Dimension in algebraic frames, II: Applications to frames of ideals in $C(X)$

Jorge Martínez, Eric R. Zenk

Abstract. This paper continues the investigation into Krull-style dimensions in algebraic frames.

Let L be an algebraic frame. $\dim(L)$ is the supremum of the lengths k of sequences $p_0 < p_1 < \cdots < p_k$ of (proper) prime elements of L. Recently, Th. Coquand, H. Lombardi and M.-F. Roy have formulated a characterization which describes the dimension of L in terms of the dimensions of certain boundary quotients of L. This paper gives a purely frame-theoretic proof of this result, at once generalizing it to frames which are not necessarily compact. This result applies to the frame $\mathcal{C}_z(X)$ of all z-ideals of $C(X)$, provided the underlying Tychonoff space X is Lindelöf. If the space X is compact, then it is shown that the dimension of $\mathcal{C}_z(X)$ is at most n if and only if X is scattered of Cantor-Bendixson index at most $n+1$.

If X is the topological union of spaces X_i, then the dimension of $\mathcal{C}_z(X)$ is the supremum of the dimensions of the $\mathcal{C}_z(X_i)$. This and other results apply to the frame of all d-ideals $\mathcal{C}_d(X)$ of $C(X)$, however, not the characterization in terms of boundaries. An explanation of this is given within, thus marking some of the differences between these two frames and their dimensions.

Keywords: dimension of a frame, z-ideals, scattered space, natural typing of open sets

Classification: Primary 06D22, 54C30; Secondary 03G10, 16P60, 54B35

Introduction

The subject of a Krull-style dimension for either distributive lattices with top and bottom, or algebraic frames in which the compact elements are closed under binary infimum, has received considerable attention in recent years. The interest in this development has come from two fairly distinct quarters. On the one hand, the subject has been investigated by researchers in real algebra, with a background in commutative algebra, and frequently employing the techniques and terminology of logic. The present authors have approached the subject from a frame-theoretic point of view, motivated by their interest and background in lattice-ordered groups and f-rings. As the reader familiar with the lattice theory involved will know, the two areas singled out in the opening sentence are equivalent if one drops the demand that the distributive lattices have a top.

Krull dimension not exceeding n signals the absence of chains $p_0 < p_1 < \cdots < p_n < p_{n+1}$ in the spectrum of the frame. A general development of the concept of 'forbidding' certain configurations in spectra is given in [BP04], by Ball and Pultr; their context is that of distributive lattices and their spectra appear as Priestley spaces.

Martínez has discussed the subject of Krull dimension from the frame-theoretic point of view, in [M04a]. In that article, a general principle was developed which allows dimension to be computed using certain finite sets of compact elements of the frame. Independently, in [CL02] and [CLR03], the authors investigated the subject in distributive lattices, and established a similar (yet more felicitous) condition for the finiteness of the dimension. What is striking about the work of Coquand, Lombardi and Roy — in [CL02] and [CLR03] — is the technique for calculating dimension which uses the notion of a 'boundary quotient', as it offers some distinct advantages over the methods of [M04a]. For one, the elementwise characterization (and prime-free methods) of the latter paper are more narrowly designed, with frames of convex ℓ-subgroups in mind, and these have an additional property — the so-called disjointification — which is not enjoyed by algebraic frames in general. For another, the account using boundary quotients lends itself to inductive arguments.

The principal and original motivating force behind our interest in a Krull-style dimension was a curiosity about the frame of z-ideals of a ring $C(X)$ of continuous real valued functions on a Tychonoff space X, and about the primes of that lattice. However, the techniques of [M04a] remained untested on z-ideals. On the other hand, the dimension of the frame of z-ideals can be computed spatially, in a sense which will be broadly explained in Section 3 and more specifically in our account of z-dimension over Section 4 and Section 5. When Krull dimension is 'spatial', there are important structural consequences, as is fully explored in the forthcoming [MZ06]. In any event, the reader will surely appreciate the advantages offered by the approach taken here, using the criteria of [CL02] and [CLR03].

We begin with a brief section which sets down the necessary background on frames, including standard material, but recalling as well some of the notation and terminology from [M04a], which we continue to adhere to in this article. In Section 2, we state the computational results from [CL02] and [CLR03] referred to above, supplying frame-theoretic proofs. As already alluded to, Section 3 explores some general categorical issues, which sets up the application of the characterization in Section 2 to z-ideals in Section 4. In Section 5 we characterize the compact spaces having finite z-dimension. We conclude the article with a discussion of the frame of d-ideals of $C(X)$, in Section 6, also as an application of the material in Section 3, although handicapped by the limitation already set forth in that section.

1. Frame-theoretic preliminaries

This section is quite simply a catalogue of background material on frames and algebraic frames, in particular. We refer the reader to [J82] and [JT84] for general background on frames, and to [MZ03] and [M04a] for additional material on closure operators.

Definition & Remarks 1.1. Throughout this commentary, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, $\uparrow x$ (resp. $\downarrow x$) stands for the set of elements $\geq x$ (resp. $\leq x$). Let us also point out to the reader that, throughout, we use the phrase 'y exceeds x' in a poset to indicate that $y \geq x$.

1. Recall that $c \in L$ is *compact* if $c \leq \bigvee_{i \in I} x_i$ implies that $c \leq \bigvee_{i \in F} x_i$, for a suitable finite subset F of I. L is *algebraic* if each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L. If 1 is compact it is said that L is *compact*.
2. L is said to have the *finite intersection property* (always abbreviated *FIP*) if for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is *coherent* if it is compact and has the FIP.
3. L is a *frame* if the following distributive law holds:
$$a \wedge \left(\bigvee S \right) = \bigvee \left\{ a \wedge s \,:\, s \in S \right\}.$$
It is well known that an algebraic lattice is a frame as long as it is distributive.
4. $p \in L$ is *prime* if $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. Note that if L is distributive then p is prime if and only if it is *meet-irreducible*: i.e., $x \wedge y = p$ implies that $x = p$ or $y = p$. Observe that if L is an algebraic frame then $p \in L$ is prime as long as
$$a \wedge b \leq p \;\Rightarrow\; a \leq p \;\text{ or }\; b \leq p$$
holds for compact a and b.

   $\mathrm{Spec}(L)$ shall denote the set of prime elements of L.
5. Let L be a frame. For each $a, b \in L$, let
$$a \to b = \bigvee \left\{ x \in L \,:\, a \wedge x \leq b \right\}.$$
When $b = 0$ we denote $a \to 0 = a^{\perp}$. $x \in L$ is a *polar* if it is of the form $x = y^{\perp}$, for some $y \in L$. It is well known that the set $P(L)$ of all polars forms a complete boolean algebra, in which infima agree with those in L.
6. Let L be a frame. Recall that $a \preceq b$ if $b \vee a^{\perp} = 1$. $x \in L$ is *regular* if
$$x = \bigvee \left\{ a \in L \,:\, a \preceq x \right\}.$$

L is *regular* if each element of L is regular.

7. (See [JT84].) Let L be a frame and suppose that $j : L \longrightarrow L$ is a closure operator; jL designates $\{x \in L : j(x) = x\}$. j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. It is well known that j is a nucleus if and only if $b \in jL$ implies that $a \to b \in jL$, for each $a \in L$. For convenience we shall call a subset with this feature *nuclear*.
8. (See [MZ03, §4].) Suppose that L is an algebraic lattice, and j is a closure operator. Say that j is *inductive* if
$$j(x) = \bigvee \Big\{ j(a) \,:\, a \in \mathfrak{k}(L),\, a \leq x \Big\}.$$
Then jL is algebraic and $\mathfrak{k}(jL) = j(\mathfrak{k}(L))$. If L is also a frame and j is a nucleus on L, then jL is an algebraic frame as well; its members are called *j-elements*.

   Observe, in addition, that if L is an algebraic frame and j is an inductive nucleus on L, then
   (a) $\mathrm{Spec}(jL) = \mathrm{Spec}(L) \cap jL$;
   (b) if L has FIP then so does jL.
9. (See [MZ03, §4].) Suppose that L is an algebraic frame with FIP and that j is a nucleus on L. Let $\mathrm{Ab}(j)$ stand for the set of all $x \in L$ such that $a \leq x$ (with a compact) implies that $j(a) \leq x$. Then $\mathrm{Ab}(j)$ is an algebraic frame with FIP. More precisely,
$$\widehat{j}(x) = \bigvee \Big\{ j(a) \,:\, a \in \mathfrak{k}(L),\, a \leq x \Big\}$$
defines an inductive nucleus such that $\widehat{j}L = \mathrm{Ab}(j)$.
10. Closure operators on L are partially ordered by defining $j_1 \leq j_2$ if $j_1(x) \leq j_2(x)$ for each $x \in L$, which, in turn, is equivalent to $j_2L \subseteq j_1L$. Under these stipulations, and using the notation of 9, $\widehat{j}$ is the largest inductive closure operator below j. The passage $j \mapsto \widehat{j}$ is referred to as *inductivization*.

    Escardó (in [Es98]) considers inductivization in a more general context. What he terms a finitary nucleus is exactly the concept of an inductive nucleus on an algebraic frame.
11. Let L be an algebraic frame. In $\mathrm{Spec}(L)$, a chain $p_0 < p_1 < \cdots < p_k$ has *length* k. The *dimension* of L, $\dim(L)$, is the maximum of lengths of chains in $\mathrm{Spec}(L)$, if such a maximum exists; otherwise, it is ∞. It is convenient to define the dimension of the trivial frame — i.e., the frame consisting of one element — to be -1.
12. The nucleus j is *dense* if $j(0) = 0$; if so we also say that L is *j-semisimple*. Note that j is dense if and only if $0 \in jL$.

Remark 1.2. It is worth underscoring that we shall assume and liberally apply Zorn's Lemma, which guarantees that all algebraic frames are spatial.

The following remark will be helpful in 3.10 of Section 3.

Remark 1.3. Throughout this discussion L denotes a complete lattice. Any closure operator j on L may be viewed as a morphism of complete join-semilattices from L to jL such that $j(\bigvee S) = \bigvee^{jL} j(S)$, for any subset S of L. ($\bigvee^{jL}$ denotes supremum in jL.)

Conversely, suppose that M too is a complete lattice and $f : L \longrightarrow M$ is a morphism of complete join-semilattices. Define $f^* : M \longrightarrow L$ by the following equivalence (which defines it unambiguously):

$$x \leq f^*(y) \quad \Leftrightarrow \quad f(x) \leq y.$$

The reader who is familiar with the relevant category theory will recognize that the relationship between f and f^* as functors on the categories L and M, respectively, is one of adjointness. As a consequence of this relationship we have the following properties, which are well known and straightforward to prove directly.

1. $x \leq f^*f(x)$, for each $x \in L$, and $ff^*(y) \leq y$, for each $y \in M$.
2. f^* preserves arbitrary infima.
3. $\ker(f) \equiv \{\, x \in L \,:\, f(x) = 0 \,\} = \downarrow f^*(0)$.
4. Assuming that f is surjective as well, we have:
   (a) $f \cdot f^* = 1_M$.
   (b) $j \equiv f^* \cdot f : L \longrightarrow L$ is a closure operator, and f^* induces a lattice isomorphism of M onto jL, the inverse of which is $f|_{jL}$.
   (c) The following are equivalent:
      i. L is j-semisimple (in the sense of 1.1.12);
      ii. $f^*(0) = 0$;
      iii. $f(x) = 0$ implies that $x = 0$.

A frame morphism $f : L \longrightarrow M$ which satisfies 4c(ii) above is said to be *dense*.

To conclude this general introduction on frame-theoretic attributes, we give a brief account of our work on regular algebraic frames, as these are, at least with the assumption of FIP, the algebraic frames of dimension 0. What follows is [MZ03, Theorem 2.4]. A version of that, without any mention of regularity appears as [M73a, Theorem 2.4].

Remark 1.4. Let L stand for an algebraic frame. The following are equivalent:

1. L is regular;
2. for each $c \in \mathfrak{k}(L)$, $c \vee c^{\perp} = 1$;
3. for each $a \leq c$, $a, c \in \mathfrak{k}(L)$ there is a $b \in \mathfrak{k}(L)$ such that $a \wedge b = 0$ and $a \vee b = c$;

4. L has the FIP and each prime of L is minimal.

Thus, if L has FIP then it is regular precisely when $\dim(L) = 0$.

For the rest of this article, we shall assume that all algebraic frames have the FIP, unless the contrary is expressly indicated.

2. Computing dimension using compact elements

It seems obvious from the start of any serious consideration of dimension in algebraic frames, that one should like to have computational devices that are either entirely or primarily couched in terms of the compact elements. Thinking of the duality between spatial frames and sober spaces, and the primes as points of the latter, this can perhaps be rephrased by saying that one should like a pointfree characterization of dimension.

In [M04a, Theorem 3.8] such a characterization is obtained. One needs to impose an additional assumption on the frame, that of 'disjointification', and in the context of [M04a] and the applications considered there, this characterization yields a substantial amount of information. Recently, in [CL02, Theorem 2.9] and [CLR03, Theorem 1.4], the authors give a primefree characterization of dimension in a coherent frame, without the additional disjointification. We will cull these two theorems into one, which we will refer to as the Coquand-Lombardi-Roy Theorem (Theorem 2.7 below). Now, these theorems are not phrased in terms of frames, rather for distributive lattices; indeed, [CL02, Theorem 2.9] is immersed in the language and notation of logic, and while the context of [CLR03, Theorem 1.4] is more transparent, the proof is sketchy. At any rate, we assumed there had to be a strictly frame-theoretic proof for the Coquand-Lombardi-Roy Theorem that would not be too long, and we believe that the proof of Theorem 2.7 succeeds on both counts.

In advance of the theorem, we need three preliminaries. The first item is a standard frame-theoretic lemma, which is basic to the proofs that come after. The second is an observation about primes of L vs primes of $\uparrow x$. The final preliminary (Lemma 2.6) will be an inductive estimation of dimension. This is also the place where boundary quotients make their appearance. It is worthwhile repeating, with regard to boundary quotients, and looking ahead at the discussion of boundaries in the upcoming section, that what makes the development of Coquand, Lombardi and Roy more tractable than the comparable criterion from [M04a], is precisely this inductive 'environment'.

The following lemma appears as [M73a, Corollary 2.5.1]. The proof involves ultrafilters of compact elements. By a *filter F of compact elements* we mean a subset of $\mathfrak{k}(L) \setminus \{0\}$, closed under finite meets and such that $c \leq d$ in $\mathfrak{k}(L)$ with $c \in F$ implies that $d \in F$. An *ultrafilter of compact elements* is a maximal filter of compact elements.

Lemma 2.1. *Suppose L is an algebraic frame. Then $p \in \mathrm{Spec}(L)$ is minimal if and only if*
$$F_p = \{\, c \in \mathfrak{k}(L) \,:\, c \not\leq p \,\}$$
is an ultrafilter on $\mathfrak{k}(L)$. In this case,
$$p = \bigvee_{c \in F_p} c^{\perp}.$$

Remark 2.2. Let $\mathrm{Min}(L)$ denote the set of all minimal prime elements of L. Zorn's Lemma easily shows that in any frame each prime element exceeds a minimal prime. It is also a routine matter to verify that, in any algebraic frame, each polar is an infimum of minimal primes.

Lemma 2.1 implies the following; this corollary amounts to half the proof of Lemma 2.6, as we shall presently see.

Corollary 2.3. *Let L be an algebraic frame. For each $a \in \mathfrak{k}(L)$ and each $p \in \mathrm{Min}(L)$, we have $a \vee a^{\perp} \not\leq p$.*

Next, we have the following observation; we leave the details of the proof to the reader.

Lemma 2.4. *Let L be an algebraic frame. For each $y \in L$, the map $j^y(x) = x \vee y$ is an inductive nucleus and $j^y L = \uparrow y$. Thus, $\mathrm{Spec}(\uparrow y)$ consists of the primes of L that exceed y.*

Proof: That j^y is actually a nucleus and inductive are routine to verify. The claim about primes then follows from 1.1.8. □

Before proceeding with Lemma 2.6, let us pause to introduce a term which, apart from being suggestive, will actually mirror the topological reality of our subsequent applications.

Definition 2.5. Let L be an algebraic frame, and $a \in \mathfrak{k}(L)$. We denote $L^a \equiv \uparrow (a \vee a^{\perp})$, and call L^a the *boundary quotient over a*.

The proof of Lemma 2.6 closely follows the spirit of the corresponding argument in [CLR03].

Lemma 2.6. *Suppose that L is an algebraic frame. Then $\dim(L) \leq k$ if and only if, for each $a \in \mathfrak{k}(L)$, the dimension of the boundary quotient L^a over a is $\leq k-1$.*

Proof: Corollary 2.3 immediately implies that, if $\dim(L) \leq k$, then $\dim(L^a) \leq k-1$, for each compact element a. To see the converse, if suffices to observe that, whenever $p < q$ are primes of L, then there is a $c \in \mathfrak{k}(L)$ such that $c \leq q$ but

$c \not\leq p$. Thus, using Lemma 2.4, if $p_0 < p_1 < \cdots < p_m$ is any maximal chain of primes in L (with p_0 necessarily minimal), there is a compact element c such that the boundary L^c over c has a chain of primes of length $m-1$. $\square$

The proof of Theorem 2.7 is now a relatively easy induction argument.

Theorem 2.7 [The Coquand-Lombardi-Roy Theorem]**.** *Let L be an algebraic frame. Then* $\dim(L) \leq k$ *if and only if*

$$1 = x_k \vee (x_k \to (\cdots (x_1 \vee (x_1 \to (x_0 \vee x_0^{\perp}))) \cdots)),$$

for all $x_0, x_1, \ldots, x_k \in \mathfrak{k}(L)$.

Proof: We induct on k. L has dimension zero precisely when every boundary quotient of L is trivial; that is, each $\dim(L^a) = -1$ (see 1.4(a)).

As to the induction step, note that if

$$1 = x_{k+1} \vee (x_{k+1} \to (\cdots (x_1 \vee (x_1 \to (x_0 \vee x_0^{\perp}))) \cdots)),$$

for all $x_0, x_1, \ldots, x_k, x_{k+1} \in \mathfrak{k}(L)$, then (in L^{x_0})

$$1 = j^{x_0}(x_{k+1}) \vee (j^{x_0}(x_{k+1}) \to (\cdots (j^{x_0}(x_1) \vee ((j^{x_0}(x_1))^{\perp})) \cdots)),$$

which according to the induction hypothesis means that each $\dim(L^{x_0}) \leq k$. By Lemma 2.6 this implies that $\dim(L) \leq k+1$. These steps are reversible, completing the proof; we leave the details to the reader. $\square$

We close the section with a local version of Theorem 2.7.

Theorem 2.8. *Let L be an algebraic frame. Then* $\dim(L) \leq k$ *if and only if for each set of compact elements $a_0, a_1, \ldots, a_k, a_{k+1}$ there exist compact elements $b_0, b_1, \ldots, b_k$ such that*

$$a_{k+1} \leq a_k \vee b_k,\ a_k \wedge b_k \leq a_{k-1} \vee b_{k-1}, \ldots, a_1 \wedge b_1 \leq a_0 \vee b_0, \quad \textit{and} \quad a_0 \wedge b_0 = 0.$$

Proof: Apply Theorem 2.7, iterating the observation that, for any compact elements a and b in L,

$$a \leq b \vee (b \to y) \ \text{ iff } \ \exists\, c \in \mathfrak{k}(L), \ \text{ with } \ a \leq b \vee c \ \text{ and } \ b \wedge c \leq y.$$

$\square$

Remark 2.9. Briefly, we make note of the fact that the condition in Theorem 2.7 coincides with the one obtained in [AB91]. The context in that article is that of clopen downsets in a Priestley space.

3. Frames of open sets of a Tychonoff space

This section is primarily expository, and designed to set up the applications of Section 2 to frames of ideals of a ring of continuous functions. Recall that our interest is principally in the frame of z-ideals of such a ring.

The scope of the discussion of this section may be extended to a general frame-theoretic setting, but we are satisfied here to restrict it to Tychonoff spaces and their rings of continuous functions. In preparation, we need to recall a fundamental correspondence between distributive lattices and algebraic frames (with the FIP, as previously announced).

First and foremost, though, we should remind the reader of the topological terminology that will be needed in the sequel.

Definition & Remarks 3.1. Let X be a topological space. $C(X)$ denotes the ring of all continuous real valued functions on X, under pointwise operations. $C(X)$ is always a commutative ring with identity, and it is *semiprime*; that is to say, there are no nonzero nilpotent elements. When considering $C(X)$ one may, without loss of generality, take X to be a *Tychonoff* space; that is, a space which is Hausdorff, and in which for each closed set K and each point p not in K, there is an $f \in C(X)$ such that $f(K) = \{0\}$ and $f(p) = 1$.

For any topological space X, we shall use $\mathfrak{O}(X)$ to denote the frame of open sets (under ordinary set-theoretic union and intersection). For each $f \in C(X)$ let

$$\operatorname{coz}(f) = \{\, x \in X \,:\, f(x) \neq 0 \,\};$$

this is the *cozeroset* of f. $\operatorname{Coz}(X)$ shall denote the set of all cozerosets of X. $\operatorname{Coz}(X)$ is a sublattice of $\mathfrak{O}(X)$, and, indeed, it is closed under countable unions. It is well known that X is a Tychonoff space if and only if $\operatorname{Coz}(X)$ is a base for $\mathfrak{O}(X)$; [GJ76, 3.2].

For each $f \in C(X)$, the *zeroset* $Z(f)$ of f is the complement of $\operatorname{coz}(f)$. $Z[X]$ stands for the set of all zerosets of X. We shall be interested in boundaries of zerosets and cozerosets in the sequel. Let $W \in \operatorname{Coz}(X)$; the *boundary* of W, denoted bW, is $bW = \operatorname{cl}_X W \cap (X \setminus W)$. For a zeroset $Z = Z(f)$ we shall also write bZ for $b\operatorname{coz}(f)$ whenever convenient.

Next, we review the categorical equivalence alluded to in the introduction to this section.

Definition & Remarks 3.2. (a) If L is an algebraic frame then $\mathfrak{k}(L)$ is a distributive lattice with bottom 0. Conversely, if B is a distributive lattice with bottom 0, then the lattice $\mathcal{I}(B)$ of all ideals of B is an algebraic frame with FIP. Recall that $J \subseteq B$ is an *ideal* of B if

(i) J is closed under finite suprema and
(ii) $0 \leq a \leq b \in J$ implies that $a \in J$.

(b) A frame homomorphism $g : L \longrightarrow M$ is *coherent* if $g(\mathfrak{k}(L)) \subseteq \mathfrak{k}(M)$. Now recall that the assignments

$$B \mapsto \mathcal{I}(B) \ \text{ and } \ L \mapsto \mathfrak{k}(L)$$

define the object portions of an equivalence between the category $\mathfrak{D}$ of all distributive lattices with bottom, together with all lattice homomorphisms which preserve bottom, and the category $\mathfrak{AFrm}$ of all algebraic frames with FIP, together with all coherent frame homomorphisms. Observe that if $g : B_1 \longrightarrow B_2$ is a morphism in $\mathfrak{D}$ then $\mathcal{I}(g)$ is defined by

$$\mathcal{I}(g)[\langle\, b_i \,:\, i \in I\,\rangle] = \langle\, g(b_i) \,:\, i \in I\,\rangle.$$

(Note: $\langle T\rangle$ denotes the ideal of B generated by $T \subseteq B$.) Conversely, if $h : L_1 \longrightarrow L_2$ is a $\mathfrak{AFrm}$-morphism, then $\mathfrak{k}(h)$ is the restriction to $\mathfrak{k}(L_1)$.

(c) The functor $\mathcal{I}$ may also be regarded as the 'free frame' over a distributive lattice with bottom. First, label the function which 'embeds' the $\mathfrak{D}$-object B in $\mathfrak{I}(B)$ in recognition of what it is: $a \mapsto \downarrow_B(a)$. Now if F is a frame and B a $\mathfrak{D}$-object, and $h : B \longrightarrow F$ is a morphism in $\mathfrak{D}$, then there is a unique frame morphism $\widetilde{h} : \mathfrak{I}(B) \longrightarrow F$ such that $\widetilde{h}\cdot \downarrow_B = h$; i.e., such that the diagram below commutes.

$$\begin{tikzcd} B \arrow[rr, "\downarrow_B"] \arrow[dr, "h"'] & & \mathcal{I}(B) \arrow[dl, "\widetilde{h}"] \\ & F & \end{tikzcd}$$

In fact, $\widetilde{h}[\langle T\rangle] = \bigvee h(T)$.

Obviously, $\mathcal{I}$ is, in this view, the left adjoint of the functor $\mathfrak{AFrm} \longrightarrow \mathfrak{D}$ which forgets the frame structure and the top, remembering merely the $\mathfrak{D}$-structure.

In the sequel B will frequently be a sublattice of F which *generates* it, in the sense that each $x \in F$ is a supremum of members of B; h will then denote the inclusion. It is an easy exercise to show that $\widetilde{h}$ is then onto F. Likewise, it is easy to prove that if h is one-to-one then $\widetilde{h}$ is dense, in the sense that if $\widetilde{h}(J) = 0$ then $J = \{0\}$ — which is the sense of 1.3.

Finally, assuming again that h is inclusion of a generating set B in F, we observe that if F is compact, then $\widetilde{h}$ is *codense*, meaning that if $\widetilde{h}(J) = 1$, then $J = B$. (Note: if B generates F then $\mathfrak{k}(F) \subseteq B$.)

Remark 3.3. One should observe, when comparing the results of the preceding section with those of Coquand, Lombardi, and Roy, that they phrase their discussion in the language of the subcategory $\mathfrak{D}_1$ of $\mathfrak{D}$ of all distributive lattices which also have a top 1. By appealing to the categorical correspondence of 3.2(b), one may view their results as results about coherent frames. Theorem 2.7 shows that the compactness of the frame in question does not play a role.

Next, we include a remark, which completes the general account of adjoint situations, as set out in 1.3; the reader will readily appreciate its relevance in questions of dimension.

Remark 3.4. As in 1.3, let $f : L \longrightarrow M$ denote a complete join-homomorphism of complete lattices. We assume throughout here that f is surjective.

1. Assume L is an algebraic lattice. By [M04b, Proposition 1.3], f is a coherent map if and only if $j = f^* \cdot f$ is an inductive closure operator. If so, then $M \cong jL$ is also algebraic.

2. Now assume L is a frame. Then f is a frame homomorphism — that is, it preserves finite infima — if and only if $j = f^* \cdot f$ is a nucleus. In this event, $M \cong jL$ is also a frame ([M04b, Proposition 1.4]).

3. Finally, assume that L and M are frames, and that f is a frame homomorphism. As is well known, and easy to prove, $f^*(\mathrm{Spec}(M)) \subseteq \mathrm{Spec}(L)$, whence f^* induces an order embedding from $\mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(L)$. Thus, if, in addition, L and M are algebraic frames with FIP and f is also coherent, we have that $\dim(M) \leq \dim(L)$.

Let us focus more closely on the problem of computing the dimension of $\mathcal{I}(\mathfrak{U})$ where $\mathfrak{U}$ is a naturally defined base of open sets of a Tychonoff space. We will motivate the usage of the phrase 'naturally defined', as we have in mind potential applications to various frames of ideals of $C(X)$, which are determined by the topology on X. Apart from the discussion of z-ideals and d-ideals contained in this article, we leave other 'potential' applications for a later writing.

Definition & Remarks 3.5. We begin with the following rather general setup. All spaces in this discussion are Tychonoff spaces.

1. Assume $l : \mathfrak{O} \longrightarrow \mathfrak{L}$ is an assignment which associates to each space X a nucleus $l_X : \mathfrak{O}(X) \longrightarrow \mathfrak{L}(X)$, with $\mathfrak{L}(X) = l_X\mathfrak{O}(X)$.

   Now let $f : Y \longrightarrow X$ be a map of Tychonoff spaces. We shall require that

f be *natural for* l; that is to say, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{O}(X) \arrow[r, "\mathfrak{O}(f)"] \arrow[d, "l_X"] & \mathfrak{O}(Y) \arrow[d, "l_Y"] \\ \mathfrak{L}(X) \arrow[r, "\mathfrak{L}(f)"] & \mathfrak{L}(Y) \end{tikzcd}$$

Note that we do not assume that $\mathfrak{L}(f)$ coincides with the action of $\mathfrak{O}(f)$ on $\mathfrak{L}(X)$. The reader would not be far off the mark in thinking of the application of l_X as a 'relative topological closure'; indeed, define, for each $U \in \mathfrak{L}(X)$,

$$\mathfrak{L}(f)(U) = l_Y(\mathfrak{O}(f)(U)).$$

$\mathfrak{T}_l$ will stand for the subcategory of Tychonoff spaces and l-natural (continuous) maps. $\mathfrak{Tch}$ denotes the category of Tychonoff spaces and all continuous maps. Note that $f : Y \longrightarrow X$ is a $\mathfrak{T}_l$-map if and only if

$$\mathfrak{L}(f) \cdot l_X = l_Y \cdot \mathfrak{O}(X).$$

It is then routine to verify that $\mathfrak{T}_l$ is a (generally, non-full) subcategory of $\mathfrak{Tch}$; we refer to the example cited in 3.6(b), as an illustration of 'non-fullness'.

2. Observe that the right adjoint of l_X — in the sense of 1.3 — is the inclusion map of $\mathfrak{L}(X)$ in $\mathfrak{O}(X)$.
3. Next is the concept of a natural base: $s : \mathfrak{b} \longrightarrow \mathfrak{L}$. By this we mean that $\mathfrak{b}(X)$ is a sublattice of $\mathfrak{L}(X)$, containing the top, X, and the bottom of $\mathfrak{L}(X)$ (*not necessarily* $\emptyset$ — see 5 below), with each $s_X : \mathfrak{b}(X) \longrightarrow \mathfrak{L}(X)$ being the inclusion, such that the diagram below commutes.

$$\begin{tikzcd} \mathfrak{L}(X) \arrow[r, "\mathfrak{L}(f)"] & \mathfrak{L}(Y) \\ \mathfrak{b}(X) \arrow[r, "\mathfrak{b}(f)"] \arrow[u, "s_X"] & \mathfrak{b}(Y) \arrow[u, "s_Y"'] \end{tikzcd}$$

for all $\mathfrak{T}_l$-maps f, with the stipulation $\mathfrak{b}(f) \equiv \mathfrak{L}(f)|_{\mathfrak{b}(X)}$.

4. Putting all these ingredients together we get the following commutative diagram, for each $\mathfrak{T}_l$-map $f : Y \longrightarrow X$:

$$\begin{tikzcd}
& \mathfrak{O}(X) \arrow[rr, "\mathfrak{O}(f)"] \arrow[d, "l_X"] & & \mathfrak{O}(Y) \arrow[d, "l_Y"] \\
& \mathfrak{L}(X) \arrow[rr, "\mathfrak{L}(f)"] & & \mathfrak{L}(Y) \\
\mathfrak{b}(X) \arrow[ur, "s_X"] \arrow[rr, bend left, "\mathfrak{b}(f)"] \arrow[dr, "\downarrow_{\mathfrak{b}(X)}"'] & & \mathfrak{b}(Y) \arrow[ur, "s_Y"] \arrow[dr, "\downarrow_{\mathfrak{b}(Y)}"'] & \\
& \mathcal{I}(\mathfrak{b}(X)) \arrow[uu, "\widetilde{s_X}"'] \arrow[rr, "\mathcal{I}(\mathfrak{b}(f))"'] & & \mathcal{I}(\mathfrak{b}(Y)) \arrow[uu, "\widetilde{s_Y}"']
\end{tikzcd} \tag{$\bullet$}$$

5. For future reference we shall think of the operators on open sets in the foregoing discussion as a triple $(l, \mathfrak{L}, \mathfrak{b})$, and refer to it as a *natural typing of open sets*. We call the natural typing $(l, \mathfrak{L}, \mathfrak{b})$ *dense* if each l_X is a dense nucleus, or, equivalently, if $\emptyset \in \mathfrak{L}(X)$.

Both of the natural typings of open sets discussed below are dense.

Examples 3.6. (a) First, let $\mathfrak{L} = \mathfrak{O}$, with $\mathfrak{b}(X) = \mathrm{Coz}(X)$, for each space X. In this case, all continuous maps are l-natural, as $l = 1$. The natural typing $(1, \mathfrak{O}, \mathrm{Coz})$ gives rise to the frame of z-ideals, as we shall see in Section 4.

(b) Let $\mathfrak{L}(X) = \mathfrak{RO}(X)$, the frame of regular open sets of X. Recall that $U \in \mathfrak{O}(X)$ is *regular* if $\mathrm{int}_X \,\mathrm{cl}_X\, U = U$. The reader should be reminded of the lattice operations in $\mathfrak{RO}(X)$:

$$\bigvee_{i\in I} U_i = \mathrm{int}_X \,\mathrm{cl}_X \Big(\bigcup_{i\in I} U_i\Big) \quad \text{and} \quad \bigwedge_{i\in I} U_i = \mathrm{int}_X \,\mathrm{cl}_X \Big(\bigcap_{i\in I} U_i\Big).$$

For brevity, we shall use ϱ_X to denote the nucleus on $\mathfrak{O}(X)$ defined by $\varrho_X U = \mathrm{int}_X \,\mathrm{cl}_X\, U$. In this connection we recall that the finite intersection of regular open sets is regular. Thus, to say that $f : Y \longrightarrow X$ is a $\mathfrak{T}_\varrho$-map means that

$$\varrho_Y f^{-1}(\varrho_X U) = \varrho_Y f^{-1}(U), \tag{$*$}$$

for each $U \in \mathfrak{O}(X)$.

In the sequel we shall examine the natural typing of open sets $(\varrho, \mathfrak{RO}, \mathfrak{b}_\varrho)$, where the natural base for $\mathfrak{RO}(X)$ is

$$\mathfrak{b}_\varrho(X) = \{\, \varrho_X \operatorname{coz}(f) \,:\, f \in C(X)\,\}.$$

The reader will readily verify that the latter is a sublattice of $\mathfrak{RO}(X)$. We shall refer to this typing as the *regular* typing of open sets.

It is an easy exercise to check that, relative to the regular typing of open sets, the retraction of the reals $\mathbb{R}$, with the usual topology, onto any closed bounded interval is not a $\mathfrak{T}_\varrho$-map.

We have the following narrower description of the $\mathfrak{T}_\varrho$-maps, in the context of the regular typing introduced above. The proposition we have in mind is preceded by a lemma, the proof of which is left to the reader.

Lemma 3.7. *Let A, B be open subsets of X, with $A \subseteq B$. The following are equivalent:*

(a) $\varrho_X A = \varrho_X B$;
(b) $\mathrm{cl}_X\, A = \mathrm{cl}_X\, B$;
(c) *for any open set U, $A \cap U = \emptyset \;\Rightarrow\; B \cap U = \emptyset$;*
(d) *$B \setminus A$ is nowhere dense.*

The maps described in Proposition 3.8(c) are called *skeletal* in the literature.

Proposition 3.8. *Let $f : Y \longrightarrow X$ be a continuous function. The following are then equivalent:*

(a) *f is ϱ-natural;*
(b) *for all $U \in \mathfrak{O}(X)$ and $W \in \mathfrak{O}(Y)$, $f^{-1}(U) \cap W = \emptyset \;\Rightarrow\; f^{-1}(\varrho_X U) \cap W = \emptyset$;*
(c) *f is skeletal; i.e., the inverse image of each dense open set is dense open.*

Proof: The equivalence of (a) and (b) is clear.

(b) $\Rightarrow$ (c): if U is dense open, then, applying (b) and the lemma to the containment of U in $X = \mathrm{cl}_X\, U$, we have that $f^{-1}(U)$ is dense open in Y.

(c) $\Rightarrow$ (b): Let $U \in \mathfrak{O}(X)$ and $W \in \mathfrak{O}(Y)$. Since $G = \varrho_X U \setminus U$ is nowhere dense and f is skeletal, we conclude that $f^{-1}(G)$ is nowhere dense. Thus, if $W \cap \varrho_X U$ is nonempty and $W \cap U = \emptyset$, then

$$\emptyset \neq f^{-1}(W \cap \varrho_X U) \subseteq f^{-1}(G),$$

contradicting that $f^{-1}(G)$ is nowhere dense. $\square$

If Y is a subspace of X and the inclusion of Y in X satisfies Proposition 3.8(c), we shall call it a *skeletal embedding*. Based on the foregoing, we have the following immediate corollary. By contrast with it, please note that a nowhere dense closed proper subspace is not skeletal.

Corollary 3.9. *Let Y be a subspace of X. $Y \subseteq X$ is a skeletal embedding if Y is dense in X or open in X.*

Proof: Apply Proposition 3.8(c). □

We wind up the section with some general observations, intended to both set up and motivate the discussion of the next section. The reader would be well served with a review of the discussion in 1.3.

Definition & Remarks 3.10. (a) Suppose that $(l, \mathfrak{L}, \mathfrak{b})$ is a dense natural typing of open sets. Given $U \in \mathfrak{b}(X)$, define

$$b_l U = X \setminus (U \vee U^{\perp}),$$

with the $U^{\perp}$ calculated in the frame $\mathfrak{L}(X)$. We call $b_l U$ the *l-boundary* of U. In view of Lemma 2.6 on boundary quotients, it seems reasonable to investigate further the l-natural l-boundaries of X. For any inclusion of a subspace $i : Y \longrightarrow X$, we appeal directly to 3.5.1, to conclude that i is l-natural precisely when, for each open set W,

$$\mathfrak{L}(i)(l_X(W)) = l_Y(W \cap Y).$$

(b) Next, assume that the inclusion $i : Y \longrightarrow X$ is l-natural, and consider the frame homomorphism $\mathcal{I}(\mathfrak{b}(i))$, which we, henceforth, will abbreviate as Φ_Y. It will be helpful to explicitly describe Φ_Y and its adjoint Φ_Y^*.

1. For an ideal $\mathcal{J}$ of $\mathfrak{b}(X)$,

$$\Phi_Y(\mathcal{J}) = \langle l_Y(W \cap Y) \,:\, W \in \mathcal{J} \rangle.$$

2. For an ideal $\mathcal{K}$ of $\mathfrak{b}(Y)$,

$$\Phi_Y^*(\mathcal{K}) = \{\, V \in \mathfrak{b}(X) \,:\, l_Y(V \cap Y) \in \mathcal{K} \,\}.$$

Of some significance is the situation in which Φ_Y is surjective; recall 1.3.4(a): $\Phi_Y \cdot \Phi_Y^* = 1_{\mathcal{I}(Y)}$, if (and only if) Φ_Y is surjective. Checking the diagram ($\bullet$), will reveal that if the restriction $\mathfrak{b}(i)$ is surjective, then so is Φ_Y. A direct calculation will verify the same thing.

But the reverse is also true. That is part of the subject of the next lemma, which also articulates where the image of Φ_Y^* lies when Φ_Y is surjective.

Lemma 3.11. *Suppose that $(l, \mathfrak{L}, \mathfrak{b})$ is a dense natural typing of open sets, and that $i : Y \longrightarrow X$ is an l-natural embedding. Then Φ_Y is surjective if and only if $\mathfrak{b}(i)$ is. In this case, for each $\mathcal{J} \in \mathcal{I}(\mathfrak{b}(X))$,*

$$\Phi_Y(\mathcal{J}) = \{\, l_Y(W \cap Y) \,:\, W \in \mathcal{J} \,\}.$$

Moreover, each $\Phi_Y^*(\mathcal{K})$ *contains the ideal* $\mathcal{J}_Y \equiv \{ V \in \mathfrak{b}(X) : V \cap Y = \emptyset \}$.

Proof: Suppose that Φ_Y is surjective. By 1.3.4(a), $\Phi_Y(\Phi_Y^*(\mathcal{K})) = \mathcal{K}$, for each ideal $\mathcal{K}$ of $\mathfrak{b}(Y)$. Thus, if $l_Y V \in \mathfrak{b}(Y)$, then

$$\langle l_Y V \rangle \subseteq \Phi_Y(\Phi_Y^*(\langle l_Y V \rangle)),$$

which implies that there is an $W \in \mathfrak{b}(X)$ such that $l_X W \in \Phi_Y^*(\langle l_Y V \rangle)$ and $l_Y V \subseteq l_Y(W \cap X)$; that is, there is an $W \in \mathfrak{b}(X)$ such that

$$l_Y V \subseteq l_Y(W \cap X) \subseteq l_Y V.$$

Therefore, $l_Y V = l_Y(W \cap X)$, for a suitable $W \in \mathfrak{b}(X)$, and $\mathfrak{b}(i)$ is indeed onto.

The final two assertions are clear. □

Definition & Remarks 3.12. Suppose that $(l, \mathfrak{L}, \mathfrak{b})$ is a dense natural typing of open sets, and that $i : Y \longrightarrow X$ is an l-natural embedding. We shall say that i is a $z(l)$-*embedding* if $\Phi \equiv \Phi_Y$ is surjective. From the remarks in 3.4.3, we are able to conclude that if Y is a $z(l)$-embedded subspace, then

$$\dim(\mathcal{I}(\mathfrak{b}(Y))) \leq \dim(\mathcal{I}(\mathfrak{b}(X))).$$

The last assertion of Lemma 3.11 may be interpreted to say that, if Y is $z(l)$-embedded in X, then the fixed set of the nucleus $\Phi^* \cdot \Phi$ is contained in the frame quotient $\uparrow \mathcal{J}_Y$. If this fixed set is *precisely* $\uparrow \mathcal{J}_Y$ then we say that Y is *optimally* l-embedded in X.

Let us now return to the consideration of an l-boundary $i : b_l U \longrightarrow X$ (with $U \in \mathfrak{b}(X)$). Note that $b_l U$ is optimally l-embedded if and only if the fixed set of $\Phi^* \cdot \Phi$ is $\uparrow ((\downarrow U) \vee (\downarrow U)^{\perp})$. (The reader will doubtless recognize the latter as the boundary quotient of $\mathcal{I}(\mathfrak{b}(X))$ over $\downarrow U$, as defined in 2.5.) When $b_l U$ is optimally l-embedded we also speak of an *optimal* l-boundary; observe that whether or not an l-boundary is optimal may in fact depend on the open set U in question.

We have the following immediate consequence of Lemma 2.6.

Theorem 3.13. *Suppose that* $(l, \mathfrak{L}, \mathfrak{b})$ *is a dense natural typing of open sets. Assume that the* l*-boundary of every* $U \in \mathfrak{b}(X)$ *is an optimal* l*-boundary. Then, for each nonnegative integer* k, $\dim(\mathcal{I}(\mathfrak{b}(X))) \leq k$ *if and only if, for each* $U \in \mathfrak{b}(X)$, $\dim(\mathcal{I}(\mathfrak{b}(b_l U))) \leq k - 1$.

To contrast, let us now identify a crucial limitation to the inductive method of Theorem 3.13. Simply put, it is an instance where reality disappoints.

Remark 3.14. Let $(l, \mathfrak{L}, \mathfrak{b})$ be a dense natural typing of open sets. It may happen that $\mathfrak{L}(X)$ is a boolean algebra, for each space X. This occurs in the natural typing $(\varrho, \mathfrak{RO}, \mathfrak{b}_\varrho)$ of 3.6(b). If this is the case then every l-boundary is empty, and it is easy to see that the empty set is always $z(l)$-embedded. However, $b_l U = \emptyset$ is optimal if and only if $X = U \vee V$ for a suitable $V \in \mathfrak{b}(X)$, with $U \cap V = \emptyset$. Thus, Theorem 3.13 applies if and only if $k = 0$.

We will say what we can about d-ideals and d-dimension in Section 6. However, as the base $\mathfrak{b}_\varrho$ appears to be crucial in getting a handle on the d-ideals of $C(X)$ (see Lemma 6.2), having empty boundaries (in this typing) probably means that d-dimension behaves differently from z-dimension in a fundamental way.

Finally, a comment which ought to invite speculation.

Remark 3.15. (a) In the introductory section we alluded to a 'spatial' dimension of topological spaces. Let us formalize this notion now. Suppose that $\mathfrak{U}$ is any base for the open sets $\mathfrak{O}(X)$ of the (not necessarily Tychonoff) space X; assume that $\mathfrak{U}$ is

1. a distributive lattice under inclusion (though not necessarily a sublattice of $\mathfrak{O}(X)$), and
2. that $\mathfrak{U}$ contains X and $\emptyset$.

Let us refer to such a base as a *lattice base of open sets*. The dimension of $\mathcal{I}(\mathfrak{U})$ is called the $\mathfrak{U}$-*dimension* of X, and denoted $\dim(X, \mathfrak{U})$.

(b) Suppose that $\mathfrak{U}$ is a lattice base of open sets. If $\mathfrak{U}$ is a sublattice of $\mathfrak{O}(X)$, then $\dim(X, \mathfrak{U}) = 0$ if and only if $\mathfrak{U}$ is also a (boolean) subalgebra of $\mathfrak{O}(X)$; that is to say, if and only if each $V \in \mathfrak{U}$ is complemented in $\mathfrak{U}$: apply Theorem 2.8 directly. The reader will then easily see that X is *zero-dimensional*, in the usual sense; namely, that the collection of all clopen sets, $\mathfrak{B}(X)$, forms a base. Conversely, if X is zero-dimensional, then $\dim(X, \mathfrak{B}(X)) = 0$.

Happily, the notions of dimension coincide.

(c) In the context of a dense natural typing $(l, \mathfrak{L}, \mathfrak{b})$, Theorem 3.13 could then be interpreted as an expression that $\mathfrak{b}(X)$-dimension (when finite) satisfies $\dim(X, \mathfrak{b}(X)) = 1 + m$, where

$$m = \sup\{ \dim(b_l U, \mathfrak{b}(b_l U)) \,:\, U \in \mathfrak{b}(X) \},$$

as long as the l-boundary of each member of the base $\mathfrak{b}(X)$ of X is an optimal l-boundary.

4. z-dimension of $C(X)$

In this section we apply the foregoing to the frame of z-ideals of a ring of continuous real valued functions on a Tychonoff space. Throughout, X denotes a fixed Tychonoff space, which is arbitrary, until we get to the point where it becomes necessary to make some assumptions in order to get any reasonable results.

We recall the definition and basic features of z-ideals from [GJ76, Chapter 2]. We shall also employ terminology from the theory of lattice-ordered groups — henceforth, ℓ-groups — and for this we refer the reader to [D95] and [BKW77].

Definition & Remarks 4.1. An ideal $\mathfrak{r}$ of $C(X)$ is a *z-ideal* if for each $f \in \mathfrak{r}$ and $g \in C(X)$, with $\mathrm{coz}(g) \subseteq \mathrm{coz}(f)$, it follows that $g \in \mathfrak{r}$. It is well known that any z-ideal is closed under the lattice operations; we shall say, in this regard, that it is an *ℓ-subgroup*. In addition, any z-ideal $\mathfrak{r}$ is (*order*) *convex*; that is, $0 \leq g \leq f \in \mathfrak{r}$ implies that $g \in \mathfrak{r}$.

The set $\mathcal{C}_z(X)$ of all z-ideals is an frame algebraic frame under the ordering of inclusion. In fact, it is shown in [M04a] that $\mathcal{C}_z(X)$ is the set of fixed elements under an inductive nucleus z — see also 1.1.9 — which assigns to each convex ℓ-subgroup A of $C(X)$ the least z-ideal zA containing A. More precisely,

$$zA = \{\, g \in C(X) \,:\, \mathrm{coz}(g) \subseteq \mathrm{coz}(f) \ \text{ for some } \ f \in A \,\}.$$

For future reference we stipulate that $\mathcal{C}(C(X))$ shall denote the frame of all convex ℓ-subgroups of $C(X)$; the latter is a frame under the operations of intersection and supremum defined as the subgroup generated by the family which is to be majorized.

It is easy to check directly that $\mathfrak{k}(\mathcal{C}_z(X))$ consists of the *principal* z-ideals; that is, the z-ideals of the form, for each $f \in C(X)$,

$$\langle f \rangle_z = z\{f\} = \{\, g \in C(X) \,:\, \mathrm{coz}(g) \subseteq \mathrm{coz}(f) \,\}.$$

For example, to prove that each compact z-ideal is of the prescribed form, note that if $\mathfrak{r}$ is a z-ideal which is generated by $f_1, f_2, \dots, f_m$, we may assume without loss of generality — by passing from f_i to $|f_i|$ — that each of the generators is positive. It is then clear that $\mathfrak{r} = \langle (f_1 + f_2 + \cdots + f_m) \rangle_z$.

Note, finally, that $\mathcal{C}_z(X)$ is compact, and therefore coherent; the top is $\langle 1 \rangle_z$.

The most immediate goal is to make the connection between the frame of z-ideals and the frame of ideals of $\mathrm{Coz}(X)$. That is the subject of the next lemma. From the comments above the proof is immediate, and we leave it to the reader.

Lemma 4.2. *Let X be a space. The map*

$$\eta^z_X(\mathrm{coz}(f)) = \langle f \rangle_z, \quad (f \geq 0),$$

is a lattice isomorphism from $\mathrm{Coz}(X)$ *onto* $\mathfrak{k}(\mathcal{C}_z(X))$.

As a consequence of Lemma 4.2 and the discussions in 3.2(b) and 3.5, we have the following, in which we feature an amalgam of the diagram in 3.5.4 with the new information.

Proposition 4.3. *Let X be a space. We have the frame isomorphism from* $\mathcal{I}(\mathrm{Coz}(X))$ *onto* $\mathcal{C}_z(X)$ *defined by extending the map* η^z_X *to the frame of ideals (calling the extension* η^z_X *as well). Moreover, we have, for any continuous function*

$f : Y \longrightarrow X$ between spaces the commutative diagram, in which $\mathcal{C}_z(f)(\mathfrak{r}) = \langle\{\, g \cdot f \, : \, g \in \mathfrak{r}\,\}\rangle_z$:

$$\begin{tikzcd}
 & \mathfrak{O}(X) \arrow[rr, "\mathfrak{O}(f)"] & & \mathfrak{O}(Y) \\
\mathrm{Coz}(X) \arrow[ur, "s_X"] \arrow[dr, "\downarrow_{\mathrm{Coz}(X)}"'] \arrow[rr, bend left, "\mathrm{Coz}(f)"] & & \mathrm{Coz}(Y) \arrow[ur, "s_Y"] \arrow[dr, "\downarrow_{\mathrm{Coz}(Y)}"'] & \\
 & \mathcal{I}(\mathrm{Coz}(X)) \arrow[uu, "\widetilde{s_X}"'] \arrow[rr, "\mathcal{I}(\mathrm{Coz}(f))"'] \arrow[d, "\eta^z_X"'] & & \mathcal{I}(\mathrm{Coz}(Y)) \arrow[uu, "\widetilde{s_Y}"'] \arrow[d, "\eta^z_Y"'] \\
 & \mathcal{C}_z(X) \arrow[rr, "\mathcal{C}_z(f)"'] & & \mathcal{C}_z(Y)
\end{tikzcd} \qquad (\bullet^z)$$

Thus, η^z is a natural equivalence between the functors $\mathcal{I} \cdot \mathrm{Coz}$ and $\mathcal{C}_z$.

It is time to turn to z-dimension of a $C(X)$ and its supporting space.

Definition & Remarks 4.4. The *z-dimension* of $C(X)$, denoted $\dim_z(C(X))$, is the dimension of the frame $\mathcal{C}_z(X)$ (or, equivalently, that of $\mathcal{I}(\mathrm{Coz}(X))$). We shall also speak of the z-dimension of X itself, and write it $\dim_z(X)$; note as well that, treating $\mathrm{Coz}(X)$ as a sublattice base of open sets, we have, in the notation of 3.15, that $\dim_z(X) = \dim(X, \mathrm{Coz}(X))$.

The definition implies immediately that $\dim_z(X) = \dim_z(\upsilon X)$, where υX denotes the realcompactification of X. We shall not comment further on this; the reader is referred to [GJ76, Chapter 8].

Let us now factor in the results of Section 2, specifically coupling Theorem 2.8 with Proposition 4.3, to give a spatial characterization of z-dimension.

Theorem 4.5. *Let X be a space. Then $\dim_z(X) \leq k$ if and only if for each sequence of cozerosets $U_0, U_1, \ldots, U_k$ there exist cozerosets $V_0, V_1, \ldots, V_k$ such that*

$$X = U_k \cup V_k, U_k \cap V_k \subseteq U_{k-1} \cup V_{k-1}, \ldots, U_1 \cap V_1 \subseteq U_0 \cup V_0, \quad \textit{and} \quad U_0 \cap V_0 = \emptyset.$$

Remarks 4.6. (a) It is easy to see from Theorem 4.5 that $\dim_z(X) = 0$ precisely when each cozeroset is closed. This is one of the many equivalent definitions of a P-space; see [GJ76, Theorem 14.29].

(b) In [HMW03], the authors studied the spaces for which $\dim_z(X) \leq 1$ (without using any of the machinery or terminology introduced here, and without mentioning dimension). Such spaces were called *quasi P* in [HMW03]; reciting Theorem 4.5 for the case $k = 1$, we have the following characterization of quasi P-spaces: *X is quasi P if and only if for each cozero sets U_0 and U_1 there exist cozerosets V_0 and V_1 such that $X = U_1 \cup V_1$, $U_1 \cap V_1 \subseteq U_0 \cup V_0$, with $U_0 \cap V_0 = \emptyset$.*

(c) At least one of the open questions of [HMW03] can now be answered with ease. Recall that if $\{ X_i : i \in I \}$ is a family of spaces, and X denotes the disjoint union of the X_i, then X is called the *topological union* if its topology is defined as follows: $V \in \mathfrak{O}(X)$ if and only if each $V \cap X_i \in \mathfrak{O}(X_i)$. Note that if X is the topological union of the X_i, then $V \in \mathrm{Coz}(X)$ precisely when $V \cap X_i \in \mathrm{Coz}(X_i)$; then also $C(X)$ is canonically isomorphic — as a ring and as an ℓ-group — to the direct product $\prod_{i \in I} C(X_i)$.

In [HMW03] it was asked whether the topological union of any number of quasi P spaces is quasi P. Several affirmative partial results were obtained, but the general question remained unresolved.

We can now settle the matter and get a more general theorem on z-dimension. The proof is a straightforward application of the topological union and Theorem 4.5; we leave the details to the reader.

Proposition 4.7. *Suppose that X is the topological union of the spaces X_i $(i \in I)$. Then*

$$\dim_z(X) = \sup_{i \in I} \dim_z(X_i).$$

It is time to discuss z-embeddings, in order to apply Theorem 3.13. For the literature on z-embedding we refer the reader to [Bl76], [BlH74] and [HJ61].

Definition & Remarks 4.8. Suppose that X is a space and Y is a subspace. We say that Y is *z-embedded* in X if every zeroset of Y is of the form $Z \cap Y$, for some $Z \in Z[X]$. It is clear from Lemma 3.11 that 'z-embedding' coincides with '$z(1)$-embedding' (with regard to the natural typing $(1, \mathfrak{O}, \mathrm{Coz})$).

For our purposes it is enough to recall the following specifics.

1. The reader is reminded that a space is *Lindelöf* if every open cover by a family of open sets has a countable subcover. Note that every closed subspace of a Lindelöf space is also Lindelöf, and that any Lindelöf space (here being Tychonoff and therefore regular) is normal ([En89, Theorem 3.8.2]).
2. Any Lindelöf space is z-embedded in any space containing it as a subspace; see [HJ61, 5.3].
3. In particular, if X is Lindelöf, then every closed subset — and hence any boundary $bU = X \setminus (U \cup \mathrm{int}_X(X \setminus U))$, with $U \in \mathrm{Coz}(X)$ — is z-embedded.
4. Suppose that X is Lindelöf. Note that, since every cozeroset is a countable union of closed sets, each $U \in \mathrm{Coz}(X)$ is Lindelöf (and also z-embedded).

Now we proceed to extract the consequences of the main theorem from Section 2, as interpreted by Theorem 3.13 and the remarks of 3.10. The first result also uses 3.4.3.

Proposition 4.9. *Suppose that Y is z-embedded in X. Then* $\dim_z(Y) \leq \dim_z(X)$.

Next, we have the best result on z-dimension, in the sense that the 'Lindelöf' hypothesis appears to be the most general one which insures that all the boundaries of cozerosets are optimally embedded. In the proof of this theorem we return to the notation of 3.10 and 3.12.

Theorem 4.10. *Suppose that X is a Lindelöf space. Then*

(a) *every boundary bU ($U \in \mathrm{Coz}(X)$) is optimally embedded;*
(b) *for each nonnegative integer k,* $\dim_z(X) \leq k$ *if and only if* $\dim_z(bU) \leq k-1$, *for each boundary bU ($U \in \mathrm{Coz}(X)$).*

PROOF: It should be obvious that (b) follows from (a). We proceed to prove (a).

Since X is Lindelöf, every boundary bU is z-embedded, according to 4.8.3. Thus, to finish the proof, it suffices to show that the restriction of Φ to induces a frame isomorphism from $\uparrow((\downarrow U) \vee (\downarrow U)^{\perp})$ onto $\mathcal{I}(\mathrm{Coz}(bU))$. (Note: we revert to the notation of 3.10, where $\Phi = \Phi_{bU}$.)

From 3.10(b), we have that $\Phi \cdot \Phi^* = 1_{\mathcal{I}(\mathrm{Coz}(bU))}$, and it is easy to calculate and show that Φ^* maps to the quotient $\uparrow((\downarrow U) \vee (\downarrow U)^{\perp})$. It will be enough then to show that, restricted to $\uparrow((\downarrow U) \vee (\downarrow U)^{\perp})$, $\Phi^* \cdot \Phi = 1$. Note that, by 1.3.1, we already have that $\mathcal{J} \subseteq \Phi^*(\Phi(\mathcal{J}))$, for any ideal $\mathcal{J}$ of $\mathrm{Coz}(X)$.

We will complete the proof by showing that $\Phi^*(\Phi(\mathcal{J})) \subseteq \mathcal{J}$, for each ideal $\mathcal{J} \in\uparrow((\downarrow U) \vee (\downarrow U)^{\perp})$. Suppose that $S \in \Phi^*(\Phi(\mathcal{J}))$; we leave it to the reader to verify that this means that there is a cozeroset $T \in \mathcal{J}$, such that $S \cap bU = T \cap bU$. Therefore, $(S \setminus T) \cap bU = \emptyset$, and so $S \setminus T \subseteq U \cup \mathrm{int}_X(X \setminus U)$, which, in turn, implies that

$$S \subseteq U \cup \mathrm{int}_X(X \setminus U) \cup (S \cap T).$$

Now, S is a cozeroset in a Lindelöf space, and therefore also Lindelöf. Hence, by a subcovering argument, there are countably many cozerosets $W_1, W_2, \ldots$, each disjoint from U, such that

$$S \subseteq U \cup W_1 \cup W_2 \cup \ldots \cup (S \cap T) = U \cup W \cup (S \cap T),$$

where $W = W_1 \cup W_2 \cup \ldots$, and W is also a cozeroset disjoint from U. Since $\mathcal{J} \in\uparrow((\downarrow U) \vee (\downarrow U)^{\perp})$, it follows that U and W are both in $\mathcal{J}$, whence $S \in \mathcal{J}$, and this completes the proof. $\square$

The foregoing has important consequences for compact spaces of finite z-dimension. We reserve that discussion for the next section.

5. Compact spaces of finite z-dimension

Throughout this section X will stand for a compact space, unless the contrary is expressly stated. The objective is Theorem 5.3, stating that $\dim_z(X) \leq k$ precisely when X is scattered of *CB*-index $\leq k+1$ (where $k \geq -1$ is an integer).

We begin the development leading up to Theorem 5.3 by briefly reviewing scattered spaces and, in particular, the so-called Cantor-Bendixson derivatives of a space.

Definition & Remarks 5.1. In this general commentary Y is an arbitrary Tychonoff space.

(a) Y is said to be *scattered* if each nonvoid subspace S has an isolated point of S. Many properties of scattered spaces are summarized in Z. Semadeni's memoir [Se59]; we also refer the reader to his book [Se71]. It is easy to see that if each nonempty closed subspace of Y has an isolated point, then Y is scattered.

A compact scattered space is necessarily zero-dimensional. The Stone dual is a *superatomic* boolean algebra: every homomorphic image has an atom. For the 'boolean algebra side' of scattered spaces the reader is referred to [Ko89, §17].

It is well known that if X is scattered, then so is any continuous image of X.

(b) If Y is a space let $\mathrm{Is}(Y)$ denote its set of isolated points, and let: $Y^{(0)} = Y$, $Y^{(1)} = Y \setminus \mathrm{Is}(Y)$. For any ordinal η, let $Y^{\eta+1} = (Y^{(\eta)})^{(1)}$, and if η is a limit ordinal, let

$$Y^{(\eta)} = \bigcap \{ Y^{(\xi)} : \xi < \eta \}.$$

The spaces $Y^{(\eta)}$ are called *Cantor-Bendixson derivatives of* Y. The reader will note that these derivatives form a decreasing transfinite sequence of closed subspaces of Y. From cardinality considerations there is an ordinal α such that $Y^{(\alpha)} = Y^{(\alpha+1)}$; then, in fact, $Y^{(\alpha)} = Y^{(\beta)}$, for each $\beta > \alpha$. Let $\mathrm{CB}(Y)$ denote the smallest ordinal for which $Y^{(\alpha)} = Y^{(\alpha+1)}$; this is the *CB-index* of a space Y.

Now, it is easily seen that Y is scattered if and only if $Y^{(\alpha)} = \emptyset$, for suitable α. If Y is scattered and $\mathrm{CB}(Y) = \alpha$, then α is also the least ordinal for which $Y^{(\alpha)} = \emptyset$. In particular, $\mathrm{CB}(Y) = 1$, with Y scattered, simply means that Y is a nontrivial discrete space.

Obviously, if Y is scattered, then any subspace S is also scattered, and $\mathrm{CB}(S) \leq \mathrm{CB}(Y)$.

If Y is compact, scattered, and $\alpha = \mathrm{CB}(Y)$, then it is clear that $\bigcap_{\eta<\alpha} Y^{(\eta)}$ is nonempty. It follows that α has a predecessor γ such that $Y^{(\gamma)}$ is finite and, hence, the last nonempty Cantor-Bendixson derivative. (To illustrate, $\mathrm{CB}(Y) = 1$ means that Y itself is finite and nonempty; $\mathrm{CB}(Y) = 2$ means that $Y \setminus \mathrm{Is}(Y)$ is finite, but nonvoid; and so on.)

Note that if Y is compact and scattered, then $\mathrm{CB}(Y) = 2$ if and only if Y is a finite topological sum of one-point compactifications of discrete spaces (of which at least one is infinite).

(c) It is well known that a compact space X is scattered if and only if the closed unit interval is not a continuous image of X.

(d) If X is scattered, with finite *CB*-index, then an easy induction argument establishes that each nonisolated point $p \in X$ is the limit of a sequence $p_1, p_2, \ldots$; moreover, if p is isolated in $X^{(i)}$, then p_n may be chosen so that it is isolated in $X^{(i_n)}$, with $i_n \leq i$.

To prove Theorem 5.3 we will apply Theorem 4.10. In order to accomplish that we shall need the following lemma. Let us state what is obvious in its assertion: that (b) implies (c). The implication '(c) ⇒ (a)' in the lemma was, so far as we know, first proved by Martínez and McGovern. It may, however, be part of the folklore of scattered spaces; to our knowledge, it is not published anywhere.

Lemma 5.2. *For any* (*compact*) *space X space the following are equivalent:*

(a) *X is scattered;*
(b) *for each open set O, bO is scattered;*
(c) *for each cozeroset U, bU is scattered.*

If X is scattered, then, for each nonnegative integer k, $\mathrm{CB}(X) \leq k$ *if and only if* $\mathrm{CB}(bU) \leq k-1$, *for each cozeroset U of X.*

Proof: (a) ⇒ (b) If O is any open set of X then it is clear that, since bO is nowhere dense, $bO \subseteq X^{(1)}$, and hence that $\mathrm{CB}(bO) \leq k-1$. This also proves the necessity in the second part of the lemma.

(c) ⇒ (a) Suppose that every cozeroset boundary of X is scattered. Suppose, by way of contradiction, that the closed unit interval I is a continuous image of X, and let $g : X \longrightarrow I$ be such a continuous surjection. Let C be the canonical copy of the Cantor set in I, and let $V = g^{-1}(I \backslash C)$. Then V is a cozeroset, and therefore bV is scattered. On the other hand, restricted to bV, g maps continuously onto the Cantor set, a contradiction. By 5.1(c), X is scattered, as claimed in (a).

Suppose, finally, that (c) holds, and each cozeroset boundary has *CB*-index $\leq k-1$, yet $X^{(k)} \neq \emptyset$. Let p be an isolated point of $X^{(k)}$. Let W be a compact neighborhood of p excluding all other isolated points of $X^{(k)}$. Since the standing hypotheses here also hold for W, and $W^{(k)} \neq \emptyset$, we may, without loss of generality, assume $W = X$.

Using 5.1(d), we may select a sequence $p_1, p_2, \ldots$, converging to p. The set $K = \{p_1, p_2, \ldots, p\}$ is C^*-embedded in X, and it should be clear that there is a cozeroset V of X such that $V \cap K = \{p_1, p_2, \ldots\}$ and $p \in bV$. Note that $p \in (bV)^{(k-1)}$, so that $\mathrm{CB}(bV) \geq k$, contradicting the assumptions.

This completes the proof of Lemma 5.2. □

It should be noted that Theorem 5.3 generalizes [HMW03, Theorem 4.1(II)].

Theorem 5.3. *Suppose X is a space. Then* $\dim_z(X) \leq k$ *if and only if X is scattered and* $\mathrm{CB}(X) \leq k+1$*; ($k \geq -1$ is an integer).*

Proof: For $k = -1$, both $\dim_z(X) \leq k$ and $\mathrm{CB}(X) \leq k+1$ are true precisely when the space $X = \emptyset$. Now suppose that $k \geq -1$, and the theorem holds for all compact spaces of z-dimension $\leq k$. Observe that $\dim_z(X) \leq k+1$ if and only if $\dim_z(bU) \leq k$, for each cozeroset U of X, which, by induction, is true if and only if each cozeroset boundary bU is scattered of CB-index $\leq k+1$. Finally, applying Lemma 5.2, the latter holds if and only if X itself is scattered and $\mathrm{CB}(X) \leq k+2$. □

Let us conclude this section with a number of corollaries and remarks. The first of these is a refinement of the statement of Lemma 5.2.

Remark 5.4. Suppose X is scattered, with $\mathrm{CB}(X) \leq k+1$ (with $k \geq -1$). Then there are at most finitely many cozeroset boundaries bU with CB-index *exactly* k. The reason for this is simply that $X^{(k)}$ is finite, and only the points of $X^{(k)}$ may lie in such cozeroset boundaries. There are some details to be worked out here, along the lines of the argument in the proof that (c) implies (a) in Lemma 5.2. We shall leave these details to the reader.

Thus, if X is compact and scattered and $\dim_z(X) \leq k$, then there are indeed at most finitely many cozeroset boundaries bU for which $\dim_z(bU) = k-1$.

We refer the reader to [Bl76] for amplification of the next remark.

Remark 5.5. (a) Recall that a Tychonoff space Y is *almost compact* if its Stone-Čech compactification βY is also its one-point compactification. Each almost compact space is C-embedded in any space containing it as a subspace ([GJ76, 6J]), and therefore also z-embedded. Thus, if Y is almost compact and Y is a subspace of X, with $\dim_z(X) \leq k < \infty$, then Y also has finite z-dimension.

On the other hand, $C(Y) = C(\beta Y)$ — see [GJ76, 6J] — for any almost compact space Y. Therefore, by Theorem 5.3, and since $\dim_z(Y) = \dim_z(\beta Y)$, $\dim_z(Y) < \infty$ implies that Y is scattered, with finite CB-index.

In particular, no space of finite z-dimension contains any copies of the ordinal line ω_1; note that this spaces is scattered, but its CB-index is ω_1.

(b) Note as well that $\beta\mathbb{N}$, the Stone-Čech compactification of the discrete natural numbers, has infinite z-dimension. Thus, any space containing a copy of $\beta\mathbb{N}$ also has infinite z-dimension. This includes all the compact F-spaces (see [GJ76, Theorem 14.25]), and all the compact SV-spaces of [MLMW94].

Finally, for locally compact spaces we can state the following.

Proposition 5.6. *Suppose that Y is locally compact and it has finite z-dimension. Then Y is scattered.*

Proof: Suppose that $K \subseteq X$ is closed and $p \in K$. Let C be a compact neighborhood of p in X; note that $C \cap K$ is compact and therefore C^*-embedded and, certainly, z-embedded. Thus, $\dim_z(C \cap K) < \infty$, and $C \cap K$ is a neighborhood of p in K, which must contain an isolated point of K. $\square$

On the other hand, for noncompact, locally compact spaces the relationship between z-dimension and CB-index is not clear. Witness the following two situations.

Example 5.7. In [Mr70, 1.2] there is an example of a space T which is first-countable, nonnormal, locally compact and scattered of CB-index 2; in fact, $T^{(1)} = \mathbb{N}$, with the discrete topology. This example is discussed in [HMW03] (Example 7.4), where it is pointed out that $\dim_z(T) \geq 2$ (although not in the language of z-dimension).

Indeed, it is not clear whether T has finite z-dimension. What is true is that $\dim_z(bU) = 0$, for each cozeroset boundary bU, since all such boundaries are discrete. Theorem 4.10 does not apply, as T is not Lindelöf. (If it were, then said theorem would show that $\dim_z(T) \leq 1$, which is not true.) Thus, this example does show, at least, that some assumptions are needed for Theorem 4.10.

Example 5.8. The reader is referred to the class of spaces Ψ discussed in [GJ76, 5I]. Each such space is scattered of CB-index 2, and, indeed, a union of the discrete natural numbers $\mathbb{N}$ and an uncountable discrete set D, the points of which are in one-to-one correspondence with members of a maximal almost-disjoint family of subsets of $\mathbb{N}$. Ψ is locally compact, but not Lindelöf. It is pseudocompact, and as is demonstrated in [HMW03, Example 4.4], every noncompact, pseudocompact scattered space of CB-index 2 has z-dimension at least 2. But as with the preceding example, the boundary of any cozeroset is discrete, and therefore has z-dimension 0.

It is unknown what $\dim_z(\Psi)$ is; since Ψ is pseudocompact this depends on whether $\beta\Psi$ is scattered or not; this too seems to be unknown.

6. d-dimension of $C(X)$

Throughout this section, the topological spaces are assumed to be Tychonoff, unless the contrary is specified. We examined the dimension of $C(X)$ associated with d-ideals, and its relationship to the natural typing of open sets $(\varrho, \mathfrak{RO}, \mathfrak{b}_\varrho)$ of Example 3.6(b).

We begin with an account of d-ideals which bypasses the more traditional one, given in more general setting, such as that of [HuP80a], [HuP80b], for Riesz spaces, or the frame-theoretic context of [MZ03], [M04a].

Definition & Remarks 6.1. An ideal $\mathfrak{r}$ of $C(X)$ is called a *d-ideal* if $f \in \mathfrak{r}$ and $\mathrm{coz}(g) \subseteq \mathrm{cl}_X \mathrm{coz}(f)$ imply that $g \in \mathfrak{r}$.

(a) To accentuate the parallels with z-ideals and the discussion about $\mathcal{C}_z(X)$ in 4.1, we note the following.

1. Every d-ideal is a z-ideal.
2. Each d-ideal is a convex ℓ-subgroup, and, clearly, the intersection of any family of d-ideals is a d-ideal.
3. It is established elsewhere — see, for example, [MZ03] — that the lattice $\mathcal{C}_d(X)$ of all d-ideals is a nuclear and inductive closure system in the frame $\mathcal{C}(C(X))$ of all convex ℓ-subgroups. Thus, there is an inductive nucleus d on $\mathcal{C}(C(X))$ for which the fixed family is precisely $\mathcal{C}_d(X)$.
4. For each $A \in \mathcal{C}(C(X))$,
$$dA = \{\, g \in C(X) \,:\, \operatorname{coz}(g) \subseteq \operatorname{cl}_X \operatorname{coz}(f) \text{ for some } f \in A \,\}.$$
5. The compact elements of $\mathcal{C}_d(X)$ are the ideals of the form $\langle f \rangle_d = d\{f\}$.

(b) Let A be a commutative ring with identity, and assume for our purposes that A contains no nonzero nilpotent elements. Recall that an ideal $\mathfrak{r}$ of A is an *annihilator ideal* if there is a subset $S \subseteq A$ such that

$$\mathfrak{r} = S^{\perp} \equiv \{\, a \in A \,:\, sa = 0,\ \forall\, s \in S \,\}.$$

Note that $\mathfrak{r}$ is an annihilator ideal if and only if $\mathfrak{r} = \mathfrak{r}^{\perp\perp}$. (Note: the $\perp$-notation is used here in accord with similar notation employed elsewhere in this paper in the context of 'pseudocomplementation'.) If $S = \{f\}$ we shall write $f^{\perp}$ for $S^{\perp}$; the meaning of the notation $f^{\perp\perp}$ ought to be clear.

In a ring of continuous functions $C(X)$ it is easy to see that $f \in S^{\perp}$ precisely when $\operatorname{coz}(f) \cap \operatorname{coz}(g) = \emptyset$, for each $g \in S$, and so $f \in S^{\perp\perp}$ if and only if $\operatorname{coz}(f) \subseteq \operatorname{cl}_X \operatorname{coz}(g)$, for each $g \in S$, which makes it clear that every annihilator ideal of $C(X)$ is a d-ideal.

Indeed, $\mathfrak{r} \in \mathcal{C}_d(X)$ if and only if $f \in \mathfrak{r}$ implies that $f^{\perp\perp} \subseteq \mathfrak{r}$. Note then that $\langle f \rangle_d = f^{\perp\perp}$, for all $f \in C(X)$.

(c) Finally, it ought to be noted that the inductive closure d is none other than $\widehat{(\cdot)^{\perp\perp}}$ — see 1.1.9 — applied to the frame $\mathcal{C}(C(X))$.

Next, we state, for completeness and without any commentary, the analogues of Lemma 4.2 and Proposition 4.3. Notice the reappearance of skeletal maps, as they are the l-natural maps in this categorical context. Recall that the map $f : Y \longrightarrow X$ between Tychonoff spaces is skeletal if the inverse image of every dense open set is dense.

Lemma 6.2. *Let X be a space. The map*

$$\eta^d_X(\varrho_X \operatorname{coz}(f)) = \langle f \rangle_d, \quad (f \geq 0),$$

is a lattice isomorphism from $\mathfrak{b}_\varrho(X)$ onto $\mathfrak{k}(\mathcal{C}_d(X))$.

Proposition 6.3. *Let X be a space. We have the frame isomorphism from $\mathcal{I}(\mathfrak{b}_\varrho(X))$ onto $\mathcal{C}_d(X)$ defined by extending the map η_X^d to the frame of ideals (calling the extension η_X^d as well). Moreover, we have, for any continuous skeletal function $f : Y \longrightarrow X$ between spaces the commutative diagram, in which $\mathcal{C}_d(f)(\mathfrak{r}) = \langle \{ g \cdot f \,:\, g \in \mathfrak{r} \} \rangle_d$:*

$$
\begin{tikzcd}
 & \mathfrak{RO}(X) \arrow[rr, "\mathfrak{RO}(f)"] & & \mathfrak{RO}(Y) \\
\mathfrak{b}_\varrho(X) \arrow[ur, "s_X"] \arrow[rr, bend left, "\mathfrak{b}_\varrho(f)"] \arrow[dr, "\downarrow_{\mathfrak{b}_\varrho(X)}"'] & & \mathfrak{b}_\varrho(Y) \arrow[ur, "s_Y"] \arrow[dr, "\downarrow_{\mathfrak{b}_\varrho(Y)}"'] & \\
 & \mathcal{I}(\mathfrak{b}_\varrho(X)) \arrow[uu, "\widetilde{s_X}"'] \arrow[rr, "\mathcal{I}(\mathfrak{b}_\varrho(f))"'] \arrow[d, "\eta_X^d"'] & & \mathcal{I}(\mathfrak{b}_\varrho(Y)) \arrow[uu, "\widetilde{s_Y}"'] \arrow[d, "\eta_Y^d"'] \\
 & \mathcal{C}_d(X) \arrow[rr, "\mathcal{C}_d(f)"'] & & \mathcal{C}_d(Y)
\end{tikzcd}
\qquad (\bullet^d)
$$

Thus, η^d is a natural equivalence between the functors $\mathcal{I} \cdot \mathfrak{b}_\varrho$ and $\mathcal{C}_d$.

Definition & Remarks 6.4. Let X be a space. The *d-dimension* of X (or of $C(X)$), denoted $\dim_d(X)$ (resp. $\dim_d(C(X))$) is the dimension of $\mathcal{C}_d(X) \cong \mathcal{I}(\mathfrak{b}_\varrho(X))$. As with z-dimension, one has, in terms of the usage in 3.15, that $\dim_d(X) = \dim(X, \mathfrak{b}_\varrho(X))$. As before, $\dim_d(X) = \dim_d(\upsilon X)$, where υX stands for the realcompactification of X.

As every d-ideal is a z-ideal, it follows that $\dim_d(X) \le \dim_z(X)$, for every space X. As to when these dimensions agree, let us combine Theorems 7.7 and 10.3(ii) from [HuP80b]. First recall that X is an *almost P-space* it has no proper dense cozerosets. The following are equivalent:

- (a) X is an almost P-space;
- (b) every cozeroset of X is regular open;
- (c) every z-ideal of $C(X)$ is a d-ideal;
- (d) every maximal ideal of $C(X)$ is a d-ideal.

Evidently, then if X is almost P, it follows that $\dim_d(X) = \dim_z(X)$.

The converse is false: let X be a disjoint union of a copy of $\alpha\mathbb{N}$ and αD, where D is an uncountable discrete space. By Proposition 4.7 (above) and Corollary 6.7 (below), $\dim_d(X) = \dim_z(X) = 1$. However, X is not almost P.

Remarks 6.5. It is easily seen that $\dim_d(X) = 0$ precisely when, for each cozeroset U of X, there is a cozeroset V of X such that $U \cap V = \emptyset$ and $U \cup V$ is dense. Such spaces are called *cozerocomplemented*; they have been extensively studied, and most recently by Henriksen and Woods ([HW04]). It is well known, that X is cozerocomplemented if and only if the space of minimal prime ideals of $C(X)$ is compact in the induced hull-kernel topology. This class of spaces includes all metric spaces and all spaces with the countable chain condition.

The one-point compactification αD of an uncountable discrete space D is not cozerocomplemented; indeed, since $\dim_z(\alpha D) = 1$, as the space is scattered with *CB*-index 2, $\dim_d(\alpha D) = 1$ as well. As $\beta\mathbb{N}$ is cozerocomplemented, but its z-dimension is infinite, one readily realizes that z-dimension and d-dimension can be quite different.

Next, we state the analogue of Theorem 4.5 for d-dimension, letting the reader sort out the details as an exercise. Suffice it to observe about the operator ϱ_X that, for any two open sets U and V of X, we have $\varrho_X(U \cap V) = \varrho_X U \cap \varrho_X V$, while $\varrho_X(U \cup V) = \varrho_X U \vee \varrho_X U$, where $\vee$ denotes the supremum in $\mathfrak{RO}(X)$.

Theorem 6.6. *Let X be a space. Then $\dim_d(X) \leq k$ if and only if for each sequence of cozerosets $U_0, U_1, \dots, U_k$ there exist cozerosets $V_0, V_1, \dots, V_k$ such that $U_k \cup V_k$ is dense in X, U_0 and V_0 are disjoint, and*

$$U_k \cap V_k \subseteq \mathrm{cl}_X(U_{k-1} \cup V_{k-1}), \dots, U_1 \cap V_1 \subseteq \mathrm{cl}_X(U_0 \cup V_0).$$

As a corollary we now have the analogue of Proposition 4.7. It follows easily, as in topological unions closure and interior are taken componentwise.

Corollary 6.7. *Suppose that X is the topological union of the spaces X_i $(i \in I)$. Then*

$$\dim_d(X) = \sup_{i \in I} \dim_d(X_i).$$

References

[AB91] Adams M.E., Beazer R., *Congruence properties of distributive double p-algebras*, Czechoslovak Math. J. **41** (1991), 395–404.

[BP04] Ball R.N., Pultr A., *Forbidden forests in Priestley spaces*, Cah. Topol. Géom. Différ. Catég. **45** (2004), no. 1, 2–22.

[BKW77] Bigard A., Keimel K., Wolfenstein S., *Groupes et anneaux réticulés*, Lecture Notes in Mathematics **608**, Springer, Berlin-Heidelberg-New York, 1977.

[Bl76] Blair R.L., *Spaces in which special sets are z-embedded*, Canad. J. Math. **28** (1976), 673–690.

[BlH74] Blair R.L., Hager A.W., *Extensions of zerosets and of real valued functions*, Math. Z. **136** (1974), 41–57.

[CL02] Coquand Th., Lombardi H., *Hidden constructions in abstract algebra: Krull dimension of distributive lattices and commutative rings*, Commutative Ring Theory and Applications (M. Fontana, S.-E. Kabbaj, S. Wiegand, Eds.), pp. 477–499; Lecture Notes in Pure and Appl. Math., 231, Marcel Dekker, New York, 2003.

[CLR03] Coquand Th., Lombardi H., Roy M.-F., *Une caractérisation élémentaire de la dimension de Krull*, preprint.

[D95] Darnel M., *Theory of Lattice-Ordered Groups*, Marcel Dekker, New York, 1995.

[En89] Engelking R., *General Topology*, Sigma Series in Pure Math. **6**, Heldermann, Berlin, 1989.

[Es98] Escardó M.H., *Properly injective spaces and function spaces*, Topology Appl. **89** (1998), 75–120.

[GJ76] Gillman L., Jerison M., *Rings of Continuous Functions*, Graduate Texts in Mathematics **43**, Springer, Berlin-Heidelberg-New York, 1976.

[HJ61] Henriksen M., Johnson D.G., *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. **50** (1961), 73–94.

[MLMW94] Henriksen M., Larson S., Martínez J., Woods R.G., *Lattice-ordered algebras that are subdirect products of valuation domains Trans. Amer. Math. Soc.* **345** (1994), no. 1, 195–221.

[HMW03] Henriksen M., Martínez J., Woods R.G., *Spaces X in which all prime z-ideals of $C(X)$ are either minimal or maximal*, Comment. Math. Univ. Carolinae **44** (2003), no. 2, 261–294.

[HW04] Henriksen M., Woods R.G., *Cozero complemented spaces: when the space of minimal prime ideals of a $C(X)$ is compact*, Topology Appl. **141** (2004), 147–170.

[HuP80a] Huijsmans C.B., de Pagter B., *On z-ideals and d-ideals in Riesz spaces, I*, Indag. Math. **42** (1980), no. 2, 183–195.

[HuP80b] Huijsmans C.B., de Pagter B., *On z-ideals and d-ideals in Riesz spaces, II*, Indag. Math. 42 (1980), no. 4, 391–408.

[J82] Johnstone P.J., *Stone Spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge Univ. Press, Cambridge, 1982.

[JT84] Joyal A., Tierney M., *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. **51** (1984), no. 309, 71 pp.

[Ko89] Koppelberg S., *Handbook of Boolean Algebras, I*, J.D. Monk, Ed., with R. Bonnet; North Holland, Amsterdam-New York-Oxford-Tokyo, 1989.

[M73a] Martínez J., *Archimedean lattices*, Algebra Universalis **3** (1973), 247–260.

[M04a] Martínez J., *Dimension in algebraic frames*, Czechoslovak Math. J., to appear.

[M04b] Martínez J., *Unit and kernel systems in algebraic frames*, Algebra Universalis, to appear.

[MZ03] Martínez J., Zenk E.R., *When an algebraic frame is regular*, Algebra Universalis **50** (2003), 231–257.

[MZ06] Martínez J., Zenk E.R., *Dimension in algebraic frames, III: dimension theories*, in preparation.

[Mr70] Mrowka S., *Some comments on the author's example of a non-R-compact space*, Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys. **18** (1970), 443–448.

[Se59] Semadeni Z., *Sur les ensembles clairsemés*, Rozprawy Mat. **19** (1959), 39 pp.

[Se71] Semadeni Z., *Banach Spaces of Continuous Functions*, Polish Scientific Publishers, Warsaw, 1971.

Department of Mathematics, University of Florida, P.O. Box 118105, Gainesville, FL 32611-8105, USA

E-mail: jmartine@math.ufl.edu

Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA

E-mail: eric.zenk@Vanderbilt.Edu

(*Received* November 2, 2004, *revised* September 20, 2005)