nearly realcompact and choose $p \in \epsilon X\setminus X$. Then M^p is an SRM ideal. There is an $f \in C_K(X)\setminus M^p$, which implies $p \in \mathrm{cl}_{\beta X}(X\setminus Z(f)) = \mathrm{cl}_X(X\setminus Z(f)) \in X$, contrary to the assumption. □

2.15 Theorem. *Every P-space is η-compact (hence nearly realcompact), ψ-compact, and μ-compact.*

PROOF: These assertions follow immediately from the fact that every pseudocompact P-space is finite and Theorem 2.14. □

3. Seemingly topological concepts preserved under isomorphism

The definition of an SRM ideal involved topological concepts, but Theorem 2.10 provides a characterization of these ideals that is purely algebraic since the real field is fixed under any isomorphism between rings of continuous real-valued, and order is preserved. (Recall that $f \geq 0$ in $C(X)$ if and only if there is a g such that $f = g^2$.) Thus:

3.1 Theorem. *If X and Y are Tychonoff spaces, then an isomorphism of $C(X)$ onto $C(Y)$ sends SRM ideals to SRM ideals.*

As noted above, there may be points p in a space X such that M^p is an SRM ideal in $C(X)$. By Theorem 2.6, the set of such points is $\beta X \backslash J \cap X$, while its complement $J \cap X$ consists of points p such that no $Z \in \mathcal{Z}(M^p)$ has p in its interior. Combining this with Theorem 3.1 enables us to prove:

3.2 Theorem. *If X and Y are nearly realcompact spaces and $C(X)$ and $C(Y)$ are isomorphic, then $X \backslash N_F(X)$ and $Y \backslash N_F(Y)$ are homeomorphic.*

Outline of proof. We give only an outline because this procedure should be familiar from Chapter 8 of [GJ76]. Let $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ denote respectively the spaces of maximal ideals of X and Y in the hull-kernel topology. Because $C(X)$ and $C(Y)$ are isomorphic, there is a homeomorphism ξ of $\mathfrak{M}(X)$ onto $\mathfrak{M}(Y)$ induced by this isomorphism ψ sending a maximal ideal of $C(Y)$ onto one of $C(X)$. Since ψ carries an SRM ideal of $C(X)$ to an SRM ideal $C(Y)$, the homeomorphism ξ carries the family of all SRM ideals in $C(X)$ onto the family of all SRM ideals of $C(Y)$. So $X \backslash N_F(X)$ is homeomorphic with $Y \backslash N_F(X)$. $\square$

Combining the last two theorems yields:

3.3 Theorem. *If X and Y are locally compact, nearly realcompact spaces, then $C(X)$ is isomorphic with $C(Y)$ if and only if X is homeomorphic with Y.*

We close with some examples of locally compact, nearly realcompact spaces that are not realcompact. Perhaps the simplest such example is a discrete space of (Ulam) measurable cardinality. (For a definition and discussion, see Chapter 12 of [GJ76].) As is noted in [BD92], every discrete space is nearly realcompact, but in models of set theory with measurable cardinals, not all discrete spaces are realcompact.

3.4 Example. The Fringed plank. Let $T = \omega_1 \times \omega \backslash \{(\omega_1, \omega)\}$ denote the Tychonoff plank with its usual topology.

The Fringed plank $\mathcal{F}T$ is obtained from T by adjoining a convergent sequence $\{x_{j,n} : n \in \omega\}$ to each point (ω_1, j) on the right edge. Thus each $\{x_{j,n} : n \in \omega\} \cup \{(\omega_1, j)\}$ is a copy of the one-point compactification of the space ω. Each point on the right edge has its usual neighborhoods together with the tails of the corresponding sequence. Thus we have added the free union of countably many copies of the one-point compactification of the space ω to T to make the space $\mathcal{F}T$, and all the added points are isolated. By considering separately the new isolated points, those on the top edge, and the points (ω_1, j) for $0 \leq j < \omega$, it is easy to verify that $\mathcal{F}T$ is locally compact.

In [SS01], J. Schommer and M.A. Swardson introduced the concept of *almost* realcompactness*, whose definition we need not repeat here. They show that $\mathcal{F}T$ is almost* realcompact and that every almost* realcompact space is ψ-compact and hence is nearly realcompact by Theorem 2.14(d). Because $\mathcal{F}T$ has the pseudocompact noncompact subspace T as a closed subspace, it cannot be realcompact.

In summary, the Fringed plank is an example of a locally compact nearly realcompact space that is not realcompact. □

Once we have an example of a locally compact nearly realcompact space, we can produce much more examples of such spaces by using the following theorem.

3.5 Theorem. *If X is a locally compact nearly realcompact nonrealcompact space, and Y is a locally compact realcompact space Y of nonmeasurable cardinality, then $X \times Y$ is a locally compact nearly realcompact space that is not realcompact.*

Two lemmas will be used to prove this theorem.

A. Lemma. *Let Y be a dense subspace of X and p is in Y. Then p has no compact neighbourhood in Y if $p \in \mathrm{cl}_X(X \setminus Y)$, and the converse is true if X is compact.*

Proof: Suppose $p \in \mathrm{cl}_X(X \setminus Y)$ and there is a compact neighbourhood V of p in Y. Then is an open set U in X such that $U \cap Y = \mathrm{int}_Y V$ and $U \subset \mathrm{cl}_X V$. Since V is compact, $\mathrm{cl}_X V = \mathrm{cl}_Y V = V$. Thus $U \subset Y$ and hence $\mathrm{int}_Y V = U \cap Y = U$ is open in X but does not meet $X \setminus Y$, contradicting the hypothesis.

Conversely, suppose that X is compact, p has no compact neighbourhood in Y, and $p \notin \mathrm{cl}_X(X \setminus Y)$. Then there exists an open set V in X containing p such that $\mathrm{cl}_X V \cap (X \setminus Y) = \emptyset$. Because X is compact, so is $\mathrm{cl}_X V$, thereby contradicting the hypothesis. □

B. Lemma. *A space X is nearly realcompact if and only if no point of $\upsilon X - X$ has a compact neighbourhood in υX.*

Proof: Since X is nearly realcompact, $\upsilon X - X \subset \mathrm{cl}_{\beta X}(\beta X - \upsilon X)$. So, by Lemma A, no point of $\upsilon X - X$ has a compact neighbourhood in υX.

Suppose conversely that no point of $\upsilon X - X$ has a compact neighbourhood in υX. Since βX is compact, it follows that each point of $\upsilon X - X$ is in the βX-closure of $(\beta X - \upsilon X)$ by Lemma A. Thus $(\beta X - \upsilon X)$ is dense in $\beta X - X$ and hence X is nearly realcompact. □

Proof of Theorem 3.5: In Corollary 2.2 of [C68], W.W. Comfort has shown that if Y is a locally compact realcompact space of nonmeasurable cardinality, then $\upsilon(X \times Y) = \upsilon X \times Y$. It is easy to verify that $\upsilon(X \times Y)\backslash(X \times Y) = (\upsilon X\backslash X) \times Y$. Clearly $X \times Y$ is locally compact as both X and Y are locally compact. Suppose $X \times Y$ is not nearly realcompact. Then, by Lemma B, there exists a point $(p, q) \in \upsilon(X \times Y)\backslash(X \times Y) = (\upsilon X - X) \times Y$, which has a compact neighborhood in $\upsilon(X \times Y)$. Because $\upsilon(X \times Y) = \upsilon X \times Y$, the projection map $P_{\upsilon X} : \upsilon(X \times Y) \to \upsilon X$ is open and continuous and any open continuous map carries a compact neighbourhood of a point to a compact neighbourhood of its image point, the compact neighborhood of the point (p, q) in $\upsilon(X \times Y)$ will be sent via the projection map to a compact neighborhood of its image point $p \in \upsilon X - X$. Thus the point p in $\upsilon X - X$ has a compact neighborhood in υX. Again by Lemma B, X is not nearly realcompact, a contradiction. That it is not real compact follows from the fact that $X \times \{y\}$ is a closed subset of $X \times Y$, which being a copy of X is not realcompact. □

Concluding remarks. If $\mathcal{P}$ is a class of Tychonoff spaces such that if whenever $X, Y \in \mathcal{P}$ and $C(X)$ and $C(Y)$ are isomorphic as rings, it follows that X any Y are homeomorphic as topological spaces, we will say that $C(X)$ *determines* X *if* $X \in \mathcal{P}$. There are many theorems that name classes of spaces within which $C(X)$ determines X. Perhaps the most famous ones are the class $\mathcal{C}$ of compact spaces and the class $\mathcal{RC}$ of realcompact spaces. Note, however, that there cannot be any such class $\mathcal{P}$ that contains the class $\mathcal{RC}$ properly. For, if $X \in \mathcal{P}\backslash\mathcal{RC}$, then $C(X)$ and $C(\upsilon X)$ are isomorphic but X and υX fail to be homeomorphic. So, any class $\mathcal{P}$ within which $C(X)$ determines X must be contained in $\mathcal{RC}$ or be oblique to it in some sense, Theorem 3.3 describes one such class, and in [Mi82] P.R. Misra shows that the class of first countable Tychonoff spaces is another one. Perhaps restricted classes of other generalizations of realcompactness will add to this list.

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