Fréchet property in compact spaces is not preserved by M-equivalence

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Abstract. An example of two M-equivalent (hence l-equivalent) compact spaces is presented, one of which is Fréchet and the other is not.

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All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We use terminology and notation as in [Eng].

Two spaces X and Y are called *M-equivalent* if their free topological groups $F(X)$ and $F(Y)$ in the sense of Markov [Mar] are topologically isomorphic. The spaces X and Y are *l-equivalent* if the spaces $C_p(X)$ and $C_p(Y)$ of real-valued continuous functions equipped with the topology of pointwise convergence are linearly homeomorphic, and *t-equivalent* if $C_p(X)$ and $C_p(Y)$ are homeomorphic (see [Arh3]); Arhangel'skii showed in [Arh1] that M-equivalence of two spaces implies their l-equivalence. We say that a topological property is preserved by an equivalence relation if whenever two spaces are in the relation, one of them has the property if and only if the other one does. Similarly, we say that a cardinal invariant is preserved by a relation if its values on two spaces are the same whenever the spaces are in the relation.

The article [Oku1] contains an example that shows that the Fréchet property, sequentiality and tightness are not preserved by the relation of M-equivalence. However, Tkachuk proved in [Tka] that the tightness is preserved by l-equivalence *in the class of compact spaces*, that is, if X and Y are l-equivalent *compact* spaces (this was later extended in [Oku2] by showing that the tightness is preserved by t-equivalence in the class of compact spaces, and the same holds for sequentiality if $2^{\mathfrak{t}} > \mathfrak{c}$.) In [Arh2, Problem 33 (1058)] Arhangel'skii asked if the Fréchet property is preserved by the relation of l-equivalence in the class of compact spaces. In this article we give an example of two M-equivalent compact spaces, one of which is Fréchet and the other is not, thus giving a negative answer to this question. The natural questions whether the tightness and sequentiality are preserved by the relations of M-equivalence, l-equivalence or t-equivalence in the classes of all σ-compact spaces, Lindelöf spaces, or normal spaces apparently remain open.

Peter Simon in [Sim] constructed two Fréchet compact spaces whose product is not Fréchet. Recall that Simon's spaces are one-point compactifications of Ψ-*spaces*, that is, the spaces of the form $\omega \cup \mathcal{A}_i$ where $\mathcal{A}_i$, $i = 1, 2$, are almost disjoint families on ω, all points of the set ω are isolated, and basic neighborhoods of a point $A \in \mathcal{A}_i$ are of the form $\{A\} \cup A \setminus F$ where F is finite. The Fréchet property in each of the one-point compactifications and its absence in their product are achieved by a special choice of the almost disjoint families $\mathcal{A}_1$, $\mathcal{A}_2$.

Thus, we have compact spaces Z_1 and Z_2 which have the following properties:

1. Z_1 and Z_2 are Fréchet spaces,
2. there are points $\infty_1 \in Z_1$ and $\infty_2 \in Z_2$ such that the subspaces $Z_1 \setminus \{\infty_1\}$, $Z_2 \setminus \{\infty_2\}$ are first-countable,
3. The product $Z_1 \times Z_2$ is not Fréchet.

Let $X = Z_1 \times Z_2$. Then X is a compact space which is not Fréchet.

Let $K_1 = Z_1 \times \{\infty_2\}$, $K_2 = \{\infty_1\} \times Z_2$ and $K = K_1 \cup K_2$. We now use the following statement (Corollary 2.8 in [Oku1]):

Proposition. *If $X = A \times B$, then X is M-equivalent to the space $(X/K, x_0) \vee (A, a_0) \vee (B, b_0)$ where $K = \big(A \times \{b_0\}\big) \cup \big(\{a_0\} \times B\big)$ and the points $x_0 \in X/K$, $a_0 \in A$ and $b_0 \in B$ are arbitrary.*

Here X/K means the partition space $\{K, \{x\} : x \in X \setminus K\}$ equipped with the real-quotient topology, that is, the finest completely regular topology that makes the natural projection $p\colon X \to X/K$ (which assigns to every element of X the element of the partition it belongs to) continuous. Note that the set K is always closed in X, and if the space X is compact as in our example, then the quotient topology on X/K is Tychonoff, so the real-quotient topology in fact coincides with the quotient topology.

The space $(X/K, x_0) \vee (A, a_0) \vee (B, b_0)$ in the formulation of the Proposition is the quotient space $\big((X/K) \oplus A \oplus B\big)/\{x_0, a_0, b_0\}$.

Thus, the space X in our example is M-equivalent to the space $Y' = (X/K, x_0) \vee (Z_1, \infty_1) \vee (Z_2, \infty_2)$ where x_0 is a point of X/K.

Note that the spaces Z_1 and Z_2 have clopen subspaces homeomorphic to the convergent sequence, so there are points $z_1 \in Z_1$ and $z_2 \in Z_2$ such that $Z_1 \setminus \{z_1\}$ is homeomorphic to Z_1 and $Z_2 \setminus \{z_2\}$ to Z_2. Hence, the space Y' is M-equivalent to the space $Y = (X/K, x_0) \vee (Z_1, z_1) \vee (Z_2, z_2) = X/K \oplus Z_1 \oplus Z_2$ by the following statement (Proposition 2.6 in [Oku1]):

Proposition. *Let $X_1, \ldots, X_n$ be spaces, x_i and y_i arbitrary points of X_i, $i = 1, \ldots, n$. Then the spaces $(X_1, x_1) \vee \cdots \vee (X_n, x_n)$ and $(X_1, y_1) \vee \cdots \vee (X_n, y_n)$ are M-equivalent.*

By the transitivity of M-equivalence, the spaces $X = Z_1 \times Z_2$ and $Y = X/K \oplus Z_1 \oplus Z_2$ are M-equivalent.

Let us verify that the space Y is Fréchet. Since Z_1 and Z_2 are Fréchet, we only need to verify that the space $Y_0 = X/(K_1 \cup K_2)$ is Fréchet. Recall that a space Y_0 is *Fréchet at a point* $y \in Y_0$ if for every set $B \subset Y_0$ such that $y \in \bar{B}$, there is a sequence in B that converges to y. Obviously, a space is Fréchet if and only if it is Fréchet at its every point.

Let $p\colon X \to Y_0 = X_0/(K_1 \cup K_2)$ be the natural projection. Denote by y_0 the point of Y_0 such that $\{y_0\} = p(K_1 \cup K_2)$.

Since X is compact, p is perfect, and the restriction of p to $X \setminus (K_1 \cup K_2) = p^{-1}(Y \setminus \{y_0\})$ is a perfect bijection of $X \setminus (K_1 \cup K_2)$ onto $Y \setminus \{y_0\}$, the spaces $Y \setminus \{y_0\}$ and $X_0 \setminus (K_1 \cup K_2) = (Z_1 \setminus \{\infty_1\}) \times (Z_2 \setminus \{\infty_2\})$ are homeomorphic. Thus, $Y_0 \setminus \{y_0\}$ has countable character, and since it is open in Y_0, the space Y_0 is Fréchet at every point of $Y_0 \setminus \{y_0\}$. To complete the proof, we are left to verify that Y_0 is Fréchet at y_0.

Let B be a set in $Y_0 \setminus \{y_0\}$ for which y_0 is a limit point. Let $A = p^{-1}(B)$. Since p is closed, the closure of A in X meets $K_1 \cup K_2$. Suppose $\bar{A} \cap K_1 \neq \emptyset$. Let $\pi_2\colon X = Z_1 \times Z_2 \to Z_2$ be the projection. Then ∞_2 is a limit point for $\pi_2(A)$, and since Z_2 is Fréchet, there is a convergent sequence S in $\pi_2(A)$ with the limit ∞_2. For every $s \in S$ pick a point $a_s \in A$ so that $\pi_2(a_s) = s$ and put $A' = \{\, a_s : s \in S \,\}$. Since the projection π_2 is closed, every neighborhood of K_1 in X contains all but finitely many points of the set A'. Hence, every neighborhood of y_0 in Y_0 contains all but finitely many points of the set $p(A')$, and $p(A')$ is a sequence in B that converges to y_0. The proof in the case $\bar{A} \cap K_2 \neq \emptyset$ is symmetric.

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