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A construction of a connection on $GY \to Y$ from a connection on $Y \to M$ by means of classical linear connections on M and Y

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Abstract. Let G be a bundle functor of order (r, s, q), $s \geq r \leq q$, on the category $\mathcal{FM}_{m,n}$ of (m, n)-dimensional fibered manifolds and local fibered diffeomorphisms. Given a general connection Γ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ we construct a general connection $\mathcal{G}(\Gamma, \lambda, \Lambda)$ on $GY \to Y$ be means of an auxiliary q-th order linear connection λ on M and an s-th order linear connection Λ on Y. Then we construct a general connection $\mathcal{G}(\Gamma, \nabla_1, \nabla_2)$ on $GY \to Y$ by means of auxiliary classical linear connections ∇_1 on M and ∇_2 on Y. In the case $G = J^1$ we determine all general connections $\mathcal{D}(\Gamma, \nabla)$ on $J^1Y \to Y$ from general connections Γ on $Y \to M$ by means of torsion free projectable classical linear connections ∇ on Y.

Keywords: general connection, classical linear connection, bundle functor, natural operator

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0. Introduction

Let $Y \to M$ be a fibered manifold. A general r-th order connection on $Y \to M$ is a section $\Gamma : Y \to J^rY$ of the r-th jet prolongation $J^rY \to Y$ of $Y \to M$. If $r = 1$, we call $\Gamma : Y \to J^1Y$ a general connection on $Y \to M$. If $r = 1$, a general connection $\Gamma : Y \to J^1Y$ can be considered as the corresponding lifting map (denoted by the same letter)

$$\Gamma : Y \times_M TM \to TY.$$

If $Y = E$ is a vector bundle, a general r-th order connection $\Gamma : E \to J^rE$ is called linear if it is a vector bundle map. If $Y = E = TM$ is the tangent bundle, a linear r-th order connection $\Gamma : TM \to J^rTM$ on $TM \to M$ is called a linear r-th order connection on M. A classical linear connection on M is a first order linear connection $\nabla : TM \to J^1TM$ on M (this definition is equivalent to the usual one by the covariant derivative), see [4].

An interesting class of geometric problems is: given a general connection on a fibered manifold, how to prolong it to a fibered manifold derived in a certain way from the original one. These problems have motivations in higher order dynamics and quantum mechanics, see [8] and [2].

Let G be a bundle functor in the sense of [4] on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibres and their fibered local diffeomorphisms of order (r,s,q), $s \geq r \leq q$. (A very important case is $G = J^r$.) Consider a general connection $\Gamma : Y \to J^1 Y$ on a fibered manifold $Y \to M$ from $\mathcal{FM}_{m,n}$.

In [3], I. Kolář constructed an induced connection

$$\mathcal{G}(\Gamma,\lambda) : GY \to J^1(GY \to M)$$

on $GY \to M$ by means of an auxiliary q-th order linear connection $\lambda : TM \to J^q TM$ on M.

In [7], using the exponential mapping of a classical linear connection ∇ on M, we construct a q-th order linear connection $\lambda^q_\nabla : TM \to J^q TM$ on M, see Section 1 of the present note. Thus we have the induced connection

$$\mathcal{G}(\Gamma,\nabla) := \mathcal{G}(\Gamma,\lambda^q_\nabla) : GY \to J^1(GY \to M)$$

on $GY \to M$, see Section 2 of the present note.

In [1], we remarked that the use of the auxiliary connection ∇ on M to obtain an induced (by Γ) connection on $GY \to M$ is unavoidable if $q \geq 1$ (for example for $G = J^r$), see Section 4 of the present note.

The purpose of this note is to construct a general connection $\mathcal{G}(\Gamma,\nabla_1,\nabla_2)$ on $GY \to Y$ by means of a classical linear connection ∇_1 on M and a classical linear connection ∇_2 on Y. We present such a construction in Section 3. First, we construct an induced connection

$$\mathcal{G}(\Gamma,\lambda,\Lambda) : GY \to J^1(GY \to Y)$$

on $GY \to Y$ by means of a linear q-th order connection $\lambda : TM \to J^q TM$ on M and an s-th order linear connection $\Lambda : TY \to J^s TY$ on Y (we will apply the connection $\mathcal{G}(\Gamma,\lambda)$ on $GY \to M$). Then we put

$$\mathcal{G}(\Gamma,\nabla_1,\nabla_2) = \mathcal{G}(\Gamma,\lambda^q_{\nabla_1},\lambda^s_{\nabla_2}) : GY \to J^1(GY \to Y).$$

In Section 4, we present some natural properties of the construction of $\mathcal{G}$.

In Section 5, we remark that the use of ∇_2 to obtain a geometric construction of an induced (by Γ and ∇_1) connection on $GY \to Y$ is unavoidable for a very large class of such G (for example for $G = J^r$).

In Section 6, we construct a general connection $\mathcal{G}(\nabla)$ on $GY \to Y$ from a projectable classical linear connection ∇ on $p : Y \to M$ (a classical linear connection

∇ on Y is projectable if there is a unique p-related to ∇ classical linear connection $\underline{\nabla}$ on M).

In the last section, in the case $G = J^1$ we determine all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D}$ corresponding to constructions of general connections $\mathcal{D}(\Gamma, \nabla)$ on $J^1Y \to Y$ from general connections Γ on $p : Y \to M$ by means of torsion free projectable classical linear connections ∇ on $Y \to M$.

All manifolds and maps are assumed to be smooth (of class $\mathcal{C}^\infty$).

1. Exponential extension of classical linear connections

In [7], we observed that a classical linear connection ∇ on a manifold P induces a linear p-th order connection

$$\lambda^p_\nabla : TP \to J^pTP$$

on P by

$$\lambda^p_\nabla(v) = j^p_x((\mathrm{Exp}^\nabla_x)_*\tilde{v}),\ v \in T_xP,\ x \in P, \tag{1}$$

where $\tilde{v} = \frac{d}{dt}_0(. + tv)$ is the constant vector field on T_xP determined by v, and $\mathrm{Exp}^\nabla_x : T_xP \supset U_{0_x} \to \tilde{U}_x \subset P$ is the exponential map (exponent) of ∇.

2. Lifting of a general connection on $Y \to M$ and a classical linear connection on M into a general connection on $GY \to M$

We recall that the definition of the order of a bundle functor G on the category $\mathcal{FM}_{m,n}$ is based on the concept of (r, s, q)-jets, $s \geq r \leq q$. Given two fibered manifolds $p : Y \to M$, $\overline{p} : \overline{Y} \to \overline{M}$, we say that two $\mathcal{FM}_{m,n}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same (r, s, q)-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y$, $p(y) = x$, if

$$j^r_y f = j^r_y g,\ j^s_y(f \,|\, Y_x) = j^s_y(g \,|\, Y_x),\ j^q_x \underline{f} = j^q_x \underline{g}\,.$$

The space of all such (r, s, q)-jets will be denoted by $J^{r,s,q}(Y, \overline{Y})$. Further, a bundle functor G on $\mathcal{FM}_{m,n}$ is said to be of order (r, s, q), if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf \,|\, G_yY = Gg \,|\, G_yY$, [4]. In this case the integer q is called the base order of G.

A projectable vector field on a fibered manifold $Y \to M$ is a fibered manifold morphism $Z : Y \to TY$ over an underlying vector field $M \to TM$. Its flow $\exp tZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Given a bundle functor G on $\mathcal{FM}_{m,n}$, the flow prolongation of Z with respect to G is a vector field $\mathcal{G}Z : GY \to TGY$ defined by $\mathcal{G}Z = \frac{\partial}{\partial t}_0 G(\exp tZ)$. By [5], if G has order (r, s, q), then the value of

$\mathcal{G}Z$ at a point of G_yY depends on $j^{r,s,q}_y Z$ only. Thus the flow prolongation of a projectable vector field can be interpreted as a map

$$\mathcal{G}_Y : GY \times_Y J^{r,s,q}TY \to TGY, \tag{2}$$

where $J^{r,s,q}TY$ denotes the vector bundle of all (r,s,q)-jets of projectable vector fields on Y. By [4], $\mathcal{G}_Y$ is linear in the second factor.

Consider a general connection $\Gamma : Y \to J^1Y$ on a fibered manifold $Y \to M$. We first recall the geometric construction of an induced connection $\mathcal{G}(\Gamma, \lambda)$ on $GY \to M$ by means of a linear q-th order connection $\lambda : TM \to J^qTM$ on M. Given a vector field X on M, its Γ-lift is a projectable vector field $\Gamma X : Y \to TY$. It is easy to see that $j^{r,s,q}_y \Gamma X$ is determined by $j^q_{p(y)}X$. Then by (2), the flow prolongation $\mathcal{G}(\Gamma X) : GY \to TGY$ can be interpreted as a map

$$\mathcal{G}\Gamma : GY \times_M J^qTM \to TGY, \tag{3}$$

which is linear in the second factor. By linearity, the composition

$$\mathcal{G}(\Gamma, \lambda) := \mathcal{G}\Gamma \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_M} \lambda) : GY \times_M TM \to TGY \tag{4}$$

is the lifting map of a general connection (denoted by the same symbol) on $GY \to M$. This construction of $\mathcal{G}(\Gamma, \lambda)$ was done in [3] by I. Kolář by using the concept of (r,q)-order of G.

Finally we recall the geometric construction of a connection $\mathcal{G}(\Gamma, \nabla)$ on $GY \to M$ by means of a classical linear connection ∇ on M. By Section 1, classical linear connection ∇ induces the linear q-th order connection $\lambda^q_\nabla : TM \to J^qTM$. Then

$$\mathcal{G}(\Gamma, \nabla) := \mathcal{G}(\Gamma, \lambda^q_\nabla) : GY \times_M TM \to TGY \tag{5}$$

is the lifting map of a general connection (denoted by the same symbol) on $GY \to M$.

3. Lifting of a general connection on $Y \to M$, a classical linear connection on M and a classical linear connection on Y into a general connection on $GY \to Y$

Let G be a bundle functor on $\mathcal{FM}_{m,n}$ of order (r,s,q), $s \geq r \leq q$. Consider a general connection $\Gamma : Y \to J^1Y$ on a fibered manifold $Y \to M$. We first introduce the geometric construction of an induced connection $\mathcal{G}(\Gamma, \lambda, \Lambda)$ on $GY \to Y$ by means of a linear q-th order connection $\lambda : TM \to J^qTM$ on M and an s-th order linear connection $\Lambda : TY \to J^sTY$ on Y.

A vertical vector field on a fibered manifold $Y \to M$ is a projectable vector field $Z : Y \to TY$ over the underlying zero vector field $0 : M \to TM$. Clearly, for vertical Z as above $j_y^{r,s,q}Z$ is determined by $j_y^s Z$. Then $\mathcal{G}_Y$ from (2) can be "restricted and then lifted" to the map

$$\mathcal{G}_Y^V : GY \times_Y J^s VY \to VGY \tag{6}$$

covering the identity of VY and linear in the second factor, where $J^s VY$ denotes the vector bundle of all s-jets of vertical vector fields on Y.

The vertical projection $p_Y^\Gamma : TY = VY \oplus_Y H^\Gamma \to VY$ of Γ sends any vector field X on Y into the vertical vector field $p_Y^\Gamma \circ X$, where $H^\Gamma \subset TY$ is the horizontal distribution of Γ. Clearly, $j_y^s(p_Y^\Gamma \circ X)$ is determined by $j_y^s X$. This defines the vector bundle map

$$(P_Y^\Gamma)^s : J^s TY \to J^s VY. \tag{7}$$

So, using an s-th order linear connection $\Lambda : TY \to J^s TY$ we produce

$$\tilde{\mathcal{G}}(\Gamma, \Lambda) := \mathcal{G}_Y^V \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_Y} ((P_Y^\Gamma)^s \circ (\Lambda \,|\, VY))) : GY \times_Y VY \to VGY. \tag{8}$$

Given a q-th order linear connection $\lambda : TM \to J^q TM$ we have the connection $\mathcal{G}(\Gamma, \lambda)$ on $Y \to M$, see (4).

We define a linear in the second factor map

$$\mathcal{G}(\Gamma, \lambda, \Lambda) : GY \times_Y TY \to TGY$$

by

$$\begin{aligned} &\mathcal{G}(\Gamma, \lambda, \Lambda) \,|\, GY \times_Y VY := \tilde{\mathcal{G}}(\Gamma, \Lambda), \\ &\mathcal{G}(\Gamma, \lambda, \Lambda) \,|\, GY \times_Y H^\Gamma := \mathcal{G}(\Gamma, \lambda) \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_Y} (Tp \,|\, H^\Gamma)) \\ &\qquad - \tilde{\mathcal{G}}(\Gamma, \Lambda) \circ (\mathrm{pr}_{GY}, p_Y^\Gamma \circ T\pi \circ \mathcal{G}(\Gamma, \lambda) \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_Y} Tp \,|\, H^\Gamma)), \end{aligned} \tag{9}$$

where $Tp : TY \to TM$ is the tangent map of the fibered manifold $p : Y \to M$, $T\pi : TGY \to TY$ is the tangent map of the bundle projection $\pi : GY \to Y$, $p_Y^\Gamma : TY \to VY$ is the (mentioned above) vertical projection of Γ, $H^\Gamma \subset TY$ is the horizontal distribution of Γ and $\mathrm{pr}_{GY} : GY \times_Y H^\Gamma \to GY$ is the canonical projection.

Proposition 1. *The map* (9) *is the lifting map of a general connection* (*denoted by the same symbol*) *on $GY \to Y$.*

Proof: It is easy to see that $T\pi \circ \mathcal{G}(\Gamma, \lambda, \Lambda)(y, v) = v$ for any $(y, v) \in GY \times_Y TY$, where $\pi : GY \to Y$ is the bundle projection. □

Finally we recall the geometric construction of a connection $\mathcal{G}(\Gamma, \nabla_1, \nabla_2)$ on $GY \to Y$ be means of classical linear connections ∇_1 on M and ∇_2 on Y. By Section 1, we have the induced q-th order linear connection $\lambda^q_{\nabla_1} : TM \to J^q TM$ on M and the induced s-th order linear connection $\lambda^s_{\nabla_2} : TY \to J^s TY$ on Y. Then

$$\mathcal{G}(\Gamma, \nabla_1, \nabla_2) := \mathcal{G}(\Gamma, \lambda^q_{\nabla_1}, \lambda^s_{\nabla_2}) : GY \times_Y TY \to TGY \tag{10}$$

is a lifting map of a general connection (denoted by the same symbol) on $GY \to Y$.

4. Some natural properties of $\mathcal{G}$

Let G be a bundle functor on the category $\mathcal{FM}_{m,n}$ (or on the category $\mathcal{FM}_m$ of all fibered manifolds with m-dimensional bases and fibered maps covering local diffeomorphisms, or on the category $\mathcal{FM}$ of all fibered manifolds and their fibered maps). Let $\mathcal{G}$ be the construction by (10). Because of the canonical character of the construction of $\mathcal{G}$ we have

Proposition 2. *Let $Y \to M$ and $\overline{Y} \to \overline{M}$ be two fibered manifolds from $\mathcal{FM}_{m,n}$ (or from $\mathcal{FM}_m$, or from $\mathcal{FM}$) and $f : Y \to \overline{Y}$ be an $\mathcal{FM}_{m,n}$-map (or $\mathcal{FM}_m$-map, or $\mathcal{FM}$-map) covering $\underline{f} : M \to \overline{M}$. Let $\Gamma : Y \to J^1Y$ and $\overline{\Gamma} : \overline{Y} \to J^1\overline{Y}$ be general connections on $Y \to M$ and $\overline{Y} \to \overline{M}$ respectively, ∇_1 and $\overline{\nabla}_1$ be classical linear connections on M and $\overline{M}$ respectively, and ∇_2 and $\overline{\nabla}_2$ be classical linear connections on Y and $\overline{Y}$ respectively. If Γ and $\overline{\Gamma}$ are $(f, \underline{f})$-related, ∇_1 and $\overline{\nabla}_1$ are $\underline{f}$-related and ∇_2 and $\overline{\nabla}_2$ are f-related, then $\mathcal{G}(\Gamma, \nabla_1, \nabla_2)$ and $\mathcal{G}(\overline{\Gamma}, \overline{\nabla}_1, \overline{\nabla}_2)$ are (Gf, f)-related.*

In other words the rule "$(\Gamma, \nabla_1, \nabla_2) \to \mathcal{G}(\Gamma, \nabla_1, \nabla_2)$" is an $\mathcal{FM}_{m,n}$-natural (or $\mathcal{FM}_m$-natural, or $\mathcal{FM}$-natural) operator in the sense of [4].

Let $\mu : G_1 \to G_2$ be a natural transformation of bundle functors on $\mathcal{FM}_{m,n}$. This means that for any $Y \to M$ from $\mathcal{FM}_{m,n}$ we have a fibered map $\mu_Y : G_1Y \to G_2Y$ covering id_Y such that $G_2f \circ \mu_Y = \mu_{\overline{Y}} \circ G_2f$ for any $\mathcal{FM}_{m,n}$-map $f : Y \to \overline{Y}$. Because of the canonical character of the construction by (10) we have

Proposition 3. *Given a general connection Γ on $Y \to M$ and classical linear connections ∇_1 and ∇_2 on M and Y respectively, the general connections $\mathcal{G}^1(\Gamma, \nabla_1, \nabla_2)$ and $\mathcal{G}^2(\Gamma, \nabla_1, \nabla_2)$ (defined by (10) for G^1 and G^2 playing the role of G) are (μ_Y, id_Y)-related.*

In particular, for $G^1 = J^r$ and $G^2 = J^k$ and $\mu = \pi^r_k : J^r \to J^k$, the jet projection, $\mathcal{J}^r(\Gamma, \nabla_1, \nabla_2)$ and $\mathcal{J}^k(\Gamma, \nabla_1, \nabla_2)$ are (π^r_k, id_Y)-related.

Quite similar properties to those of Propositions 2 and 3 hold for the construction $\mathcal{G}(\Gamma, \lambda, \Lambda)$ given by (9) instead of $\mathcal{G}(\Gamma, \nabla_1, \nabla_2)$, and for the constructions $\mathcal{G}(\Gamma, \lambda)$ and $\mathcal{G}(\Gamma, \nabla)$ given by (4) and (5), see also [3].

5. Remarks

Let G be a bundle functor on $\mathcal{FM}_{m,n}$ of order (r, s, q), $s \geq r \leq q$. Consider a general connection $\Gamma : Y \to J^1Y$ on a fibered manifold $Y \to M$.

Remark 1. In [1], we studied the existence problem of geometric construction of an induced connection $\mathcal{D}(\Gamma)$ on $GY \to M$ from Γ. We proved that such a construction exists if and only if $q = 0$. For example, the r-jet prolongation functor J^r on $\mathcal{FM}_{m,n}$ is with (minimal) $q = r \geq 1$. So, the use of an auxiliary classical linear connection ∇ on M to obtain an induced (by Γ) connection on $J^rY \to M$ is unavoidable.

Remark 2. In [6], we studied the existence problem of geometric construction of an induced connection on $GY \to Y$ from Γ by means of some type of geometric objects on M. In particular, we can easily obtain that if there exists a construction of an induced connection $\mathcal{D}(\Gamma, \nabla_1)$ from Γ by means a classical linear connection ∇_1 on M, then the natural bundle G^2 on $\mathcal{M}f_n$ given by $G^2N = G(\mathbb{R}^m \times N)$ and $G^2\psi = G(\mathrm{id}_{\mathbb{R}^m} \times \psi)$ is of order 0. For example, the r-jet prolongation functor J^r on $\mathcal{FM}_{m,n}$ is with $(J^r)^2$ of (minimal) order $r \geq 1$. So, the use of an auxiliary classical linear connection ∇_2 on Y to obtain an induced (by Γ and ∇_1) connection on $J^rY \to Y$ is unavoidable.

Open problem: In [2], J. Janyška and M. Modugno constructed a general connection $\chi(\nabla)$ on $J^1Y \to Y$ by means of a classical linear connection ∇ on Y. Does there exist for every bundle functor G on $\mathcal{FM}_{m,n}$ a construction of a general connection $\mathcal{D}(\Gamma, \nabla)$ on $GY \to Y$ from a general connection Γ on $Y \to M$ by means of a classical linear connection ∇ on Y?

6. Construction of a general connection $\mathcal{G}(\nabla)$ on $GY \to Y$ from a projectable classical linear connection ∇ on $Y \to M$

Let ∇ be a projectable classical linear connection on $p^Y : Y \to M$. Then $\mathrm{Exp}^{\nabla}_y : V_{\{0_y\}} \subset T_yY \to U_y \subset Y$ is an $\mathcal{FM}_{m,n}$-map between fibered manifolds $T_yp^Y \to T_{p^Y(y)}M$ and $p^Y : Y \to M$. Then we can define $\lambda^p_\nabla : TY \to J^{p,p,p}TY$ by the formula (1) (we see that $\tilde{v}$ is projectable on $T_yY \to T_{p^P(y)}M$).

Let G be a bundle functor on $\mathcal{FM}_{m,n}$ of order (p, p, p). Define a general connection $\mathcal{G}(\nabla) : GY \times_Y TY \to TGY$ on $GY \to Y$ by

$$\mathcal{G}(\nabla)(v, w) = \mathcal{G}X(v) \in T_vGY, \tag{11}$$

where X is a projectable vector field on Y such that $j^{p,p,p}_y(X) = \lambda^p_\nabla(w)$, $v \in G_yY$, $w \in T_yY$, $y \in Y$ and $\mathcal{G}X$ is the flow lifting of X to G.

7. The case $G = J^1$

The classification of $\mathcal{FM}_{m,n}$-natural operators (constructions) $\mathcal{D}$ transforming general connections Γ on $Y \to M$ and classical torsion free linear connections ∇ on M into general connections $\mathcal{D}(\Gamma, \nabla)$ on $J^1Y \to M$ can be found in [4].

In [2], the authors presented a complete description of all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D}$ transforming a classical linear connection ∇ on Y into general connections on $J^1Y \to Y$.

In this section we present a complete description of $\mathcal{FM}_{m,n}$-natural operators (constructions) $\mathcal{D}$ transforming general connections Γ on $Y \to M$ and projectable torsion free classical linear connections ∇ on Y covering $\underline{\nabla}$ on M into general connections $\mathcal{D}(\Gamma,\nabla)$ on $J^1Y\to Y$.

To describe all operators in question we need long preparations.

From now on $\mathbb{R}^{m,n}$ denotes the trivial bundle $\mathbb{R}^m\times\mathbb{R}^n$ over $\mathbb{R}^m$. Let $x^1,\dots,x^m$, $y^1,\dots,y^n$ be the usual coordinates in $\mathbb{R}^{m,n}$.

Let

$$\begin{aligned}A :&((\mathbb{R}^m)^*\otimes\mathbb{R}^n)\times((\mathbb{R}^n)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)\times\\ &\times((\mathbb{R}^m)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)\to((\mathbb{R}^m)^*\times(\mathbb{R}^n)^*)\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n\end{aligned}\tag{12}$$

be a $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant map between the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-spaces with the usual actions. (See Remark 3 for the classification of A.)

Clearly A can be (in an obvious way) considered as the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant map

$$A : J^1_{(0,0)}(C(\mathbb{R}^{m,n}))\to T^*_{(0,0)}\mathbb{R}^{m,n}\otimes T^*_0\mathbb{R}^m\otimes V_{(0,0)}\mathbb{R}^{m,n}\tag{13}$$

between the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-spaces with the usual actions, where

$$J^1_{(0,0)}(C(\mathbb{R}^{m,n}))=\{j^1_{(0,0)}\Gamma\,|\,\Gamma\text{ is a general connection on }\mathbb{R}^{m,n}\}.$$

Here we use the obvious $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant identification

$$T^*_{(0,0)}(\mathbb{R}^{m,n})\otimes T^*_0\mathbb{R}^m\otimes V_{(0,0)}\mathbb{R}^{m,n}=((\mathbb{R}^m)^*\times(\mathbb{R}^n)^*)\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n$$

and the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant identification

$$((\mathbb{R}^m)^*\otimes\mathbb{R}^n)\times((\mathbb{R}^n)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)\times((\mathbb{R}^m)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)=J^1_{(0,0)}(C(\mathbb{R}^{m,n}))$$

given by

$$\begin{aligned}((a^j_i),(b^j_{\tilde{j}i}),(c^j_{\tilde{i}i}))\to j^1_{(0,0)}\Big(&\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}+\sum_{i=1}^m\sum_{j=1}^n a^j_i dx^i\otimes\frac{\partial}{\partial y^j}\\ &+\sum_{i=1}^m\sum_{\tilde{j},j=1}^n b^j_{\tilde{j}i}y^{\tilde{j}}dx^i\otimes\frac{\partial}{\partial y^j}+\sum_{i,\tilde{i}=1}^m\sum_{j=1}^n c^j_{\tilde{i}i}x^{\tilde{i}}dx^i\otimes\frac{\partial}{\partial y^j}\Big).\end{aligned}$$

Next, define

$$\overline{A} : J^1_{(0,0)}(C(\mathbb{R}^{m,n}))\times(J^1_0\mathbb{R}^{m,n})_0\to T^*_{(0,0)}\mathbb{R}^{m,n}\otimes T^*_0\mathbb{R}^m\otimes V_{(0,0)}\mathbb{R}^{m,n}$$

by

$$\overline{A}(j^1_{(0,0)}\Gamma, j^1_0(\mathrm{id}_{\mathbb{R}^m},\sigma))=B^{-1}.A(B.j^1_{(0,0)}\Gamma),\tag{14}$$

where $B\in GL(\mathbb{R}^{m,n})$ is a corresponding to $j^1_0(\mathrm{id}_{\mathbb{R}^m},\sigma)$ map given by $B(x,y)=(x,y-\tilde{\sigma}(x))$, $\tilde{\sigma}:\mathbb{R}^m\to\mathbb{R}^n$ is the linear map such that $j^1_0\sigma=j^1_0\tilde{\sigma}$, and by the dots we denote the actions of $GL(\mathbb{R}^{m,n})$.

Because of the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariance of A we obtain that

Lemma 1. *$\overline{A}$ is $GL(\mathbb{R}^{m,n})$-equivariant.*

Proof: Let $C \in GL(\mathbb{R}^{m,n})$. It is of the form $C(x,y) = (\underline{C}(x), C_1(x)+C_2(y))$. Let $j^1_0(\mathrm{id}_{\mathbb{R}^m}, \sigma) \in (J^1_0\mathbb{R}^{m,n})_0$ and $j^1_{(0,0)}\Gamma \in J^1_{(0,0)}(C(\mathbb{R}^{m,n}))$. Let $B \in GL(\mathbb{R}^{m,n})$ correspond to $j^1_0(\mathrm{id}_{\mathbb{R}^m}, \sigma)$ and $B_1 \in GL(\mathbb{R}^{m,n})$ correspond to $C.j^1_0(\mathrm{id}_{\mathbb{R}^m}, \sigma)$. Then $B_1 \circ C \circ B^{-1} \in GL(\mathbb{R}^m) \times GL(\mathbb{R}^n)$. (For, one can compute $B_1 \circ C \circ B^{-1}(x,y) = (\underline{C}(x), C_2(y))$.) Then

$$
\begin{aligned}
&\overline{A}(C.j^1_{(0,0)}\Gamma)(C.j^1_0(\mathrm{id}_{\mathbb{R}^m}, \sigma)) = B_1^{-1}.A(B_1.(C.j^1_{(0,0)}\Gamma))\\
&= B_1^{-1}.A((B_1 \circ C \circ B^{-1}).(B.j^1_{(0,0)}\Gamma)) =^*= B_1^{-1}.((B_1 \circ C \circ B^{-1}).A(B.j^1_{(0,0)}\Gamma))\\
&= C.(B^{-1}.A(B.j^1_{(0,0)}\Gamma)) = C.\overline{A}(j^1_{(0,0)}\Gamma)(j^1_0(\mathrm{id}_{\mathbb{R}^m}, \sigma)),
\end{aligned}
$$

where the equality $=^*=$ is by the $GL(\mathbb{R}^m) \times GL(\mathbb{R}^n)$-equivariance of A. □

Let Γ be a general connection on $p : Y \to M$ and ∇ be a projectable classical linear connection on Y. It means that there is a unique p-related with ∇ classical linear connection $\underline{\nabla}$ on M.

Define

$$\Delta^A(\Gamma, \nabla) : J^1Y \to T^*Y \otimes T^*M \otimes VY$$

as follows. Let $v \in (J^1Y)_y$, $y \in Y$. Let Ψ be a normal coordinate system on Y of ∇ with centrum y, $\Psi(y) = (0,0)$. Then Ψ is an $\mathcal{FM}_{m,n}$-map covering $\underline{\Psi}$. We put

$$\Delta^A(\Gamma, \nabla)(y) = T^*\Psi^{-1} \otimes T^*\underline{\Psi}^{-1} \otimes V\Psi^{-1}(\overline{A}(j^1_{(0,0)}(\Psi_*\Gamma))(J^1\Psi(y))). \tag{15}$$

Since any other normal coordinate system of ∇ with centre y is of the form $\Psi' = C \circ \Psi$ for some $C \in GL(\mathbb{R}^{m,n})$, using the equivariance of $\overline{A}$ (Lemma 1) we see that $\Delta^A(\Gamma, \nabla)(y)$ is well defined (the definition is independent of the choice of Ψ).

We shall not indicate the pull-backs with respect to obvious projections. We know that $J^1Y \to Y$ is the affine bundle over Y with the corresponding vector bundle $T^*M \otimes VY$. Then $V(J^1Y \to Y) = T^*M \otimes VY$ over J^1Y. Similarly $J^1(J^1Y \to Y)$ is the affine bundle over J^1Y with the corresponding vector bundle $T^*Y \otimes V(J^1Y \to Y) = T^*Y \otimes T^*M \otimes VY$.

We have the general connection

$$\mathcal{J}^1(\nabla) : J^1Y \to J^1(J^1Y \to Y) \tag{16}$$

on $J^1Y \to Y$ by (11) with J^1 playing the role of G.

Example 1. We have the family of general connections

$$\mathcal{D}^A(\Gamma,\nabla)=\mathcal{J}^1(\nabla)+\Delta^A(\Gamma,\nabla):J^1Y\to J^1(J^1Y\to Y) \tag{17}$$

on $J^1Y\to Y$ for all $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant maps A as in (12). (One can eventually use $\mathcal{J}^1(\Gamma,\underline{\nabla},\nabla)$ given by (10) with J^1 playing the role of G instead of $\mathcal{J}^1(\nabla)$ given by (11).)

Theorem 1. *All $\mathcal{FM}_{m,n}$-natural operators constructing general connections on $J^1Y\to Y$ from general connections on $Y\to M$ by means of projectable torsion free classical linear connections on Y are of the form* (17) *for all $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant maps A of the type* (12) *(see Remark* 3 *for the classification of such A).*

Proof: Consider a natural operator $\mathcal{D}$ in question. Denote

$$\Delta(\Gamma,\nabla):=\mathcal{D}(\Gamma,\nabla)-\mathcal{J}^1(\nabla):J^1Y\to T^*Y\otimes T^*M\otimes VY. \tag{18}$$

Using the invariance of Δ with respect to normal coordinates for ∇ we see that Δ is determined by the values

$$\Delta(\Gamma,\nabla)(j^1_0(\mathrm{id}_{\mathbb{R}^m},\sigma))\in T^*_{(0,0)}\mathbb{R}^{m,n}\otimes T^*_0\mathbb{R}^m\otimes V_{(0,0)}\mathbb{R}^{m,n} \tag{19}$$

for all connections Γ on $\mathbb{R}^{m,n}$, all projectable torsion free classical linear connections ∇ on $\mathbb{R}^{m,n}$ with vanishing Christoffell symbols at $(0,0)$ and linear $\sigma:\mathbb{R}^m\to\mathbb{R}^n$.

Then using the invariance of Δ with respect to linear $\mathcal{FM}_{m,n}$-map $(x,y)\to(x,y-\sigma(x))$ we can additionally assume that in (19) we have $\sigma=0$.

Then using the regularity and the invariance of Δ with respect to the homotheties $t\,\mathrm{id}_{\mathbb{R}^{m,n}}$ for $t\neq 0$ and apply the homogeneous function theorem and the non-linear Peetre theorem [4] we see that $\Delta(\Gamma,\nabla)(j^1_0(\mathrm{id}_{\mathbb{R}^m},0))$ depends only on $j^1_{(0,0)}\Gamma$ and ∇^o, the flat connection on $\mathbb{R}^{m,n}$ with vanishing Christoffell symbols. Then we can assume that in (19) we have $\nabla=\nabla^o$.

Then Δ is determined by the $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant map of the type (13) given by

$$A(j^1_{(0,0)}\Gamma)=\Delta(\Gamma,\nabla^o)(j^1_0(\mathrm{id}_{\mathbb{R}^n},0)), \tag{20}$$

which can be considered as an $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-map of the type (12). $\square$

Remark 3. Any $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant map A of type (12) is the system of $GL(\mathbb{R}^m)\times GL(\mathbb{R}^n)$-equivariant maps

$$\begin{aligned}A_1:&((\mathbb{R}^m)^*\otimes\mathbb{R}^n)\times((\mathbb{R}^n)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)\\&\times((\mathbb{R}^m)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n)\to(\mathbb{R}^m)^*\otimes(\mathbb{R}^m)^*\otimes\mathbb{R}^n\end{aligned} \tag{21}$$

and

$$A_2 :((\mathbb{R}^m)^* \otimes \mathbb{R}^n) \times ((\mathbb{R}^n)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^n) \times ((\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^n) \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^n. \tag{22}$$

We have three canonical projections from $((\mathbb{R}^m)^* \otimes \mathbb{R}^n) \times ((\mathbb{R}^n)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^n) \times ((\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^n)$ onto the three factors. Using the methods of Chapter VI in [4] one can deduce that all $GL(\mathbb{R}^m) \times GL(\mathbb{R}^n)$-equivariant maps of the type (21) (or (22)) can be obtained by the following steps:

(1) We compose with the tensor product the respective systems induced by three canonical projections or $\mathrm{id}_{\mathbb{R}^m}$ or $\mathrm{id}_{\mathbb{R}^n}$.
(2) We compose with respective contractions and permutations of indices.
(3) We take linear combinations with real coefficients.

In this way we obtain that the vector space of the $GL(\mathbb{R}^m) \times GL(\mathbb{R}^n)$-maps of the type (21) is five dimensional. (We can compose the system of the first and the second projections with the tensor product, then compose with two respective contractions, then compose with two respective permutations of indices for the first case, and compose with the respective permutation of indices and compose with two respective permutations of indices for the second case, and we can take the third projection.) We also obtain that the vector space of $GL(\mathbb{R}^m) \times GL(\mathbb{R}^n)$-maps of the type (22) is two dimensional. (We can compose the second projection with the respective contraction, then tensor by $\mathrm{id}_{\mathbb{R}^n}$, and compose with the respective permutation of indices, and we can take the second projection.)

Remark 4. An interesting example of an $\mathcal{FM}_{m,n}$-natural operator Δ transforming connections Γ on $p : Y \to M$ into fibred maps

$$\Delta(\Gamma) : J^1Y \to T^*Y \otimes T^*M \otimes VY$$

is the respective composition of the target projection $\beta : J^1Y \to Y$ and $Tp : TY \to TM$ with the curvature operator $K^\Gamma : Y \to \wedge^2 T^*M \otimes VY$.

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