# Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces

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*Abstract.* We introduce the classes of nearly contraction mappings and nearly asymptotically nonexpansive mappings. The class of nearly contraction mappings includes the class of contraction mappings, but the class of nearly asymptotically nonexpansive mappings contains the class of asymptotically nonexpansive mappings and is contained in the class of mappings of asymptotically nonexpansive type. We study the existence of fixed points and the structure of fixed point sets of mappings of these classes in Banach spaces. Our results improve various celebrated results of fixed point theory in the context of demicontinuity.

*Keywords:* asymptotically nonexpansive mapping, Banach contraction principle, fixed point, Lipschitzian mapping, nearly Lipschitzian mapping, nearly asymptotically nonexpansive mapping, uniformly convex Banach space

*Classification:* 47H10, 47H15, 47H09, 47H17, 65J15

## 1. Introduction

Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to C$ a nonlinear mapping. The mapping $T$ is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \text{for all} \quad x, y \in C.$$

A Lipschitzian mapping $T$ is called *uniformly $k$-Lipschitzian* if $k_n = k$ for all $n \in \mathbb{N}$ and *asymptotically nonexpansive* if $\lim_{n\to\infty} k_n = 1$, respectively.

By an asymptotic fixed point theorem for the operator $T$, we mean a theorem which guarantees the existence of a fixed point of $T$, if the iterative $T^n$ possesses certain properties. The well known Browder [5]- Göhde [8]- Kirk [12] theorem is a central asymptotic fixed point theorem for nonexpansive mappings in Banach spaces, which was established in 1965.

The asymptotic fixed point theory has a fundamental role in nonlinear functional analysis (cf. [6]). This theory has been studied by many authors (see, e.g. [3], [4], [10], [11], [13], [15], [22]) for various classes of nonlinear mappings (e.g. Lipschitzian, uniformly $k$-Lipschitzian and non-Lipschitzian mappings). A branch of this theory related to asymptotically nonexpansive mappings has been developed by many authors (see, e.g. [7], [9], [12], [14]–[20], [22]) in Banach spaces with suitable geometrical structures. Asymptotic nonexpansiveness is an interesting aspect of asymptotically nonexpansive mappings. It is well known that for certain applications the continuity assumption becomes a rather strong condition. In view of this, the following natural question arises: Does there exist a class of (not necessarily continuous) mappings more general than the class of asymptotically nonexpansive mappings (which have asymptotic nonexpansiveness)?

Motivated and inspired by the above question, we will consider now a more general situation:

Let $C$ be a nonempty subset of a Banach space $X$ and fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x-y\| + a_n) \quad \text{for all} \quad x, y \in C. \tag{1.1}$$

The infimum of constants $k_n$ for which (1.1) holds will be denoted by $\eta(T^n)$ and called *nearly Lipschitz constant*. Notice that

$$\eta(T^n) = \sup\left\{ \frac{\|T^n x - T^n y\|}{\|x-y\| + a_n} : x, y \in C, x \neq y \right\}.$$

A nearly Lipschitzian mapping $T$ with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(ii) *nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,
(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,
(iv) *nearly uniformly $k$-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,
(v) *nearly uniform $k$-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

**Example 1.1.** Let $X = \mathbb{R}$, $C = [0,1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, $T$ is discontinuous and non-Lipschitzian. However, it is nearly nonexpansive. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$\|Tx - Ty\| \leq \|x-y\| + a_1 \quad \text{for all} \quad x, y \in C$$

and

$$\|T^n x - T^n y\| \leq \|x-y\| + a_n \quad \text{for all} \quad x, y \in C \quad \text{and} \quad n \geq 2,$$

since

$$T^n x = \frac{1}{2} \quad \text{for all} \quad x \in [0,1] \quad \text{and} \quad n \geq 2.$$

**Remark 1.2.** If $C$ is a bounded domain of an asymptotically nonexpansive mapping $T$, then $T$ is nearly nonexpansive. In fact, for all $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\|T^n x - T^n y\| &\leq k_n \|x - y\| \\ &\leq \|x - y\| + (k_n - 1)\|x - y\| \\ &\leq \|x - y\| + (k_n - 1)\ \operatorname{diam}(C).\end{aligned}$$

**Remark 1.3.** If $C$ is a bounded domain of a nearly asymptotically nonexpansive mapping $T$, then $T$ is mapping of asymptotically nonexpansive type. To see this, let $T$ be a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$. Then

$$\|T^n x - T^n y\| \leq \eta(T^n)(\|x - y\| + a_n) \quad \text{for all} \quad x, y \in C,\ n \in \mathbb{N}$$

which implies that

$$\sup_{x \in C}[\|T^n x - T^n y\| - \|x - y\|] \leq (\eta(T^n) - 1)\operatorname{diam}(C) + a_n \eta(T^n) \qquad \text{for all} \quad y \in C \quad \text{and} \quad n \in \mathbb{N}.$$

Hence $\overline{\lim}_{n\to\infty}(\sup_{x\in C}[\|T^n x - T^n y\| - \|x - y\|]) \leq 0$ for all $y \in C$.

We observe from Remarks 1.2 and 1.3 that the classes of nearly nonexpansive mappings and nearly asymptotically nonexpansive mappings are intermediate classes between the class of asymptotically nonexpansive mappings (cf. [7]) and that of mappings of asymptotically nonexpansive type (cf. [13]). Indeed, we have the following implications:

*asymptotically nonexpansive*

$$\Downarrow$$

*nearly nonexpansive*

$$\Downarrow$$

*nearly asymptotically nonexpansive*

$$\Downarrow$$

*asymptotically nonexpansive type.*

The purpose of this paper is to develop asymptotic fixed point theory for a more general class of demicontinuous nearly Lipschitzian mappings in Banach spaces. It is shown here that the Banach contraction principle ([1]) and result of Weissinger ([21]) can be improved for demicontinuous nearly contraction mappings in Banach spaces. It is also shown that the continuity of nonexpansive mappings from well known results of Browder [5]- Göhde [8]- Kirk [12] can be weakened to demicontinuity for nearly nonexpansive mappings in uniformly convex Banach spaces.

## 2. Preliminaries

Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to C$ a mapping. $T$ is said to be *demicontinuous* if, whenever a sequence $\{x_n\}$ in $C$ converges strongly to $x \in C$ then $\{Tx_n\}$ converges weakly to $Tx$.

Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$, $\{x_n\}$ a bounded sequence in $X$ and $r : C \to [0, \infty)$ a functional defined by

$$r(x) = \overline{\lim_{n\to\infty}} \|x_n - x\|, x \in C.$$

The infimum of $r(\cdot)$ over $C$ is called *asymptotic radius* of $\{x_n\}$ with respect to $C$ and is denoted by $r(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic centre* of the sequence $\{x_n\}$ with respect to $C$ if

$$r(z) = \min\{r(x) : x \in C\}.$$

The set of all asymptotic centers is denoted by $A(C, \{x_n\})$.

It is well known that every bounded sequence $\{x_n\}$ in a uniformly convex Banach space $X$ has a unique asymptotic centre with respect to any closed convex subset $C$ of $X$, i.e.,

$$A(C, \{x_n\}) = \{z\}$$

and

$$\overline{\lim_{n\to\infty}} \|x_n - z\| < \overline{\lim_{n\to\infty}} \|x_n - x\| \text{ for all } x \neq z.$$

The following lemma is well known (see [2]):

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$, $\{x_n\}$ a bounded sequence in $X$ and $A(C, \{x_n\}) = \{z\}$. Then we have*

$$(\{y_n\} \subset C \text{ and } r(y_m) \to r(C, \{x_n\}) \text{ as } m \to \infty) \Rightarrow (y_m \to z \text{ as } m \to \infty).$$

The following lemma is crucial for our main results:

**Lemma 2.2.** *Let $C$ be a nonempty subset of a Banach space and let $T : C \to C$ be demicontinuous. Suppose that $T^n u \to x^*$ as $n \to \infty$ for some $u, x^* \in C$. Then $x^*$ is an element of $F(T)$, the set of fixed points of $T$.*

Proof: Let $u_n = T^n u$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ and $\{Tu_n\}$ converge strongly to $x^*$. By demicontinuity of $T$, $\{Tu_n\}$ converges weakly to $Tx^*$. By uniqueness of weak limit of $\{Tu_n\}$, we have $x^* = Tx^*$. □

**Lemma 2.3.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \le 1$. If $\{y_n\}$ is a bounded sequence in $C$ such that*

$$\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0 \quad \text{and} \quad A(C, \{y_n\}) = \{x^*\},$$

*then $x^*$ is a fixed point of $T$.*

PROOF: We define a sequence $\{x_n\}$ in $C$ by

$$x_m = T^m x^*, \ m \in \mathbb{N}.$$

For $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_m - y_n\| &\le \|T^m x^* - T^m y_n\| + \|T^m y_n - y_n\| \\ &\le \eta(T^m)(\|x^* - y_n\| + a_m) + \|T^m y_n - y_n\|. \end{aligned} \tag{2.1}$$

Define a functional $r : C \to \mathbb{R}^+$ by

$$r(y) = \overline{\lim_{n\to\infty}} \|y_n - y\|, y \in C.$$

Then from (2.1)

$$\begin{aligned} r(x_m) &= \overline{\lim_{n\to\infty}} \|y_n - x_m\| \\ &\le \eta(T^m)(r(x^*) + a_m) + \overline{\lim_{n\to\infty}} \|T^m y_n - y_n\| \to r(x^*) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Hence Lemma 2.1 gives that $T^m x^* \to x^*$. By Lemma 2.2 we conclude that $x^* \in F(T)$. □

## 3. The principle of nearly contraction mappings for existence of fixed points

We develop existence and uniqueness of fixed points of demicontinuous nearly Lipschitzian mappings in a general Banach space.

**Theorem 3.1.** *Let $C$ be a nonempty closed subset of a Banach space $X$ and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$. Suppose $\eta_\infty(T) = \overline{\lim}_{n\to\infty}[\eta(T^n)]^{1/n} < 1$. Then we have the following:*

(a) *$T$ has a unique fixed point $x^* \in C$;*
(b) *for each $x_0 \in C$, the sequence $\{T^n x_0\}$ converges strongly to $x^*$;*
(c) *$\|T^n x_0 - x^*\| \le (\|x_0 - T x_0\| + M) \sum_{i=n}^{\infty} \eta(T^i)$ for all $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$.*

Proof: (a) Fix $x_0 \in X$ and let $x_n = T^n x_0$, $n \in \mathbb{N}$. Set $d_n := \|x_n - x_{n+1}\|$. Hence

$$d_n = \|T^n x_0 - T^{n+1} x_0\| \leq \eta(T^n)(\|x_0 - Tx_0\| + a_n)$$

which implies that

$$\sum_{n=1}^{\infty} d_n \leq (d_0 + M) \sum_{n=1}^{\infty} \eta(T^n)$$

for some $M > 0$, since $\lim_{n\to\infty} a_n = 0$. By the Root Test for convergence of series, if $\eta_\infty(T) = \overline{\lim}_{n\to\infty}[\eta(T^n)]^{1/n} < 1$, then $\sum_{n=1}^{\infty} \eta(T^n) < \infty$. It follows that $\sum_{n=1}^{\infty} d_n < \infty$ and hence $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n\to\infty} x_n$ exists (say $x^* \in C$). By Lemma 2.2, $x^*$ is a fixed point of $T$. Let $w$ be another fixed point of $T$. Then

$$\begin{aligned}\infty = \sum_{n=1}^{\infty} \|x^* - w\| &= \sum_{n=1}^{\infty} \|T^n x^* - T^n w\| \\ &\leq \sum_{n=1}^{\infty} \eta(T^n)(\|x^* - w\| + a_n) \\ &\leq (\|x^* - w\| + M) \sum_{n=1}^{\infty} \eta(T^n) < \infty,\end{aligned}$$

a contradiction. Thus, $T$ has a unique fixed point $x^* \in C$.

(b) It follows from part (a).

(c) If $m \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n - x_{n+m}\| &= \|T^n x_0 - T^{n+m} x_0\| \\ &\leq \sum_{i=n}^{n+m-1} \|T^i x_0 - T^{i+1} x_0\| \\ &\leq \sum_{i=n}^{n+m-1} \eta(T^i)(\|x_0 - Tx_0\| + a_i) \\ &\leq (\|x_0 - Tx_0\| + M) \sum_{i=n}^{n+m-1} \eta(T^i).\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\|x_n - x^*\| \leq (\|x_0 - Tx_0\| + M) \sum_{i=n}^{\infty} \eta(T^i).$$

$\square$

**Remark 3.2.** In case of a nearly uniformly $k$-Lipschitzian mapping, we have

$$\overline{\lim_{n\to\infty}}[\eta(T^n)]^{\frac{1}{n}} = \overline{\lim_{n\to\infty}}[k]^{\frac{1}{n}} = 1.$$

Therefore, the assumptions of Theorem 3.1 do not hold for nearly uniformly $k$-Lipschitzian (non-Lipschitzian) mappings.

**Remark 3.3.** Theorem 3.1 generalizes Banach contraction principle ([1]). Particularly, it generalizes the result of Weissinger ([21]) in the following ways:

(1) $T$ is more general than the mapping considered by Weissinger [21],
(2) $T$ may not be continuous.

The well known Banach contraction principle asserts that every contraction mapping has a unique fixed point. But it is not true for nearly contraction mappings. Let us consider another condition which only guarantees the uniqueness of fixed points of nearly contractions.

**Theorem 3.4.** *Let $C$ be a nonempty closed subset of a Banach space and $T : C \to C$ a nearly contraction with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1}-1} = 0$. Then $F(T)$ has at most one element.*

Proof: Let $x$ and $y$ be two distinct elements in $F(T)$. Then

$$\|x - y\| = \|T^n x - T^n y\| \le \eta(T^n)(\|x - y\| + a_n)$$

which implies that

$$\|x - y\| \le \frac{a_n}{\eta(T^n)^{-1} - 1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $T$ has at most one fixed point. □

**Corollary 3.5.** *Let $C$ be a nonempty closed subset of a Banach space and $T : C \to C$ a nearly uniform $k$-contraction. Then $F(T)$ has at most one element.*

A convex subset $C$ of a Banach space $X$ is said to have the *approximate fixed point property* (*AFPP*) for a nonexpansive mapping $T : C \to C$ if

$$\inf\{\|x - Tx\| : x \in C\} = 0,$$

i.e., there exists a sequence $\{x_n\}$ in $C$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Such a sequence $\{x_n\}$ is called *approximate fixed point sequence* of $T$.

It is well known that every closed convex bounded subset $C$ of $X$ has approximating fixed point property (*AFPP*) for nonexpansive mappings. However, no analog of this fact is known for nearly nonexpansive mappings. More precisely, let $T$ be a nonexpansive self-mapping of a closed convex bounded subset $C$ of a Banach space with fixed element $x_0 \in C$, $\{\lambda_n\}$ a sequence in $(0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$ and let the mapping $T_n : C \to C$ be defined by

$$T_n x = \lambda_n T^n x + (1-\lambda_n)x_0, \ x \in C.$$

Then for each $n \in \mathbb{N}$, $T_n$ has exactly one fixed point by Banach contraction principle.

The following result (see Theorem 3.7) shows that Banach spaces do not have *AFPP* even for nearly contraction mappings.

**Proposition 3.6.** *If $T : C \to C$ is a nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$, then*

$$\|T^n x - T^n y\| \le \max\left\{\|x-y\|, \frac{a_n}{\eta(T^n)^{-1}-1}\right\} \quad \text{for all} \ \ x, y \in C \ \ \text{and} \ \ n \in \mathbb{N}.$$

Proof: Note that

$$\eta(T^n)(\|x-y\| + a_n) \le \|x-y\| \Leftrightarrow \|x-y\| \ge \frac{a_n}{\eta(T^n)^{-1}-1}\,.$$

If $\|x-y\| \ge \frac{a_n}{\eta(T^n)^{-1}-1}$, then

$$\|T^n x - T^n y\| \le \eta(T^n)(\|x-y\| + a_n) \le \|x-y\|.$$

If $\|x-y\| \le \frac{a_n}{\eta(T^n)^{-1}-1}$, then

$$\|T^n x - T^n y\| \le \eta(T^n)(\|x-y\| + a_n) \le \frac{a_n}{\eta(T^n)^{-1}-1}.$$

The proposition follows. □

**Theorem 3.7.** *Let $C$ be a nonempty closed convex bounded subset of a Banach space $X$ and $T : C \to C$ a nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1}-1} = 0$. Then $\lim_{n\to\infty}(\inf\{\|x - T^n x\| : x \in C\}) = 0$, i.e., there exists a sequence $\{x_m\}$ in $C$ such that*

$$\lim_{n\to\infty}\big(\lim_{m\to\infty} \|x_m - T^n x_m\|\big) = 0.$$

Proof: Let $\{t_n\}$ be a real sequence in (0,1) such that $\lim_{n\to\infty} t_n = 1$. For each $n \in \mathbb{N}$ and some $u \in C$, define

$$T_n x = (1-t_n)u + t_n T^n x, \ x \in C.$$

Observe that

$$\begin{aligned}\|T_n x - T_n y\| &\le t_n \|T^n x - T^n y\| \\ &\le \max\left\{t_n\|x-y\|, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \text{ for all } x,y\in C \text{ and } n\in\mathbb{N}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|T_n^\ell x - T_n^\ell y\| &\le \max\left\{t_n\|T_n^{\ell-1}x - T_n^{\ell-1}y\|, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\ &\le \max\left\{t_n \max\left\{t_n\|T_n^{\ell-2}x - T_n^{\ell-2}y\|, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\}, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\ &= \max\left\{t_n^2\|T^{\ell-2}x - T^{\ell-2}y\|, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\ &\le \dots \\ &\le \max\left\{t_n^\ell\|x-y\|, \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\}.\end{aligned}$$

Since $C$ is bounded and $\lim_{\ell\to\infty} t_n^\ell = 0$ for $n\in\mathbb{N}$,

$$\begin{aligned}\|T_n^\ell x - T_n^{\ell+1}x\| &\le \max\left\{t_n^\ell \ \operatorname{diam}(C), \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\ &\to \frac{t_n a_n}{\eta(T^n)^{-1}-1} \quad \text{as} \quad \ell\to\infty.\end{aligned}$$

This gives that

$$\inf\left\{\|x - T_n x\| : x\in C\right\} \le t_n \frac{a_n}{\eta(T^n)^{-1}-1}.$$

Thus

$$\begin{aligned}\|x - T^n x\| &\le t_n^{-1}\|x - T_n x\| + t_n^{-1}(1-t_n)\|x-u\| \\ &\le t_n^{-1}\|x - T_n x\| + t_n^{-1}(1-t_n)\operatorname{diam}(C)\end{aligned}$$

which implies that

$$\inf\left\{\|x - T^n x\| : x\in C\right\} \le \frac{a_n}{\eta(T^n)^{-1}-1} + t_n^{-1}(1-t_n)\operatorname{diam}(C).$$

Therefore, the conclusion follows. □

The next result is more general in nature. It demonstrates that every demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty}\eta(T^n)\le 1$ has a fixed point in a Banach space under geometric structure "uniform convexity".

**Theorem 3.8.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \leq 1$. Then the following statements are equivalent:*

(a) *$T$ has a fixed point;*
(b) *there exists a bounded sequence $\{T^n x_0\}$ in $C$;*
(c) *there exists a bounded sequence $\{y_n\}$ in $C$ such that*

$$\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0.$$

Proof: (a) $\Rightarrow$(b) and (a) $\Rightarrow$(c) follow easily.
(b) $\Rightarrow$(a). Let us assume that the sequence $\{T^n x_0\}$ is bounded and $A(C, \{T^n x_0\}) = \{z\}$. Since, for $n \geq m \geq 1$

$$\|T^n x_0 - T^m z\| \leq \eta(T^m)(\|T^{n-m} x_0 - z\| + a_m),$$

it follows that

$$\overline{\lim_{n\to\infty}} \|T^n x_0 - T^m z\| \leq \eta(T^m)(\overline{\lim_{n\to\infty}} \|T^n x_0 - z\| + a_m).$$

Hence $r(T^m z) \to r(C, \{T^n x_0\})$ as $m \to \infty$. By applying Lemma 2.1 we get $T^m z \to z$ as $m \to \infty$. This shows by Lemma 2.2 that $z$ is a fixed point of $T$.
(c) $\Rightarrow$(a). Let $\{y_n\}$ be a bounded sequence in $C$ such that $\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0$. The result follows from Lemma 2.3. □

Combining Theorems 3.7 and 3.8, we show that the condition $\frac{a_n}{\eta(T^n)^{-1}-1} \to 0$ also assures the existence of fixed points of nearly contraction mappings in uniformly convex Banach space.

**Theorem 3.9.** *Let $C$ be a nonempty closed convex bounded subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1}-1} = 0$. Then $T$ has a unique fixed point.*

Proof: By Theorem 3.7, there exists a sequence $\{y_n\}$ in $C$ such that $\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0$. We conclude the result by Theorem 3.8. □

The following result generalizes the result of Goebel and Kirk [7].

**Corollary 3.10.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly asymptotically nonexpansive mapping. If there is a point $x_0 \in C$ such that $\{T^n x_0\}$ is bounded, then $T$ has a fixed point in $C$.*

## 4. Structure of set of fixed points of demicontinuous nearly Lipschitzian mappings

**Theorem 4.1.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) = 1$. Then $F(T)$ is closed and convex.*

Proof: Closedness of $F(T)$: Let $\{z_n\}$ be a sequence in $F(T)$ such that $z_n \to z$. Then it remains to show that $z \in F(T)$. Note that

$$\|z_n - T^n z\| = \|T^n z_n - T^n z\| \le \eta(T^n)(\|z_n - z\| + a_n),$$

which implies that

$$\overline{\lim_{n\to\infty}} \|z_n - T^n z\| = 0.$$

Since

$$\|z - T^n z\| \le \|z - z_n\| + \|z_n - T^n z\|,$$

we have

$$\overline{\lim_{n\to\infty}} \|z - T^n z\| = 0.$$

By Lemma 2.2, we conclude that $z \in F(T)$, i.e., $F(T)$ is closed.

Convexity of $F(T)$: Let $x, y \in F(T)$ such that $x \ne y$. Let $z = \frac{1}{2}(x + y)$. Then

$$\|T^n z - x\| = \|T^n z - T^n x\| \le \eta(T^n)(\|z - x\| + a_n) \le \eta(T^n)(\frac{1}{2}\|x - y\| + a_n)$$

and

$$\|T^n z - y\| \le \eta(T^n)(\frac{1}{2}\|x - y\| + a_n).$$

Thus,

$$\begin{aligned}\|T^n z - z\| &= \|\frac{1}{2}(T^n z - x) + \frac{1}{2}(T^n z - y)\| \\ &\le \eta(T^n)(\frac{1}{2}\|x - y\| + a_n)\left\{1 - \delta\left(\frac{\|x - y\|}{\eta(T^n)(\frac{1}{2}\|x - y\| + a_n)}\right)\right\}\end{aligned}$$

for all $n \in \mathbb{N}$, where $\delta$ is modulus of convexity of $X$. It follows that $\lim_{n\to\infty} \|T^n z - z\| = 0$ and hence $z \in F(T)$ by Lemma 2.2. □

**Corollary 4.2.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$. Then $F(T)$ is closed and convex.*

**Corollary 4.3.** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a demicontinuous nearly asymptotically nonexpansive mapping. Then $F(T)$ is closed and convex.*

Recall that a subset $F \subseteq C$ is said to be a 1-*local retract* of $C$ if every family $\{B_i : i \in I\}$ of closed balls centered at points of $F$ has the property:

$$\left(\bigcap_{i\in I} B_i\right) \cap C \neq \emptyset \Longrightarrow \left(\bigcap_{i\in I} B_i\right) \cap F \neq \emptyset.$$

It is easy to see that a 1-local retract of a convex set is metrically convex and 1-local retract of a closed set must itself be closed.

Finally, we show that the fixed point set of a nearly Lipschitzian mapping is a 1-local retract of its domain.

**Theorem 4.4.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T : C \to C$ a nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\overline{\lim}_{n\to\infty}\eta(T^n) \leq 1$. Suppose that each closed convex subset $D$ of $C$ has the fixed point property for $T$. Then $F(T)$ is a (nonempty) 1-local retract of $C$.*

Proof: By assumption $F(T) \neq \emptyset$. Let $\{B(x_i, r_i) : i \in I\}$ be a family closed balls centered at points $x_i \in F(T)$.

Suppose

$$S_0 = \left(\bigcap_{i\in I} B(x_i, r_i)\right) \cap C \neq \emptyset,\ \ r(\{T^n x\}, x_i) = \overline{\lim_{n\to\infty}} \|T^n x - x_i\|, x \in C$$

and

$$S_1 = \{x \in C : r(\{T^n x\}, x_i) \leq r_i\}.$$

Let $x \in S_0$; then

$$\begin{aligned} r(\{T^n x\}, x_i) &= \overline{\lim_{n\to\infty}} \|T^n x - x_i\| \\ &= \overline{\lim_{n\to\infty}} \|T^n x - T^n x_i\| \\ &\leq \overline{\lim_{n\to\infty}} \eta(T^n)(\|x - x_i\| + a_n) \\ &\leq \|x - x_i\| \\ &\leq r_i. \end{aligned}$$

It follows that $x \in S_1$, i.e., $S_0 \subseteq S_1 \neq \emptyset$.

Let $x, y \in S_1$ be such that $z = (1-t)x + ty$, $t \in [0,1]$. Now

$$\begin{aligned} r(\{T^n z\}, x_i) &\leq \overline{\lim_{n\to\infty}} \|T^n z - x_i\| \leq \|z - x_i\| \\ &\leq (1-t)\|x - x_i\| + t\|y - x_i\| \\ &\leq r_i, \end{aligned}$$

hence $z \in S_1$. So $S_1$ is convex.

Now, let $\{y_m\}$ be a sequence in $S_1$ such that $y_m \to y$ as $m \to \infty$. We claim that $y \in S_1$. We have

$$r(\{T^n y_m\}, x_i) \leq r_i \quad \text{for all} \quad i \in I.$$

Hence for each $m \in \mathbb{N}$,

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \|T^n y - x_i\| &\leq \overline{\lim_{n\to\infty}} \|T^n y - T^n y_m\| + \overline{\lim_{n\to\infty}} \|T^n y_m - x_i\| \\
&\leq \overline{\lim_{n\to\infty}} \eta(T^n)(\|y - y_m\| + a_n) + r_i \\
&\leq \|y - y_m\| + r_i \\
&\leq r_i \quad \text{as} \quad m \to \infty,
\end{aligned}$$

it follows that $y \in S_1$. Hence $S_1$ is a nonempty closed and convex subset of $C$. Moreover, $T$ is self-mapping on $S_1$. Hence $S_1 \cap F(T)$ is nonempty by assumption. Let $p$ be an element of $S_1 \cap F(T)$. Then

$$r(\{T^n p\}, x_i\}) = r(\{p\}, x_i) = \|p - x_i\| \leq r_i,$$

it infers that

$$S_1 \cap F(T) \subseteq S_0 \cap F(T).$$

We note that $S_0 \cap F(T) \subset S_1 \cap F(T)$ since $S_0 \subseteq S_1$. Thus, $S_0 \cap F(T) = S_1 \cap F(T) \neq \emptyset$. □

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