The maximal regular ideal of some commutative rings

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Abstract. In 1950 in volume 1 of Proc. Amer. Math. Soc., B. Brown and N. McCoy showed that every (not necessarily commutative) ring R has an ideal $\mathfrak{M}(R)$ consisting of elements a for which there is an x such that $axa = a$, and maximal with respect to this property. Considering only the case when R is commutative and has an identity element, it is often not easy to determine when $\mathfrak{M}(R)$ is not just the zero ideal. We determine when this happens in a number of cases: Namely when at least one of a or $1 - a$ has a von Neumann inverse, when R is a product of local rings (e.g., when R is $\mathbb{Z}_n$ or $\mathbb{Z}_n[i]$), when R is a polynomial or a power series ring, and when R is the ring of all real-valued continuous functions on a topological space.

Keywords: commutative rings, von Neumann regular rings, von Neumann local rings, Gelfand rings, polynomial rings, power series rings, rings of Gaussian integers (mod n), prime and maximal ideals, maximal regular ideals, pure ideals, quadratic residues, Stone-Čech compactification, $C(X)$, zerosets, cozerosets, P-spaces

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1. Introduction

Throughout R will denote a commutative ring with identity element 1 unless the contrary is stated explicitly, and the notation of [AHA04] will be followed.

1.1 Definition. An element $a \in R$ is called *regular* if there is a $b \in R$ such that $a = a^2b$. Let $\mathrm{vr}(R) = \{a \in R : a \text{ is regular}\}$ and $\mathrm{nvr}(R) = R \setminus \mathrm{vr}(R)$. An ideal I of R is called a *regular ideal* if $I \subset \mathrm{vr}(R)$. The element a is called *m-regular* if the ideal generated by a is a regular ideal. Let $\mathfrak{M}(R) = \{a \in R : a \text{ is m-regular}\}$. A ring R is called *von Neumann regular ring* (VNR ring) if $R = \mathrm{vr}(R)$.

This terminology is motivated in part by a theorem of Brown and McCoy in which they show that $\mathfrak{M}(R)$ is a regular ideal. Indeed it is the largest regular ideal or R. See [BM50]. R may contain regular elements which are not m-regular, as one can see easily that $3 \in \mathrm{vr}(\mathbb{Z}_4) \setminus \mathfrak{M}(\mathbb{Z}_4)$. (As usual, $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}$ of integers mod n for a positive integer n.)

If $S \subset R$, then $\mathrm{Ann}(S)$ denotes $\{a \in R : aS = \{0\}\}$, the set of maximal ideals of R is denoted by $\mathrm{Max}(R)$, and their intersection $J(R)$ is the Jacobson radical of R. In [BM50], the following is also established.

1.2 Lemma.

$\mathfrak{M}(R/\mathfrak{M}(R)) = \{0\}$.
$\mathfrak{M}(R) \cap J(R) = \{0\}$.
$\mathfrak{M}(R) \subset \mathrm{Ann}(J(R))$.
$\mathfrak{M}(R) \cap \mathrm{Ann}(\mathfrak{M}(R)) = \{0\}$.

If $R/J(R)$ is VNR-ring, then $\mathfrak{M}(R) = \{0\}$ if and only if $\mathrm{Ann}(J(R)) \subset J(R)$.

If R satisfies the descending chain condition on ideals, then $R = \mathfrak{M}(R) + \mathrm{Ann}(\mathfrak{M}(R))$.

For each ideal I of R, let $mI = \{a \in I : a \in aI\} = \{a \in R : I + \mathrm{Ann}(a) = R\}$. Then mI is called the *pure part* of I. An ideal I is called a *pure ideal* if $I = mI$. It is clear that $a \in mM$ for an $M \in \mathrm{Max}(R)$, if and only if $\mathrm{Ann}(a)$ is not contained in M.

The following description of $\mathfrak{M}(R)$ will be used frequently below.

1.3 Theorem. *If R is not a von Neumann regular ring, then $\mathfrak{M}(R) = \bigcap\{mM : M \in \mathrm{Max}(R)$ and $M \neq mM\}$ is the intersection of the pure parts of those maximal ideals M of R that are not pure.*

PROOF: If $a \notin \mathfrak{M}(R)$, then there is an $x \in R$ such that $ax \notin \mathrm{vr}(R)$. So by Theorem 2.4 of [AHA04], there is an $N \in \mathrm{Max}(R)$ such that $ax \in N \setminus mN$. It follows that N is not pure and $a \notin \bigcap\{mM : M \in \mathrm{Max}(R)$ and $M \neq mM\}$. Thus $\bigcap\{mM : M \in \mathrm{Max}(R)$ and $M \neq mM\} \subset \mathfrak{M}(R)$.

If instead $a \in \mathfrak{M}(R)$ and there is an $M \in \mathrm{Max}(R)$ and an $x \in M \setminus mM$, then $ax \in mM$ and so as noted above, there is a $b \notin M$ such that $bax = 0$. So $ba \in \mathrm{Ann}(x)$ which is contained in M because this maximal ideal in not pure. But M is a prime ideal, so $a \in M$. Thus $\mathfrak{M}(R) \subset mM$. Hence $\mathfrak{M}(R) \subset \bigcap\{mM : M \in \mathrm{Max}(R)$ and $M \neq mM\}$. □

In this article, we determine when $\mathfrak{M}(R)$ is not the zero ideal for a number of classes of rings. In Section 2, we study rings in which at least one of a or $1-a$ has a von Neumann inverse. Section 3 is devoted to the study of products of local rings (e.g., the ring $\mathbb{Z}_n$ of integers modulo an integer $n \geq 2$ and to $\mathbb{Z}_n[i]$). The complicated conditions needed to describe when $\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}$ hint at why it may be quite difficult to describe when the maximal regular ideal of a finite ring is nonzero. In Section 4, it is shown that the maximal regular ideal of a polynomial or powers series ring is the zero ideal, and in Section 5, it is determined when the maximal regular ideal of the ring of all continuous functions on a topological space is nonzero.

2. Von Neumann local and strong von Neumann local rings

Recall from [AHA04] that R is called a *von Neumann local* (*VNL*) ring if $a \in \mathrm{vr}(R)$ or $1-a \in \mathrm{vr}(R)$ for each $a \in R$. It is easy to see that VNR rings and local rings are VNL rings. R is called a *strong von Neumann local* (*SVNL*) ring if whenever the ideal $\langle S\rangle$ generated by a subset S of R is all of R, then some element of S is in $\mathrm{vr}(R)$, or equivalently if $\langle \mathrm{nvr}(R)\rangle \neq R$. Clearly every SVNL ring is a VNL ring, but the validity of the converse remains an open problem. R is called a *Gelfand ring* or a *PM ring* if each of its proper prime ideals is contained in a unique maximal ideal. If M is a maximal ideal of R, then O_M denotes intersection of all of the (minimal) prime ideals of R that are contained in M.

2.1 Lemma. *Every VNL ring R is a Gelfand ring and if R is also reduced, then $mM = O_M$ whenever $M \in Max(R)$.*

PROOF: The first assertion is shown in [C84]. (Combine in that paper Proposition 4.4, Theorems 3.2 and 2.4 with Proposition 1.1.) The second assertion is shown in Proposition 3 of [H77]. □

See also [DO71].

Next, we make use of Theorem 1.1 above.

In Theorem 2.6 of [AHA04] it is shown that R is an SVNL ring that is not a VNR ring if and only if it has exactly one maximal ideal that fails to be pure. Combining this with Theorem 1.3 yields:

2.2 Theorem. *If R is an SVNL ring that is not a VNR ring, then it has a unique maximal N that is not pure. Moreover $\mathfrak{M}(R) = mN = O_M$.*

PROOF: The first assertion is part of Theorem 2.6 of [AHA04], and the second is immediate from Theorem 1.3 and Lemma 2.1. □

Next we begin to exhibit a class of rings whose maximal regular ideal is not the zero ideal.

2.3 Lemma. *If R and S are commutative rings with identity whose direct sum $R \oplus S$ is a VNL ring, then at least one of R and S is a VNR ring.*

PROOF: Suppose instead that there are $r \in R$ and $s \in S$ that are not von Neumann regular. Then neither $(r, 1-s)$ nor $(1,1) - (r, 1-s) = (1-r, s)$ are von Neumann regular in $R \oplus S$, so the conclusion follows. □

2.4 Theorem. *If R is a VNL ring that is neither local nor a VNR ring, then $\mathfrak{M}(R)$ contains fR for some idempotent f not in $\{0,1\}$ and hence is not the zero ideal.*

PROOF: By Theorem 4.6 of [AHA04], a nonlocal VNL ring has an idempotent $e \notin \{0,1\}$, so $R = eR \oplus (1-e)R$. Thus by Lemma 2.3, exactly one of these two summands must be a VNR ring, which is a nonzero ideal included in $\mathfrak{M}(R)$. □

3. Products of local rings

In this section, it will be determined when a direct product of local rings has a nonzero maximal regular ideal.

It is an exercise to show that a local VNR ring is a field. Moreover, if M is the unique maximal ideal of R, and $a = am \in mM$ for some $m \in M$, then $a = 0$ since $1 - m$ in invertible. Because each element of $\mathfrak{M}(R)$ is in mM, we conclude from Theorem 1.3 that:

3.1 Lemma. *If R is a local ring, then R is a field or $\mathfrak{M}(R) = \{0\}$.*

3.2 Lemma. *If $R = \prod_{i \in I} R_i$ is the direct product of rings R_i with identity, then*

(1) *$(r_i)_{i \in I} \in \mathrm{vr}(R)$ if and only if $r_i \in \mathrm{vr}(R_i)$ for each $i \in I$, and*

(2) *$(r_i)_{i \in I} \in \mathfrak{M}(R)$ if and only if $r_i \in \mathfrak{M}(R_i)$ for each $i \in I$.*

Proof: (1) $(r_i)_{i \in I} \in \mathrm{vr}(R)$ if and only if there exists $(x_i)_{i \in I} \in R$ such that $(r_i)_{i \in I} = ((r_i)_{i \in I})^2\ (x_i)_{i \in I} = (r_i^2 x_i)_{i \in I}$ if and only if $r_i = r_i^2 x_i$ for each $i \in I$ if and only if $r_i \in \mathrm{vr}(R_i)$ for each $i \in I$.

(2) Suppose that $(r_i)_{i \in I} \in \mathfrak{M}(R)$. Pick $r_k \in R_k$ and let $x \in R_k$.

Define $x_i = \begin{cases} x & i=k \\ 0 & i \neq k \end{cases}$.

Now, $(r_i)_{i \in I}(x_i)_{i \in I} \in \mathrm{vr}(R)$, so there exists $(y_i)_{i \in I} \in R$ such that $(r_i)_{i \in I}(x_i)_{i \in I} = ((r_i)_{i \in I}(x_i)_{i \in I})^2(y_i)_{i \in I} = ((r_i x_i)^2 y_i)_{i \in I}$. In particular $r_k x = (r_k x)^2 y_k$. Thus $r_k \in \mathfrak{M}(R_k)$. Conversely, suppose that $r_i \in \mathfrak{M}(R_i)$ for each $i \in I$. Let $(x_i)_{i \in I} \in R$. Then $r_i x_i \in \mathrm{vr}(R_i)$ for each $i \in I$, which implies that there exists $y_i \in R_i$ such that $r_i x_i = (r_i x_i)^2 y_i$ for each $i \in I$. Hence $(r_i)_{i \in I}(x_i)_{i \in I} = ((r_i x_i)^2 y_i)_{i \in I} = ((r_i)_{i \in I}(x_i)_{i \in I})^2 (y_i)_{i \in I}$ which implies that $(r_i)_{i \in I} \in \mathfrak{M}(R)$. □

It follows that:

3.3 Theorem. *If $R = \prod_{i \in I} R_i$ is the direct product of rings R_i with identity, then $\mathfrak{M}(R) = \prod_{i \in I} \mathfrak{M}(R_i)$.*

Because a local VNR ring is a field and if R is a field, then $R = \mathfrak{M}(R)$, it follows that:

3.4 Corollary. *If $R = \prod_{i \in I} R_i$ is the direct product of local rings R_i with identity, then $\mathfrak{M}(R) \neq \{0\}$ if and only if R_j is a field for at least one $j \in I$.*

In Chapter VI of [M74], it is shown that every finite commutative ring with identity element is a direct product of local rings. Hence we have

3.5 Theorem. *If R is finite, then $\mathfrak{M}(R) \neq \{0\}$ if and only if R is a direct product of local rings at least one of which is a field.*

Much more is said about finite local rings in [M74]. If R is such a ring then its unique maximal ideal M is nilpotent and $\mathfrak{M}(R) = \{0\}$ by Lemma 3.1. Indeed, every element of R is either nilpotent or invertible.

Next, some examples are considered.

It is well known that if $n > 1$ is in $\mathbb{Z}$, then $\mathbb{Z}_n$ is local if and only if $n = p^k$ for some prime p and positive integer k, and is a field if and only if $k = 1$.

3.6 Corollary. *If $n = \prod_{i=1}^{s} p_i^{k_i}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n$ is the direct product of the local rings $\mathbb{Z}_{p_i^{k_i}}$ and $\mathfrak{M}(R) \neq \{0\}$ if and only if $k_j = 1$ for at least one $j \in \{1, \ldots, s\}$.*

3.7 Definition. If $i^2 = -1$ and $Z[i] = \{a + ib : a, b \in Z\}$ is the ring of Gaussian integers, then for any integer $n > 1$, $\mathbb{Z}_n[i] = \mathbb{Z}[i]/n\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}_n\}$ denotes the ring of *Gaussian integers* mod n.

3.8 Lemma. (a) *If an element $a + ib$ of $\mathbb{Z}_n[i]$ is nilpotent [resp. idempotent] then $a^2 + b^2$ is nilpotent [resp. idempotent] in $\mathbb{Z}_n$.*

(b) *$a + ib$ is a unit in $\mathbb{Z}_n[i]$ if and only if $a^2 + b^2$ is a unit of $\mathbb{Z}_n$.*

(c) *$(a + ib)^2 = a + ib$ is a nontrivial idempotent if and only if $a^2 - b^2 = a$ and $2ab = b$ in $\mathbb{Z}_n$ and neither a nor b is zero in $\mathbb{Z}_n$.*

Proof: (a) If $a + ib$ is nilpotent, then so is $(a - ib)(a + ib) = a^2 + b^2$ because complex conjugation is an automorphism of $\mathbb{Z}_n[i]$. The proof for idempotents is similar.

(b) follows because $(a - ib)(a + ib) = a^2 + b^2$ and any divisor of a unit is a unit.

(c) is an exercise. □

As in Corollary 3.6, if $n = \prod_{i=1}^{s} p^{k_i}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{p_i^{k_i}}[i]$. So by Theorem 3.3, $\mathfrak{M}(\mathbb{Z}_n[i]) = \prod_{i=1}^{s} \mathfrak{M}(\mathbb{Z}_{p_i^{k_i}}[i]) \neq \{0\}$ if and only if at least one of the ideals in this latter product is nonzero. This motivates the question:

(∗) If p and k are positive integers and p is prime, when is $\mathfrak{M}(\mathbb{Z}_{p^k}[i]) \neq \{0\}$?

While it is true that $\mathbb{Z}_n$ is a local ring whenever n is a power of a prime, this is not the case for $\mathbb{Z}_n[i]$ as will be shown next. Recall that if a ring R is finite, then R is local if and only if its only idempotents are 0 and 1 (which are called *trivial idempotents*).

3.9 Theorem. *If $m = p^k$ for some prime p and positive integer k, then $\mathbb{Z}_m[i]$ is local if and only if $p = 2$ or $p \equiv -1 (\text{mod } 4)$.*

Proof: We will show that if $a + ib$ is a nontrivial idempotent of $\mathbb{Z}_m[i]$, then

(i) $2a \equiv 1 (\text{mod } p^k)$, and

(ii) there is a c such that $c^2 \equiv -1 (\text{mod } p^k)$.

To see (i), recall from Lemma 3.8 that if $a + ib$ is an nontrivial idempotent, then $a^2 - b^2 = a$ and $2ab = b$ in $\mathbb{Z}_m$ and neither a nor b is $0 (\text{mod } p^k)$. This latter equation says $b(2a - 1) \equiv 0 (\text{mod } p^k)$. By Lemma 3.8, $a^2 + b^2$ is an idempotent in $\mathbb{Z}_m$ and hence is congruent to 0, so if $p \mid b$, then $p \mid a$. It follows that $p^2 \mid b$ because $2ab = b$. A routine induction yields $p^k \mid b$ and hence that $b \equiv 0 (\text{mod } p^k)$; contrary to the assumption that $a + ib$ is a nontrivial idempotent. Hence p is not a divisor of b, i.e. b is a unit in $\mathbb{Z}_m$, but $b(2a - 1) \equiv 0 (\text{mod } p^k)$. So (i) holds.

This shows that there are no nontrivial idempotents in $\mathbb{Z}_{2^k}[i]$. So this ring is local and is never a field because it contains the nonzero nilpotent ideal $(1+i)\mathbb{Z}_{2^k}[i]$. Thus $\mathfrak{M}(\mathbb{Z}_{2^k}) = \{0\}$ for all k.

Assume next that p is odd and note that by (i) and its proof $(2b)^2 = 4(a^2-a) \equiv (2a)^2 - 2(2a) = (p^k+1)^2 - 2(p^k+1) \equiv -1 (\mathrm{mod}\ p^k)$. So $c = 2b$ is the solution of the equation in (ii). Thus $\mathbb{Z}_m[i]$ has a nontrivial idempotent exactly when the equation in (ii) has a solution in which case $\frac{1}{2} + i\frac{c}{2}$ is such an idempotent.

It is noted in Chapter 5 of [L58] that for p odd, the congruence $c^2 \equiv -1 (\mathrm{mod}\ p^k)$ has a solution, i.e. -1 is a quadratic residue mod p^k, when p is odd if and only if it has one for $k = 1$. It is shown that -1 is a quadratic residue mod p if and only if $p \equiv 1 (\mathrm{mod}\ 4)$. This completes the proof of the theorem. □

For a more thorough discussion of the topic of the last paragraph, see Section 5.8 of [L58].

3.10 Corollary. *If p is an odd prime, then $\mathbb{Z}_p[i]$ is a VNR ring.*

PROOF: If $p \equiv -1 (\mathrm{mod}\ 4)$, then $\mathbb{Z}_p[i]$ is a field because by Theorem 7.2 of [L58], the congruence $a^2 + b^2 \equiv 0 (\mathrm{mod}\ p)$ has no solution.

Assume next that $p \equiv 1 (\mathrm{mod}\ 4)$. It follows by Theorem 3.9 that $\mathbb{Z}_p[i]$ is not local, thus $\mathbb{Z}_p[i]$ (which has p^2 elements) is product of exactly two local rings, each isomorphic to $\mathbb{Z}_p$. Hence $\mathbb{Z}_p[i]$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ a product of two VNR rings. □

3.11 Corollary. *If $m = p^k$ for some odd prime p and positive integer k, then $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$ if and only if $k = 1$.*

PROOF: As noted in the proof of Theorem 3.9, $\mathfrak{M}(\mathbb{Z}_{2^k}[i]) = \{0\}$ for all k. By the last corollary, if p is an odd prime and $k = 1$, then $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$.

Now if $k > 1$ and $p \equiv -1 (\mathrm{mod}\ 4)$ or if $p = 2$, then by Theorem 3.9, $\mathbb{Z}_m[i]$ is a local ring which is not a field. So $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$ by Lemma 3.1.

If $k > 1$, $p \equiv 1 (\mathrm{mod}\ 4)$, and $a+ib$ is a nonunit of $\mathbb{Z}_m[i]$, then $a^2+b^2 \equiv 0 (\mathrm{mod}\ p)$. If $p \mid a$, or $p \mid b$, then p divides the other, so $p \mid (a+ib)$. Thus $a+ib$ is a nonzero nilpotent element of $\mathbb{Z}_m[i]$ since $k > 1$. If, instead p fails to divide a or b, then it is easy to verify that $p(a+ib)$ is a nonzero nilpotent in $\mathbb{Z}_m[i]$. Thus no nonzero nonunit of R can be m-regular, and the existence of the nonzero nilpotent ideal pR shows that no unit of $\mathbb{Z}_m[i]$ can be m-regular. Hence $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$ and the proof is complete. □

In summary we have using Theorem 3.3 and the above:

3.12 Corollary. *If $n = \prod_{i=1}^{s} p_i^{k_i}$ is the prime power decomposition of the positive integer n, then $\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}$ if and only if p_j is an odd prime and $k_j = 1$ for at least one $j \in \{1, \ldots, s\}$.*

4. Polynomial and power series rings

For each ring R, we write the *polynomial ring* as $R[x] = \{\sum_{i=0}^{n} a_i x^i : a_i \in R\}$ and the *power series ring* by $R[[x]] = \{ \sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$ where addition is coefficientwise, and in each case $(\sum a_i x^i)(\sum b_j x^j) = \sum c_k x^k$, where $c_k = \sum_{i+j=k} a_i b_j$. The coefficient of x^k in $c(x) = \sum c_k x^k$ is denoted by c_k. Both of these rings are commutative and have an identity. The next lemma is well known. See the first set of exercises in [AM69] and Section 1 of [B81].

4.1 Lemma. (a) *$u(x)$ is invertible in $R[x]$ if and only if u_0 is invertible and the coefficient of each nonzero power of x is nilpotent.*
(b) *$u(x)$ is invertible in $R[[x]]$ if and only if u_0 is invertible in R.*

Note that if $e^2 = e$ is an idempotent, then $(1-2e)^2 = 1$, so:

4.2 Lemma. *If e is an idempotent, then $(1-2e)$ is a unit of R.*

We combine these two lemmas to obtain:

4.3 Lemma. *If $a(x)$ is an idempotent in $R[x]$ or $R[[x]]$, then $a(x) = a_0 \in R$.*

PROOF: If $a(x) = \sum_{i=0}^{\infty} a_i x^i$ and $a(x) = (a(x))^2$, then $\sum_{i+j=n} a_i a_j = a_n$ for $n = 0, 1, 2, \dots$. If $n = 0$, then $a_0 = a_0^2$, so $(1-2a_0)$ is a unit by the last lemma. Equating coefficients of x yields $a_1(1-2a_0) = 0$, which implies that $a_1 = 0$. Doing the same with the coefficients of x^2 yields $a_2(1-2a_0) = -a_1 a_1 = 0$, which implies that $a_2 = 0$. Proceeding inductively, if $a_1 = a_2 = \cdots = a_{n-1} = 0$, then $a_n(1-2a_0) = -\sum_{i+j=n} a_i a_j = 0$. Thus $a_n = 0$ for each $n \geq 1$ and hence $a(x) = a_0 \in R$. □

We now characterize von Neumann regular elements in $R[x]$ and $R[[x]]$. In the proof of the next theorem, we need the fact that if a is a von Neumann regular element of a commutative ring, then there is unit u such that $a^2 u = a$, and hence that au is an idempotent. See, for example [AHA04].

4.4 Theorem. *Let $a(x) = \sum_{i=0}^{n} a_i x^i$. Then $a(x)$ is von Neumann regular in $R[x]$ if and only if $a(x)$ is a product of a von Neumann regular element in R and a unit in $R[x]$.*

PROOF: If $a(x) \in \mathrm{vr}(R[x])$, then there exists a unit $u(x) = \sum_{i=0}^{m} u_i x^i \in R[x]$ such that $a(x) = (a(x))^2 u(x)$. Hence by Lemmas 4.1 and 4.3, we have
(iii) $a(x)u(x) = a_0 u_0 = (a_0 u_0)^2$ and
(iv) $\sum_{i+j=k} a_i u_j = 0$ for $k = 1, 2, 3, \dots, n$.

By Lemma 4.1, u_j is nilpotent if $j \geq 1$ and by the equation in (iv) for $k = 1$, $a_1 = -u_0^{-1} a_0 u_1$, which implies that a_1 is nilpotent. Similarly, $a_2 = -u_0^{-1}(a_0 u_2 + a_1 u_1)$, which implies that a_2 is nilpotent. Proceeding inductively, if $a_1, a_2, \dots, a_{n-1}$ are nilpotents, then $a_n = -u_0^{-1} \sum_{i+j=n} a_i u_j$. So a_k is nilpotent for each $k \geq 1$, while $a_0 \in \mathrm{vr}(R)$ and $a(x) = a(x)a(x)u(x) = a(x)a_0u_0$. Let $v(x) = u_0 + a_1u_0^2x + a_2u_0^2x^2 + \cdots$ and note that it is a unit of $R[x]$ by Lemma 4.1. Then:

$$\begin{aligned} a(x) &= \sum_{i=0}^{n} a_i a_0 u_0 x^i = a_0^2 u_0 + a_1 a_0 u_0 x + a_2 a_0 u_0 x^2 + \cdots \\ &= a_0^2 u_0 + a_1 a_0^2 u_0^2 x + a_2 a_0^2 u_0^2 x^2 + \cdots = a_0^2 v(x) \end{aligned}$$

is the product of an element of $\mathrm{vr}(R)$ and a unit of $R[x]$.

The converse is clear. $\square$

A similar argument will establish:

4.5 Theorem. *If $a(x) = \sum_{i=0}^{\infty} a_i x^i$, then $a(x)$ is von Neumann regular in $R[[x]]$ if and only if $a(x)$ is a product of a von Neumann regular element in R and a unit in $R[[x]]$.*

By the last two theorems, $xa(x) \in \mathrm{vr}(R[x])$ implies $a(x) = \mathbf{0}$, so we conclude this section with:

4.6 Corollary. *For each ring R, $\mathfrak{M}(R[x]) = \{0\}$ and $\mathfrak{M}(R[[x]]) = \{0\}$.*

5. The ring $C(X)$

All topological spaces X are assumed to be Tychonoff spaces, βX the Stone-Čech compactification of X and $C(X)$ will denote the algebra of continuous real-valued functions under the usual pointwise operations. For each $f \in C(X)$, we denote the *zeroset* of f by $Z(f) = \{x \in X : f(x) = 0\}$, and the *cozeroset* $\mathrm{coz}(f) = X - Z(f)$. A point $p \in X$ such that for every $f \in C(X)$, $f(p) = 0$ implies $p \in \mathrm{int}\, Z(f)$ is called a *P-point*, and X is called a *P-space* if each of its points is a P-point. If $x \in \beta X$, let $M^x = \{f \in C(X) : x \in \mathrm{cl}_{\beta X} Z(f)\}$ and $O^x = \{f \in C(X) : x \in \mathrm{int}_{\beta X}[\mathrm{cl}_{\beta X} Z(f)]\}$. The notation and terminology of [GJ76] is used. In this section we will characterize m-regular elements in $C(X)$, we will find for what spaces X, $\mathfrak{M}(C(X))$ contains non zero elements.

Recall from Section 2 that R is a VNL ring if for each $a \in R$, one of a or $1 - a$ is von Neumann regular.

The next proposition is established in [AHA04] and in [GJ76].

5.1 Proposition. (a) *$C(X)$ is a VNR ring if and only if X is a P-space if and only if every G_δ-set of X is open.*

(b) *$C(X)$ is VNL ring if and only if at most one point of X is not a P-point (in which case X is said to be* essentially a P-space).

The next simple lemma will be used below.

5.2 Lemma. *If $f \in \mathrm{vr}(C(X))$, then $Z(f)$ is clopen.*

Proof: As is noted just above Theorem 4.4, there is a unit u in $C(X)$ such that $f = f(fu)$ and fu is idempotent. Because the zeroset of an idempotent is clopen, the conclusion follows. □

Thus we obtain:

5.3 Theorem. *A function f is in $\mathfrak{M}(C(X)) \setminus \{0\}$ if and only if $\mathrm{coz}(f)$ is a nonempty clopen P-space.*

Proof: Suppose that $f \in \mathfrak{M}(C(X)) \setminus \{0\}$, then $f \in \mathrm{vr}(C(X))$ and so $\mathrm{coz}(f)$ is a nonempty clopen set by Lemma 5.2. Let $G = \bigcap_{n=1}^{\infty} G_n$ be a G_δ-set of X contained in $\mathrm{coz}(f)$ and suppose $x \in G$. For each n there exists $g_n \in C(X)$ such that $g_n(x) = 0$ and $g_n(X \setminus G_n) = 1$. Let $g = \sum_{n=1}^{\infty}(|g_n|/2^n)$, then $g \in C(X)$ and $Z(g) = G \subset \mathrm{coz}(f)$. Since $fg \in \mathrm{vr}(C(X))$, its zeroset is clopen by Lemma 5.2. So, because $Z(fg) = Z(f) \cup Z(g)$, $Z(f) \cap Z(g) = \varnothing$, and $Z(f)$ is clopen, it follows that $Z(g)$ and hence $\mathrm{coz}(g)$ is clopen. Thus, by Proposition 5.1, $\mathrm{coz}(f)$ is a P-space.

Suppose conversely that $\mathrm{coz}(f)$ is a nonempty clopen P-space. Then $C(X)$ is the direct product of $C(\mathrm{coz}(f))$ and $C(Z(f))$, so $f \in \mathfrak{M}(C(X)) \setminus \{0\}$. □

5.4 Corollary. *$\mathfrak{M}(C(X)) \neq \{0\}$ if and only if X contains a nonempty clopen P-space.*

By making use of Theorem 1.3, we can describe $\mathfrak{M}(C(X))$ more precisely.

If Y is a subset of X, we let $O^Y = \bigcap_{y \in Y} O^y$. Let $P(X)$ be the set of all P-points in X, then it is clear that $O^{X-P(X)} = \bigcap_{y \notin P(X)} O^y \subseteq \mathrm{vr}(C(X))$ and so, $O^{X-P(X)} \subseteq \mathfrak{M}(C(X))$. For each $x \in \beta X$, $mM^x = O^x$, using this together with Theorem 1.3 above we conclude that:

5.5 Corollary. *$\mathfrak{M}(C(X)) = O^{X-P(X)}$ for any space X.*

We conclude with an interesting example.

5.6 Example. Let $X_1 = (0,1)$ with its usual topology and $X_2 = \mathbb{N}$ with its discrete topology. Let $X = X_1 \bigoplus X_2$ and define $f(x) = \left\{ \begin{smallmatrix} 0 & x \in X_1 \\ 1 & x \in X_2 \end{smallmatrix} \right.$, then $f \in \mathfrak{M}(C(X)) \setminus \{0\}$, while $C(X)$ is not a VNR ring.

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