G_δ-modification of compacta and cardinal invariants

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Abstract. Given a space X, its G_δ-subsets form a basis of a new space X_ω, called the G_δ-modification of X. We study how the assumption that the G_δ-modification X_ω is homogeneous influences properties of X. If X is first countable, then X_ω is discrete and, hence, homogeneous. Thus, X_ω is much more often homogeneous than X itself. We prove that if X is a compact Hausdorff space of countable tightness such that the G_δ-modification of X is homogeneous, then the weight $w(X)$ of X does not exceed 2^ω (Theorem 1). We also establish that if a compact Hausdorff space of countable tightness is covered by a family of G_δ-subspaces of the weight $\leq c = 2^\omega$, then the weight of X is not greater than 2^ω (Theorem 4). Several other related results are obtained, a few new open questions are formulated. Fedorchuk's hereditarily separable compactum of the cardinality greater than $c = 2^\omega$ is shown to be G_δ-homogeneous under CH. Of course, it is not homogeneous when given its own topology.

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Let $\mathcal{T}$ be a topology on a set X. Then the family of all G_δ-subsets of X is a base of a new topology on X, denoted by $\mathcal{T}_\omega$ and called the *G_δ-modification* of $\mathcal{T}$. The space $(X, \mathcal{T}_\omega)$ is also denoted by X_ω and is called the *G_δ-modification* of the space $(X, \mathcal{T})$. Clearly, the G_δ-modification X_ω of any topological space is a P-space, that is, every G_δ-subset of X_ω is open in X_ω.

In general, the space $(X, \mathcal{T}_\omega)$ is very different from the space $(X, \mathcal{T})$. Many properties of $(X, \mathcal{T})$, such as compactness, Lindelöfness, paracompactness are easily lost under G_δ-modifications. On the other hand, properties of the space can greatly improve under the operation of G_δ-modification. For example, if $(X, \mathcal{T})$ is first countable, then the space $(X, \mathcal{T}_\omega)$ is discrete. Thus, no matter which first countable space $(X, \mathcal{T})$ we take, the resulting space $(X, \mathcal{T}_\omega)$ will be metrizable, zero-dimensional, Čech-complete and homogeneous! We see that the difference in properties between the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}_\omega)$ can indeed be tremendous!

Some interesting facts on G_δ-modifications and on P-spaces were established in [12], where also a survey of what is known in this direction is given. See also [11].

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It is our goal in this article to show that homogeneity of G_δ-modification has a deep influence on the structure of the space itself and on the relationship between its cardinal invariants. Our main result in this direction (Theorem 1 below) is inspired by R. de la Vega's recent result that the weight of any homogeneous compact Hausdorff space of countable tightness is $\leq 2^\omega$. We generalize de la Vega's theorem as follows:

Theorem 1. *Let X be a compact Hausdorff space of countable tightness such that the G_δ-modification X_ω of X is homogeneous. Then the weight $w(X)$ of X, as well as the weight of X_ω, is not greater than 2^ω.*

PROOF: We claim that there is a non-empty open subspace U of X_ω such that $w(U) \leq 2^\omega$. Indeed, since X is a non-empty compact Hausdorff space of countable tightness, there exists a non-empty G_δ-subset U of X such that the weight of the subspace U of X is not greater than 2^ω ([2], [1]). Then U is an open subspace of X_ω and the weight of the subspace U of X_ω is also not greater than 2^ω. Since X_ω is homogeneous, it follows that every point in X_ω has an open neighbourhood Ox in X_ω such that $w(Ox) \leq 2^\omega$.

According to a result of E.G. Pytkeev [14], the Lindelöf degree of the G_δ-modification of any compact Hausdorff space of countable tightness does not exceed 2^ω (see Theorem 4 in [14]). Therefore, $l(X_\omega) \leq 2^\omega$. Since the local weight of X_ω does not exceed 2^ω, it follows that there exists an open covering γ of X_ω such that $w(U) \leq 2^\omega$, for each $U \in \gamma$, and $|\gamma| \leq 2^\omega$. Fixing a base of cardinality $\leq 2^\omega$ in each $U \in \gamma$, and taking the union of these bases, we obtain a base of cardinality $\leq 2^\omega$ in X_ω. Thus, $w(X_\omega) \leq 2^\omega$. Since, X is a continuous image of X_ω, we have $nw(X) \leq w(X_\omega) \leq 2^\omega$. However, since X is compact, $w(X) = nw(X) \leq 2^\omega$ ([9]). $\square$

This theorem immediately implies that the cardinality of every first countable compact Hausdorff space does not exceed 2^ω [Arh2]. Indeed, the tightness of first countable spaces is countable, and, obviously, if the weight of a first countable Hausdorff space is $\leq 2^\omega$, then the cardinality of X is also not greater than 2^ω. Theorem 1 also implies de la Vega's result that the weight of any homogeneous compact Hausdorff space of countable tightness is $\leq 2^\omega$, since the G_δ-modification of a homogeneous space is homogeneous.

A space Y is *power-homogeneous* if Y^τ is homogeneous, for some $\tau > 0$ (see [4]). Weakening one of the assumptions in Theorem 1, we arrive at a weaker conclusion:

Theorem 2. *Let X be a compact Hausdorff space of countable tightness such that the G_δ-modification of X is power-homogeneous. Then the character of X is not greater than 2^ω.*

PROOF: Take any non-empty G_δ-subset Y of X. There exists a non-empty G_δ-subset U of Y such that the weight of the subspace U of the space X is not greater than 2^ω ([2], [1]). Then U is an open subspace of X_ω and the weight of the subspace U of X_ω is also not greater than 2^ω. It follows that the set Z of all $x \in X$ such that the character of x in X_ω is not greater than 2^ω is dense in the space X_ω. Since X_ω is power-homogeneous and $Z \neq \emptyset$, it follows from Theorem 7 in [4] that the set M of all G_c-points in X_ω is closed. Obviously, $Z \subset M$. Therefore, $M = X$; thus, each $x \in X$ is a G_c-point in X_ω.

Fix an arbitrary $a \in X$. According to Pytkeev's theorem (see the proof of Theorem 1), the Lindelöf degree of X_ω is not greater than $c = 2^\omega$. Put $A = X \setminus \{a\}$. Since a is a G_c-point in X_ω, it follows that $l(A) \leq 2^\omega$, where A is considered as a subspace of X_ω. Since the identity mapping of X_ω onto X is continuous, we conclude that the Lindelöf degree of A, considered as a subspace of X, does not exceed 2^ω as well. This implies that a is a G_c-point in X. Since X is compact and Hausdorff, it follows that the character of X at a is not greater than 2^ω ([9]). □

Theorem 3. *Let X be a sequential Hausdorff compact space such that the G_δ-modification of X is power-homogeneous. Then $|X| \leq 2^\omega$.*

Proof: It follows from Theorem 2 that $\chi(X) \leq 2^\omega$. However, the cardinality of every sequential Hausdorff compact space such that $\chi(X) \leq 2^\omega$ does not exceed 2^ω (see [2]). □

The last result generalizes Corollary 3.8 in [5] and an earlier result on the cardinality of homogeneous compact sequential spaces in [2].

The technique of G_δ-modification can be used to obtain some addition theorems for the weight that do not involve the assumption of homogeneity. In particular, we have:

Theorem 4. *Let X be a compact Hausdorff space of countable tightness, and suppose that X is covered by a family γ of G_δ-subsets such that the weight of P is not greater than 2^ω, for each $P \in \gamma$. Then the weight of X is not greater than 2^ω.*

Proof: The proof is close to the proof of Theorem 1. Consider the G_δ-modification X_ω of X. The family γ is an open covering of X_ω, and the weight of each $P \in \gamma$, interpreted as a subspace of X_ω, is not greater than 2^ω. By Pytkeev's theorem (see the proof of Theorem 1), the Lindelöf degree of X_ω is not greater than $c = 2^\omega$. Therefore, the weight of X_ω is not greater than 2^ω (to get an appropriate base of X_ω, just take the union of the bases of cardinality $\leq 2^\omega$ of elements of γ). Since X is a continuous image of X_ω, we have $nw(X) \leq w(X_\omega) \leq 2^\omega$. However, X is compact. Hence, $w(X) = nw(X) \leq 2^\omega$. □

For some results related to Theorem 4 see [15] and [6].

The assumption of countable tightness in the last statement can be replaced by some other conditions.

Theorem 5. *Let X be a scattered compact Hausdorff space covered by a family γ of G_δ-subsets such that the weight of P is not greater than 2^ω, for each $P \in \gamma$. Then the weight of X does not exceed 2^ω.*

Proof: The Lindelöf degree of the G_δ-modification X_ω of the space X does not exceed ω ([13]). Since γ is an open covering of X_ω, we can assume that γ is countable. It follows that $w(X_\omega) \leq 2^\omega$, which implies that $nw(X) \leq w(X_\omega) \leq 2^\omega$. Finally, since X is compact, we have $w(X) = nw(X) \leq 2^\omega$. □

The proof of the next result should be clear by now:

Theorem 6. *Let X be a scattered space. Then the G_δ-modification X_ω of X is power-homogeneous if and only if the pseudocharacter of X is countable* (*that is, if and only if the G_δ-modification of X is discrete*).

Problem 7. *Suppose that X is a compact Hausdorff space covered by a family γ of G_δ-subsets P such that the weight of P is not greater than 2^ω, for each $P \in \gamma$. Is the weight of X not greater than 2^ω?*

Problem 8 (Arhangel'skii, Buzyakova)**.** *Let X be a compact Hausdorff space of countable tightness such that the character of X does not exceed 2^ω. Is the weight of X not greater than 2^ω?*

Consistently the answer to the last question is "yes". Indeed, it was shown in [7] to be consistent with ZFC to assume that every compact Hausdorff space of countable tightness is sequential. It remains to apply the following result from [2]: the cardinality of every sequential Hausdorff compact space such that $\chi(X) \leq 2^\omega$ does not exceed 2^ω.

Closely related to Problem 8 is the following question: Let X be a compact Hausdorff space of countable tightness such that the G_δ-modification of X is homogeneous. Is $|X| \leq 2^\omega$? The answer to this question is independent of ZFC. Under Proper Forcing Axiom (PFA) (for the discussion of (PFA) see [8]) the answer is "yes". In fact, we can prove a stronger statement:

Theorem 9. *Assume* (*PFA*)*, and let X be a Hausdorff compact space of countable tightness such that the G_δ-modification of X is power-homogeneous. Then X is first countable* (*and hence, $|X| \leq 2^\omega$ and $w(X) \leq 2^\omega$*).

Proof: A. Dow has shown in [Dow] that under (PFA) every non-empty compact Hausdorff space of countable tightness has a point of first countability. It follows easily from this result that, under (PFA), the set of isolated points is dense in the G_δ-modification X_ω of the compactum X.

Since X_ω is power-homogeneous, it follows from Theorem 7 in [4] that the set M of all G_δ-points in X_ω is closed. Therefore, $M = X$, that is, each $x \in X$ is a G_δ-point in X_ω. Since X_ω is a P-space, we conclude that the space X_ω is discrete. Hence, the pseudocharacter of the space X is countable. Since X is compact and Hausdorff, it follows that X is first countable. □

On the other hand, we have the following result:

Theorem 10 (CH)**.** *Let X be a hereditarily separable compact Hausdorff space without points of first countability. Then the G_δ-modification of X is homogeneous.*

This theorem will follow from a more general result below. Notice that Fedorchuk has constructed [10] a consistent example of a hereditarily separable, nowhere first countable, compact Hausdorff space X such that the cardinality of X is greater than 2^ω. In the model of Set-theory he considered (CH) was also satisfied.

Theorem 11 (CH)**.** *Let X be a compact Hausdorff space of the weight ω_1 such that the character of X at each point is exactly ω_1. Then the G_δ-modification X_ω of X is homeomorphic to the G_δ-modification of the compactum D^{ω_1}.*

Fix a set A of the cardinality $\omega_1 = c = 2^\omega$, give A the discrete topology, and let B be the G_δ-modification of the product space A^{ω_1}.

Claim 1: The G_δ-modification of D^{ω_1} is homeomorphic to the space B.

This is obvious.

By Claim 1, it is enough to prove that X_ω is homeomorphic to B. For that, we need the following lemma:

Lemma 12. *Let X be a non-scattered compact Hausdorff space. Then there exists a disjoint covering γ of X by non-empty closed G_δ-sets such that $|\gamma| = 2^\omega$.*

Proof: Since X is not scattered, there exists a continuous mapping f of X onto the closed interval $I = [0,1]$ (see [9]). Then $\gamma = \{f^{-1}(y) : 0 \le y \le 1\}$ is, clearly, the covering we are looking for. □

Below we will need the following slightly stronger version of Lemma 12:

Lemma 13. *Let X be a non-scattered compact Hausdorff space and F_0 be a closed G_δ-subset of X. Then there exists a disjoint covering γ_1 of X by non-empty closed G_δ-sets such that $|\gamma_1| = 2^\omega$ and $F_0 = \bigcup \eta$, for some subfamily η of γ_1.*

Proof: We can fix a continuous real-valued function g on X such that $g^{-1}(0) = F_0$, since X is normal. Take also a disjoint covering γ of X by closed G_δ-subsets such that $|\gamma| = 2^\omega$ (this is possible by Lemma 12). Now let γ_1 be the family $\{g^{-1}(a) \cap P : a \in \mathbb{R}, P \in \gamma\} \setminus \{\emptyset\}$, where $\mathbb{R}$ is the set of reals. Obviously, γ_1 is the covering we are looking for. □

Proof of Theorem 11: A standard construction by transfinite recursion along ω_1, using (CH) and Lemmas 12 and 13, provides us with a transfinite sequence $\{\gamma_\alpha : \alpha < \omega_1\}$ of disjoint coverings of X by closed non-empty G_δ-subsets of X such that the following conditions are satisfied:

1) γ_β refines γ_α, whenever $\alpha < \beta < \omega_1$;

2) for each $P \in \gamma_\alpha$, the cardinality of the family $\eta_P = \{F \in \gamma_{\alpha+1} : F \subset P\}$ is ω_1;
3) the family $S = \bigcup\{\gamma_\alpha : \alpha < \omega_1\}$ is a network of the space X.

Observe that compactness of X and the above conditions ensure that the following condition is satisfied:

4) for every uncountable centered family ξ of elements of S, the intersection of ξ consists of exactly one point x_ξ, ξ is a network of X at x, and ξ is a base of the G_δ-modification X_ω at x.

Note, that elements of S are open-closed subsets of X_ω, and that if $\xi \subset S$ is countable, then either $\bigcap \xi = \emptyset$ or the cardinality of $\bigcap \xi$ is $c = \omega_1$.

The above properties of the family $\{\gamma_\alpha : \alpha < \omega_1\}$ allow to establish a homeomorphism between the space X_ω and the space B in an obvious routine way. □

Corollary 14 (CH). *Let X be a compact Hausdorff space of the weight ω_1 such that the character of X at each point is exactly ω_1. Then the G_δ-modification X_ω of X is homogeneous. Furthermore, X_ω is homeomorphic to a topological group.*

PROOF: Indeed, by Theorem 11 X_ω is homeomorphic to the G_δ-modification B of the compactum D^{ω_1}. However, the space B is homogeneous, since D^{ω_1} is homogeneous. Hence, X_ω is homogeneous as well. In fact, B is homeomorphic to a topological group, since D^{ω_1} is a topological group. □

Problem 15. *Can (CH) be dropped in the above statement?*

The following long standing problems posed in [3], [1], [2] remain open:

Problem 16. *Is it true in ZFC that every homogeneous compact sequential space is first countable?*

Problem 17. *Is it true in ZFC that every homogeneous compact space of countable tightness is first countable?*

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