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Groups with the weak minimal condition for non-subnormal subgroups II

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Abstract. Let G be a group with the property that there are no infinite descending chains of non-subnormal subgroups of G for which all successive indices are infinite. The main result is that if G is a locally (soluble-by-finite) group with this property then either G has *all* subgroups subnormal or G is a soluble-by-finite minimax group. This result fills a gap left in an earlier paper by the same authors on groups with the stated property.

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A group G is said to satisfy the weak minimal condition for non-subnormal subgroups (denoted min-∞-$\overline{sn}$) if there is no infinite descending chain $H_1 > H_2 > \dots$ of non-subnormal subgroups H_i of G such that every index $|H_i : H_{i+1}|$ is infinite. If G is a group that has *all* of its subgroups subnormal then of course G has the property min-∞-$\overline{sn}$, and the same conclusion holds if G satisfies min-∞, the weak minimal condition for *all* subgroups (where there is no infinite descending chain of subgroups H_i each of infinite index in the previous one). Groups with this latter property and the corresponding property max-∞ have been discussed by Zai'cev (see [Z] for the generalization "min-max-∞"), one of the main results being that a group G that is locally (soluble-by-finite) and satisfies either min-∞ or max-∞ for all subgroups has a soluble minimax subgroup of finite index. Theorem C of the article [KS1] states the following.

Theorem A. *Let G be a generalized radical group with min-∞-$\overline{sn}$. Then either G is soluble-by-finite and minimax or every subgroup of G is subnormal.*

We recall that a group is *generalized radical* if it has an ascending series of normal subgroups where the successive factor groups are either locally finite or locally nilpotent. It was remarked in [KS1] that we were unable to obtain a similar result for the class of locally soluble groups, one obstacle being that a locally soluble group with all abelian subgroups minimax need not be minimax ([Me]) — in the proof of Theorem A (above) we used the fact that both locally finite groups and locally nilpotent groups (and even radical groups) inherit the minimax property from abelian subgroups. This omission turned out to be somewhat more unfortunate, in our view, when we were later able to provide in [KS2] a reasonably satisfactory classification of groups G with max-∞-$\overline{sn}$, subject only to the further hypothesis that G have an ascending chain of normal subgroups where the successive factors are locally (soluble-by-finite); we refer the reader to that paper for a full description of this classification (see Theorems C and D of [KS2]), though it is worth pointing out here that there is not the simple dichotomy in the "max" case that there is for groups with min-∞-$\overline{sn}$. The purpose of the present article is to remedy the omission referred to above, and we shall establish the following result.

Theorem B. *Let G be a locally (soluble-by-finite) group that satisfies min-∞-$\overline{sn}$. Then either G is soluble-by-finite and minimax or every subgroup of G is subnormal.*

It is perhaps worth stating explicitly the contrapositive form of this result, namely: If G is a locally (soluble-by-finite) group that is not minimax, and if G has a subgroup H that is not subnormal, then there is an infinite descending chain $H_1 > H_2 > \dots$ of non-subnormal subgroups of G such that each index $|H_i : H_{i+1}|$ is infinite. A famous result of Möhres [Mö] tells us that a group with all subgroups subnormal is in any case soluble, and thus we see that the property min-∞-$\overline{sn}$ in a locally (soluble-by-finite) group implies "almost solubility". In the spirit of Theorem C of [KS1] and Theorem D of [KS2] we present an easily established generalization of the above result. (We shall use, here and elsewhere, the fact that min-∞-$\overline{sn}$ is inherited by subgroups and by homomorphic images.) It is well-known that a group with all subgroups subnormal is locally nilpotent, and since a soluble-by-finite group has a (finite) characteristic series with factors locally finite or locally nilpotent, Theorems B and A combine to give us the following.

Theorem B*. *Let G be a group that has an ascending series of normal subgroups where each successive factor is locally (soluble-by-finite), and suppose that G satisfies min-∞-$\overline{sn}$. Then either G is soluble-by-finite and minimax or every subgroup of G is subnormal.*

We now present a few preliminary results that will assist us with the proof of Theorem B. Recall that a group G has *finite rank* (that is, finite special rank, or finite Mal'cev-Prüfer rank) if there is a positive integer r such that every finitely generated subgroup of G is r-generated.

Lemma 1. *Let G be a countable, locally (soluble-by-finite) group of infinite rank that satisfies min-∞-$\overline{sn}$, and suppose that every locally soluble subgroup of G is either minimax or locally nilpotent. Then every subgroup of G is subnormal.*

Proof: Suppose the result false. By Theorem A, G is not generalized radical. Write G as the ascending union $F_1 < F_2 < \dots$ of finitely generated subgroups and let S_n denote the soluble radical of F_n for each n. For $n \leq m$ we see that

F_n normalizes S_m, and it follows easily that the subgroup S generated by all the S_n is the product of these subgroups and is locally soluble. Furthermore, S has finite index in $\langle S, F_n \rangle = SF_n$ for each n. If S is not minimax then, by hypothesis, it is locally nilpotent and SF_n is generalized radical for each n and hence, by Theorem A, locally nilpotent, which gives the contradiction that G is locally nilpotent. Thus S is minimax and therefore has finite rank r, say. If C is a finitely generated free abelian subgroup of G then $C \leq F_m$ for some m and C has the same rank as $S_m \cap C$, namely r. It follows that every torsion-free abelian subgroup of G has rank at most r, and since G has infinite rank we deduce from Corollary 3.7 of [DES] that G has a periodic abelian subgroup Q of infinite rank. Since Q does not satisfy min-∞ it has a subgroup R of infinite rank that is subnormal in G. In particular, the maximal normal torsion subgroup T of G has infinite rank, and by Theorem A every subgroup of T is subnormal. So T is locally nilpotent, and TSF_n is generalized radical for all n and hence, again by Theorem A, locally nilpotent. This gives the contradiction once more that G is locally nilpotent, and the result follows. $\square$

Lemma 2. *Let G be a finitely generated abelian-by-finite group, H a subgroup of G, and let K be a normal abelian subgroup of H having finite index in H. Then G has a normal abelian subgroup L of finite index such that $L \cap H \leq K$.*

Proof: Let A be a normal abelian subgroup of G with G/A finite, and let $T/(K \cap A)$ be the periodic component of $A/(K \cap A)$, so that $(H \cap A)/(K \cap A) \leq T/(K \cap A)$. Since $T/(K \cap A)$ is finite we have $A/(K \cap A) = T/(K \cap A) \times S/(K \cap A)$ for some subgroup S of A. In particular, $S \cap T = K \cap A$, so that $S \cap H = S \cap A \cap H \leq S \cap T = K \cap A$, that is, $S \cap H \leq K$. Since both $|G : A|$ and $|A : S|$ are finite we may set $L = \mathrm{Core}_G(S)$ to complete the proof. $\square$

Lemma 3. *Suppose that the group G is the ascending union of finitely generated subgroups $F_1 < F_2 < \dots$, where each F_n is soluble-by-finite and minimax. Then, for each n, there are normal subgroups H_n, K_n of F_n satisfying the following conditions.*

(i) *H_n is locally nilpotent.*
(ii) *$H_n \leq K_n$, K_n/H_n is abelian and F_n/K_n is finite.*
(iii) *$K_{n+1} \cap F_n \leq K_n$.*

Furthermore, if K is the subgroup generated by all of the K_n then $K \cap F_n = K_1 K_2 \dots K_n$ for each n.

Proof: Let H_n denote the locally nilpotent radical of F_n for each n. Then F_n/H_n has a normal abelian subgroup K_n/H_n of finite index — see, for example, Theorem 3.25 of [R]. Supposing that K_m has been chosen for some m, we may apply Lemma 2 to deduce that F_{m+1}/H_{m+1} has a normal abelian subgroup K_{m+1}/H_{m+1} of finite index such that $(K_{m+1}/H_{m+1}) \cap (F_m H_{m+1}/H_{m+1}) \leq K_m H_{m+1}/H_{m+1}$, whence $K_{m+1} \cap F_m \leq K_m H_{m+1} \cap F_m = K_m(H_{m+1} \cap F_m) \leq$

$K_mH_m = K_m$. It follows that the K_n may be chosen so that (ii) and (iii) are satisfied. Since K_m is normalized by F_n whenever $m \geq n$ it is clear that K is the product of the subgroups K_n. For $m > n$ we have $(K_1K_2 \dots K_m) \cap F_n = (K_1K_2 \dots K_m) \cap F_{m-1} \cap F_n = (K_1K_2 \dots K_{m-1})(K_m \cap F_{m-1}) \cap F_n = (K_1K_2 \dots K_{m-1}) \cap F_n$, since $K_m \cap F_{m-1} \leq K_{m-1}$. Repeating this argument as often as necessary, we see that $(K_1K_2 \dots K_m) \cap F_n = (K_1K_2 \dots K_n) \cap F_n = K_1K_2 \dots K_n$, and hence that $K \cap F_n = K_1K_2 \dots K_n$ for each n. The lemma is therefore proved. □

Proof of Theorem B. Suppose that the conclusion of the theorem does not hold, so that G is not generalized radical, by Theorem A. If L is a locally (soluble-by-finite) group of finite rank then we obtain from Theorem 3.6 of [DES] that L is (locally soluble)-by-finite and hence, using Lemma 10.39 of [R], generalized radical (in fact, (locally nilpotent)-by-soluble-by-finite). By Theorem A, it follows that every section of G that has finite rank is either minimax or locally nilpotent. Certainly therefore G has infinite rank and hence contains a countable subgroup of infinite rank (which cannot be minimax). Since G also has a countable subgroup that is not locally nilpotent (and hence does not have all subgroups subnormal), we may assume that G itself is countable. By Lemma 1 we may further assume that G is locally soluble.

Let J denote the iterated locally nilpotent radical of G, and suppose that G/J has a nontrivial locally finite normal subgroup J_1/J. Since J_1 is not locally nilpotent it is soluble-by-finite, by Theorem A, and hence soluble, giving the contradiction that $J_1 = J$. Since $J < G$ we deduce that G/J is also a counterexample to the statement of the theorem, and we may therefore factor by J and hence assume that G has trivial locally finite radical and trivial locally nilpotent radical, hence no nontrivial subnormal subgroups that are either locally nilpotent or locally finite.

Let F be an arbitrary finitely generated subgroup of G. Then F is soluble and so, by Theorem A, either F is minimax or F has all subgroups subnormal and is therefore locally nilpotent and hence nilpotent. Thus in any case F is minimax.

Next, we note that G satisfies the weak minimal condition for non-subnormal non-minimax subgroups and hence has a non-minimax subgroup G_1 such that every subgroup of infinite index in G_1 is either minimax or subnormal in G. If G_1 has an abelian subgroup A that is not minimax then A has a nontrivial subgroup B of infinite index, which is therefore subnormal in G, contradicting the fact that G has trivial locally nilpotent radical. Thus every abelian subgroup of G_1 is minimax and so, by a result of Baer and Zai'cev (see Theorem 10.35 of [R]), every radical subgroup of G_1 is minimax. Certainly therefore G_1 is not locally nilpotent, and so G_1 is also a counterexample to the statement of the theorem. Note that G_1 therefore has infinite rank.

Write G_1 as the ascending union of finitely generated subgroups $F_1 < F_2 < \dots$. Each F_n is soluble minimax and hence has subgroups H_n, K_n such that the conditions of Lemma 3 hold. If there is an upper bound t for the derived lengths of the factors F_n/K_n then $F_n^{(t+1)} \leq H_n$ for each n and so $G_1^{(t+1)}$ is locally nilpotent, and Theorem A gives a contradiction. Passing to an appropriate subsequence of the F_n if necessary, we may therefore assume that $F_n^{(n)} \not\leq K_1K_2 \dots K_n$ for each positive integer n. If there is a positive integer s such that $G_1^{(s)} \leq K$ then $F_s^{(s)} \leq K \cap F_s = K_1K_2 \dots K_s$, a contradiction. Certainly therefore K has infinite index in G_1 and is either minimax or subnormal in G_1. If K is subnormal then there is a finite subnormal series from K to G_1 whose factors are periodic and hence locally finite, since F_n/K_n is finite for all n. By Theorem A and the main result of [Mö] each of these factors is soluble, and we obtain the contradiction that $G_1^{(s)}$ is contained in K for some integer s. Therefore K is minimax and hence of finite rank. As in the proof of Lemma 1 (with K replacing S), it follows that there is an upper bound for the ranks of the torsion-free abelian subgroups of G_1 and hence that some periodic abelian subgroup of G_1 has infinite rank. But this contradicts the fact that every abelian subgroup of G_1 is minimax, and the proof of the theorem is complete.

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