Comment.Math.Univ.Carolin. 47,1 (2006)35–46

Two weight norm inequalities for fractional one-sided maximal and integral operators

Liliana de Rosa

Abstract. In this paper, we give a generalization of Fefferman-Stein inequality for the fractional one-sided maximal operator:

$$\int_{-\infty}^{+\infty} M_\alpha^+(f)(x)^p w(x)\,dx \le A_p \int_{-\infty}^{+\infty} |f(x)|^p M_{\alpha p}^-(w)(x)\,dx,$$

where $0 < \alpha < 1$ and $1 < p < 1/\alpha$. We also obtain a substitute of dual theorem and weighted norm inequalities for the one-sided fractional integral I_α^+.

Keywords: one-sided fractional operators, weighted inequalities

Classification: Primary 26A33; Secondary 42B25

1. Introduction

For each $0 < \alpha < 1$ and f locally integrable on the real line $\mathbb{R}$ the fractional one-sided maximal operators are defined by

$$M_\alpha^+(f)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(y)|\,dy \text{ and } M_\alpha^-(f)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x} |f(y)|\,dy.$$

In the case $\alpha = 0$ we have $M_0^+ = M^+$ and $M_0^- = M^-$ the one-sided maximal Hardy-Littlewood operators.

The fractional one-sided integral operators are defined by

$$I_\alpha^+(f)(x) = \int_x^{+\infty} \frac{f(y)}{(y-x)^{1-\alpha}}\,dy \quad\text{and}\quad I_\alpha^-(f)(x) = \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}}\,dy.$$

For each x in $\mathbb{R}$ we consider the family of intervals $A_x = \{I = [a,b) : I \text{ is dyadic and } 0 < a - x \le b - a\}$. For each locally integrable function f and $0 < \alpha < 1$, its one-sided dyadic fractional maximal operator is given by

$$M_{\alpha,D}^+(f)(x) = \sup\left\{ \frac{1}{|I|^{1-\alpha}} \int_I |f| : I \in A_x \right\}.$$

Similarly, $M^-_{\alpha,D}(f)$ was introduced.

By Proposition 2.5 in [7] for each $0 < \alpha < 1$, there exist two constants P_α and Q_α such that

$$Q_\alpha\, M^+_{\alpha,D}(f)(x) \le M^+_\alpha(f)(x) \le P_\alpha\, M^+_{\alpha,D}(f)(x). \tag{1.1}$$

Let X be a Banach function space on $\mathbb{R}$. We recall that generalized Hölder inequality

$$\int_{\mathbb{R}} |f(y)g(y)|\, d\mu(y) \le \|f\|_X \|g\|_{X'} \tag{1.2}$$

holds, where X' is the associated space.

The X-average of a measurable function f over a bounded interval I is given by

$$\|f\|_{X,I} = \|\delta_{|I|}(f\chi_I)\|_X,$$

where δ_s is the dilation operator $\delta_s f(x) = f(sx)$, $s > 0$.

As a consequence of (1.2) we have that for every interval I the inequality

$$\frac{1}{|I|}\int_I |f(y)g(y)|\, d\mu(y) \le \|f\|_{X,I}\|g\|_{X',I} \tag{1.3}$$

holds. The one-sided maximal Hardy-Littlewood operators associated to X were defined by

$$M^+_X f(x) = \sup_{b>x} \|f\|_{X,(x,b)} \quad\text{and}\quad M^-_X f(x) = \sup_{a<x} \|f\|_{X,(a,x)}.$$

We refer the reader to [1] for a complete study of Banach function spaces.

Given an interval $I = [a, b)$ we will denote by I^- the interval $[a - (b - a), a)$. If $p > 1$ its conjugate exponent will be denoted by p'.

A weight w is a non negative and locally integrable function defined on $\mathbb{R}$.

The following theorem gives us a weak type boundedness for the one-sided dyadic fractional maximal operator $M^+_{\alpha,D}$ with respect to a pair of weights. It will be proved in Section 2.

Theorem 1.1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let X be a Banach function space satisfying the following property: there exists a constant $C > 0$ such that for every dyadic interval $J = [b, c)$ and each $y \in J^-$ the inequality*

$$\|f\|_{X,J} \le C\|f\|_{X,(y,c)} \tag{1.4}$$

holds, and the operator $M_X^+ : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ *is bounded, that is, there exists a constant* C_p *such that for every* f

$$\|M_X^+(f)\|_p \le C_p\|f\|_p.$$

Suppose that the pair of weights (w, v) *satisfies the condition*

$$|J|^\alpha \left[\frac{1}{|J|}\ w^p(J^-)\right]^{1/p} \|v^{-1}\|_{X',J} \le K \tag{1.5}$$

for every dyadic interval J.

Then, if for every $t > 0$ *we denote*

$$E_t = \{x : M^+_{\alpha,D}(f)(x) > t\}$$

we have that,

$$w^p(E_t) \le \frac{2K^pC_pC}{t^p}\int_{-\infty}^{+\infty} |f(y)|^p v(y)^p\, dy.$$

In this paper, every theorem has a corresponding one reversing the orientation of the real line.

For each $0 \le \alpha < n$, we consider the maximal operator

$$M_\alpha(f)(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes and $|Q|$ denotes its Lebesgue measure. The inequality

$$\int_{\mathbb{R}^n} M_\alpha(f)(x)^p w(x)\, dx \le A_p \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p}(w)(x)\, dx,$$

where $1 < p < n/\alpha$ and w is any weight, for $\alpha = 0$ was obtained by C. Fefferman and E.M. Stein in [3] and for $0 < \alpha < 1$ was proved by D. Cruz-Uribe, in Theorem 1.7 of [2]. We study the one-sided problem and give a proof of the following result in Section 2.

Theorem 1.2. *Let* $0 \le \alpha < 1$ *and* $1 < p < 1/\alpha$. *There exists a constant* A_p *such that for every weight* w *the inequality*

$$\int_{-\infty}^{+\infty} M_\alpha^+(f)(x)^p w(x)\, dx \le A_p \int_{-\infty}^{+\infty} |f(x)|^p M^-_{\alpha p}(w)(x)\, dx$$

holds, for every measurable function f *and every weight* w.

The one-sided fractional maximal operator M_α^+ is not a linear operator. As a dual version of Theorem 1.2 we will prove the following result in Section 3.

Theorem 1.3. *Let $1 < p < \infty$ and $0 < \alpha < 1/p'$. There exists a constant $C > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} M_\alpha^+(f)(x)^p \, [M_{\alpha p'}^+(M^{[p']}w)(x)]^{1-p} \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p \, w(x)^{1-p} \, dx$$

holds, for every measurable function f and every weight w where $M^{[p']}$ is the maximal Hardy-Littlewood operator iterated $[p']$ times.

For the one-sided fractional integral operator I_α^+ we have the following weighted norm inequality which will be proved in Section 3.

Theorem 1.4. *Let $1 < p < \infty$ and $0 < \alpha < 1/p'$. There exists a constant $C > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} |I_\alpha^+(f)(x)|^p \, [M_{\alpha p'}^+(M^{[p']}w)(x)]^{1-p} \, dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p \, w(x)^{1-p} \, dx$$

holds, for every measurable function f and every weight w where $M^{[p']}$ is the maximal Hardy-Littlewood operator iterated $[p']$ times.

Throughout this paper, the letters A, B and C will denote positive constants, not necessarily the same at each occurrence.

2. Proofs of Theorem 1.1 and Theorem 1.2

The following proposition is a fractional version of Calderon-Zygmund decomposition. It will be applied in the proof of Theorem 1.1.

Proposition 2.1. *Let f belong to $L^1(\mathbb{R})$, $0 < \alpha < 1$ and $t > 0$. There exists a countable family $\{J_k\}_{k \geq 1}$ of dyadic disjoint intervals such that for every $k \geq 1$*

$$t < \frac{1}{|J_k|^{1-\alpha}} \int_{J_k} |f| \leq 2^{1-\alpha} t.$$

Moreover,

$$E_t = \{x : M_{\alpha,D}^+(f)(x) > t\} = \Omega^- \cup A,$$

where

$$\Omega^- = \bigcup_{k \geq 1} J_k^- \quad \text{and} \quad A = \bigcup_{k \geq 1} A_k$$

with $A_k = (E_t \backslash \Omega^-) \cap J_k$ and for each x in A_k there exists a dyadic interval I_j satisfying

$$I_j^- \cup I_j \subseteq J_k, \qquad x \in I_j^- \quad \text{and} \quad t < \frac{1}{|I_j|^{1-\alpha}} \int_{I_j} |f|.$$

PROOF: Let $\mathcal{D} = \{I = [a,b) : I \text{ is dyadic}\}$. Given an interval I in $\mathcal{D}$ such that

$$t < \frac{1}{|I|^{1-\alpha}} \int_I |f| \tag{2.1}$$

we have that

$$|I| < \left(\frac{\|f\|_1}{t}\right)^{\frac{1}{1-\alpha}},$$

hence, the measure $|I|$ is finite and there exist maximal dyadic intervals satisfying (2.1). Let

$$C_t = \left\{ J \in \mathcal{D} : J \text{ is maximal with the property } t < \frac{1}{|J|^{1-\alpha}} \int_J |f| \right\}.$$

Let J belong to C_t. There exists an interval $H \in \mathcal{D}$ such that $J \subset H$ and $|H| = 2|J|$. Taking into account that J is maximal with respect to the property (2.1) then $H \notin C_t$ and,

$$t < \frac{1}{|J|^{1-\alpha}} \int_J |f| \leq \frac{2^{1-\alpha}}{|H|^{1-\alpha}} \int_H |f| \leq 2^{1-\alpha} t.$$

Since the family of dyadic intervals $\mathcal{D}$ is countable we can denote $C_t = \{J_k\}_{k \geq 1}$. By the definition of $M^+_{\alpha,D}$ we have that $\Omega^- \cup A \subseteq E_t$.
We shall prove that

$$E_t \subseteq \Omega^- \cup A$$

where

$$\Omega^- = \bigcup_{k \geq 1} J_k^- \quad \text{and} \quad A = \bigcup_{k \geq 1} A_k \quad \text{with} \quad A_k = (E_t \backslash \Omega^-) \cap J_k.$$

Suppose that $x \in E_t$ and $x \notin \Omega^-$. We shall prove that $x \in A_k$ for some $k \geq 1$. Since $x \in E_t$, there exists an interval $I \in \mathcal{D}$ such that

$$x \in I^- \quad \text{and} \quad t < \frac{1}{|I|^{1-\alpha}} \int_I |f|$$

and the definition of C_t implies that $I \subseteq J_k$ for some $k \geq 1$.

It must be $I \neq J_k$, because if $I = J_k$ then $x \in J_k^-$ and $x \notin \Omega^-$. Thus, $I \neq J_k$ which implies that $I^- \subset J_k^-$ or $I^- \subset J_k$. Necessarily $I^- \subset J_k$, because in the other case $x \in J_k^-$ and $x \notin \Omega^-$, a contradiction. In consequence, $I^- \cup I \subseteq J_k$.

Since the family of dyadic intervals is countable, there exists a sequence $\{I_j\}_{j\geq 1}$ of disjoint dyadic intervals satisfying

$$A_k = \bigcup_{j\geq 1} I_j^-, \qquad I_j^- \cup I_j \subseteq J_k \quad \text{and} \quad t < \frac{1}{|I_j|^{1-\alpha}} \int_{I_j} |f|.$$

□

Proof of Theorem 1.1: By a standard argument it will be sufficient to consider bounded functions f with compact support. Applying Proposition 2.1

$$E_t = \Omega^- \cup A$$

where

$$\Omega^- = \bigcup_{k\geq 1} J_k^- \quad \text{and} \quad A = \bigcup_{k\geq 1} A_k$$

with $A_k = (E_t \backslash \Omega^-) \cap J_k$.

For each $k \geq 1$ by the inequality (3.1), condition (1.5) and hypothesis (1.4) we have that

$$\begin{aligned}
w^p(J_k^-) &< \frac{w^p(J_k^-)}{t^p} \frac{1}{|J_k|^{(1-\alpha)p}} \left[\int_{J_k} |f|\right]^p \\
&= \frac{w^p(J_k^-)}{t^p} |J_k|^{\alpha p} \left[\frac{1}{|J_k|} \int_{J_k} |f| v v^{-1}\right]^p \\
&\leq \frac{w^p(J_k^-)}{t^p} |J_k|^{\alpha p} \, \|f v \chi_{J_k}\|^p_{X,J_k} \, \|v^{-1}\|^p_{X',J_k} \\
&\leq \frac{K^p}{t^p} |J_k| \, \|f v \chi_{J_k}\|^p_{X,J_k} \\
&\leq \frac{K^p}{t^p} \int_{J_k^-} \|f v \chi_{J_k}\|^p_{X,J_k} \, dy \\
&\leq \frac{K^p C^p}{t^p} \int_{J_k^-} M_X^+(f v \chi_{J_k})(y)^p \, dy.
\end{aligned}$$

Taking into account that the operator M_X^+ is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, we obtain

$$w^p(J_k^-) \leq \frac{K^p C_p C^p}{t^p} \int_{J_k} |f|^p v^p.$$

In consequence,

$$w^p(\Omega^-) \leq \sum_{k\geq 1} w^p(J_k^-) \leq \frac{K^p C_p C^p}{t^p} \int_{\bigcup_{k\geq 1} J_k} |f|^p v^p. \tag{2.2}$$

By Proposition 2.1, for each $k \geq 1$ it follows that

$$A_k = \bigcup_{j\geq 1} I_j^-,$$

where

$$t < \frac{1}{|I_j|^{1-\alpha}} \int_{I_j} |f| \quad \text{and} \quad I_j^- \cup I_j \subseteq J_k$$

for every $j \geq 1$. Then,

$$\begin{aligned} w^p(A_k) &\leq \sum_{j\geq 1} w^p(I_j^-) \\ &\leq \frac{1}{t^p} \sum_{j\geq 1} w^p(I_j^-) \left[\frac{1}{|I_j|^{1-\alpha}} \int_{I_j} |f|\right]^p \\ &= \frac{1}{t^p} \sum_{j\geq 1} w^p(I_j^-)|I_j|^{\alpha p} \left[\frac{1}{|I_j|} \int_{I_j} |f|vv^{-1}\right]^p. \end{aligned}$$

By the inequality (1.3), condition (1.5), hypothesis (1.4) and keeping in mind that $\{I_j^-\}_{j\geq 1}$ is a family of disjoint dyadic intervals contained in J_k,

$$\begin{aligned} w^p(A_k) &\leq \frac{1}{t^p} \sum_{j\geq 1} w^p(I_j^-)|I_j|^{\alpha p} \|fv\chi_{J_k}\|^p_{X,I_j} \|v^{-1}\|^p_{X',I_j} \\ &\leq \frac{K^p}{t^p} \sum_{j\geq 1} |I_j| \|fv\chi_{J_k}\|^p_{X,I_j} \\ &\leq \frac{K^p}{t^p} \sum_{j\geq 1} \int_{I_j^-} \|fv\chi_{J_k}\|^p_{X,I_j}\, dy \\ &\leq \frac{K^pC^p}{t^p} \sum_{j\geq 1} \int_{I_j^-} M_X^+(fv\chi_{J_k})(y)^p\, dy \\ &\leq \frac{K^pC^p}{t^p} \int_{J_k} M_X^+(fv\chi_{J_k})(y)^p\, dy. \end{aligned}$$

Since M_X^+ is bounded from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ and $\{J_k\}_{k\geq 1}$ is a family of disjoint dyadic intervals,

$$w^p(A) = \sum_{k\geq 1} w^p(A_k) \leq \frac{K^pC_pC^p}{t^p} \int_{\bigcup_{k\geq 1} J_k} |f(y)|^p v(y)^p\, dy.$$

Then, by (2.2)

$$w^p(E_t) \le w^p(\Omega^-) + w^p(A) \le \frac{2K^pC_pC^p}{t^p} \int_{\bigcup_{k\ge 1} J_k} |f(y)|^p v(y)^p \, dy.$$

□

As a consequence of Theorem 1.1 we obtain the next two corollaries.

Corollary 2.2. *Let $1 \le r < p < \infty$, $0 < \alpha < 1$ and assume that the pair of weights (w, v) satisfies the following condition: there exists a constant K such that for every dyadic interval J,*

$$|J|^\alpha \left[\frac{1}{|J|} \, w^p(J^-)\right]^{1/p} \left[\frac{1}{|J|} \int_J v^{-r'}\right]^{1/r'} \le K. \tag{2.3}$$

Then, for every $t > 0$ we have

$$w^p\left(\left\{x : M^+_{\alpha,D}(f)(x) > t\right\}\right) \le \frac{2^{1+\frac{p}{r}} K^p C_{p/r}}{t^p} \int_{-\infty}^{+\infty} |f(x)|^p v(x)^p \, dx,$$

where $C_{p/r}$ is the constant of the strong type $(p/r, p/r)$ of the one-sided maximal Hardy-Littlewood operator M^+.

Proof: Suppose that X is the Orlicz space defined by the Young function $B(t) = t^r$, its associated space X' is given by $\overline{B}(t) \approx t^{r'}$. Since $1 \le r < p < \infty$ then $M^+_X = M^+_r : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ is bounded. Taking into account that

$$\|v^{-1}\|_{X',J} = \left[\frac{1}{|J|} \int_J v^{-r'}\right]^{1/r'}$$

holds for every dyadic interval J, the pair of weights (w, v) satisfies the condition (1.5). □

Corollary 2.3. *Let $1 < p < 1/\alpha$ and w be a weight. Then, for every measurable function f and every $t > 0$ we have that*

$$w\left(\left\{x : M^+_{\alpha,D}(f)(x) > t\right\}\right) \le \frac{B_p}{t^p} \int_{-\infty}^{+\infty} |f(x)|^p M^-_{\alpha p}(w)(x) \, dx$$

where $B_p = 2^{2+p-\alpha p} C_p$ and C_p is the constant of the strong type (p, p) of the one-sided maximal Hardy-Littlewood operator M^+.

Proof: Let $r = 1$. Given a dyadic interval $J = [b, c)$ if $J^- = [a, b)$ for each $x \in J$ we have that

$$\begin{aligned} M^-_{\alpha p}(w)(x) &= \sup_{h>0} \frac{1}{h^{1-\alpha p}} \int_{x-h}^{x} w(y)\, dy \\ &\geq \frac{1}{(2|J|)^{1-\alpha p}} \int_a^b w(y)\, dy = \frac{1}{2^{1-\alpha p}} \frac{1}{|J|^{1-\alpha p}} w(J^-). \end{aligned}$$

Thus,

$$\begin{aligned} &|J|^\alpha \left[\frac{1}{|J|}\, w(J^-) \right]^{1/p} \|M^-_{\alpha p}(w)^{-1/p} \chi_J\|_\infty \\ &\qquad \leq |J|^\alpha \left[\frac{1}{|J|}\, w(J^-) \right]^{1/p} \left[\frac{1}{2^{1-\alpha p}} \frac{1}{|J|^{1-\alpha p}} w(J^-) \right]^{-1/p} = 2^{(1/p)-\alpha}. \end{aligned}$$

Then, the pair of weights $(w^{1/p}, M^-_{\alpha p}(w)^{1/p})$ satisfies the condition (2.3) in Corollary 2.2. $\square$

Proof of Theorem 1.2: If $\alpha = 0$, the pair $(w, M^-(w))$ is independent of p and this result is a consequence of the weak type $(1, 1)$ with respect to $(w, M^-(w))$ proved by F.J. Martín-Reyes in Theorem 1 of [5], the strong type (∞, ∞) and the Marcinkiewicz interpolation theorem.

Using (1.1) and Corollary 2.3, the proof in the case $0 < \alpha < 1$ and $1 < p < 1/\alpha$ is similar to Theorem 1.7 in [2]. $\square$

3. Proofs of Theorem 1.3 and Theorem 1.4

Following the techniques employed by C. Pérez in Corollary 1.12 of [8] we will prove the next result.

Proof of Theorem 1.3: We will choose X a Banach function space with the following property: there exists a constant $C > 0$ such that for all $a < b < c$ with $b - a < c - b$ we have that

$$\|f\|_{X,(b,c)} \leq C \|f\|_{X,(a,c)}$$

and the operator $M^+_X : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ is bounded. We will apply Theorem 1 in [9]. For this, it will be sufficient to show that there exists a constant K such that

$$(3.1) \qquad (c - b)^\alpha \left(\frac{1}{b - a} \int_a^b [M^+_{\alpha p'}(M^{[p']} w)(x)]^{1-p}\, dx \right)^{1/p} \|w^{1/p'}\|_{X',(b,c)} \leq K$$

for every $a < b < c$ with $b - a < c - b$. Let X' be the Orlicz space associated to Young function $B(t) \approx t^{p'}(\log^+ t)^{[p']}$.

Since $[p'](p-1) > 1$, the integral

$$\int_e^{+\infty} \left(\frac{t^{p'}}{B(t)} \right)^{p-1} \frac{dt}{t}$$

is convergent and applying Theorem 4 in [9] we obtain that the operator $M^+_{\overline{B}}$ is bounded from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ where $\overline{B}$ is the associated Young function to B.

If $A(t) = B(t^{1/p'}) \approx t(\log^+ t)^{[p']}$, it is easy to check that

$$\|w^{1/p'}\|_{B,(b,c)} = \|w\|^{1/p'}_{A,(b,c)}.$$

For each $x \in [a, b]$ since $c - x \leq c - a \leq 2(c - b)$ we have that

$$\begin{aligned} M^+_{\alpha p'}(M^{[p']}w)(x) &\geq \frac{1}{(c-x)^{1-\alpha p'}} \int_x^c M^{[p']}(w)(z)\, dz \\ &\geq \frac{1}{[2(c-b)]^{1-\alpha p'}} \int_b^c M^{[p']}(w)(z)\, dz. \end{aligned}$$

Then, (3.1) is bounded by

$$\begin{aligned} I &= (c-b)^{\alpha} \left[\frac{1}{[2(c-b)]^{1-\alpha p'}} \int_b^c M^{[p']}(w)(z)\, dz \right]^{\frac{1-p}{p}} \|w\|^{1/p'}_{A,(b,c)} \\ &= 2^{1/p'} \left[\frac{1}{c-b} \int_b^c M^{[p']}(w)(z)\, dz \right]^{-\frac{1}{p'}} \|w\|^{1/p'}_{A,(b,c)}. \end{aligned}$$

Taking into account that $A(t) \approx t(\log^+ t)^{[p']}$ and using the estimate (24) in [8] we obtain that

$$\|w\|_{A,(b,c)} \leq K \frac{1}{c-b} \int_b^c M^{[p']}(w)(z)\, dz$$

and, it follows that

$$I \leq 2^{1/p'} K^{1/p'},$$

which proves that (3.1) holds. □

We recall that a weight w belongs to the class $A^+_p, 1 < p < \infty$, introduced by E. Sawyer in [10] if

$$\sup_{a\in\mathbb{R},\ h>0} \left(\frac{1}{h} \int_{a-h}^{a} w(y)\, dy \right) \left(\frac{1}{h} \int_a^{a+h} w(y)^{-\frac{1}{p-1}}\, dy \right)^{p-1} < \infty.$$

We shall say that w belongs to A_1^+ if there exists a constant $C > 0$ such that

$$M^-(w)(x) \le Cw(x) \ \text{ a.e.}$$

A weight w is in A_∞^+ if there exist two positive constants C, δ such that for all $a < b < c$ and every measurable set $E \subset (b, c)$ the inequality

$$\frac{|E|}{(c-a)} \le C\left(\frac{w(E)}{w(a,b)}\right)^\delta$$

holds. Similarly the classes A_p^-, $1 \le p \le \infty$, were defined.

If $1 \le p < q \le \infty$, then $A_p^+ \subset A_q^+$ and $A_p^+ = (A_1^+)(A_1^-)^{1-p}$. The study of these classes of weights can be found in [5] and [10].

The following proposition extends Theorem 3.4 on page 158 of [4]. Its proof will be omitted.

Proposition 3.1. *Let $0 \le \alpha < 1$, $0 < \gamma < 1/(1-\alpha)$ and let μ be a positive Borel measure on $\mathbb{R}$ such that $M_\alpha^-(\mu)(x) < \infty$ almost everywhere. Then, $[M_\alpha^-(\mu)(x)]^\gamma \in A_1^+$ with a constant depending only on γ.*

Proof of Theorem 1.4: For each $0 < \beta < 1$, from Proposition 3.1 it follows that $M_\beta^+(\mu) \in A_1^-$. Then, $M_\beta^+(\mu)^{1-p} \in A_p^+ \subset A_\infty^+$. Applying Theorem 3 in [6] and Theorem 1.3 we have that

$$\begin{aligned}\int_{-\infty}^{+\infty} &|I_\alpha^+(f)(x)|^p \ [M_{\alpha p'}^+(M^{[p']}w)(x)]^{1-p}\,dx \\ &\le C_1 \int_{-\infty}^{+\infty} M_\alpha^+(f)(x)^p \ [M_{\alpha p'}^+(M^{[p']}w)(x)]^{1-p}\,dx \\ &\le C_1C_2 \int_{-\infty}^{+\infty} |f(x)|^p \ w(x)^{1-p}\,dx,\end{aligned}$$

and the proof is complete. □

Corollary 3.2. *Let $1 < p < \infty$ and $0 < \alpha < 1/p'$. There exists a constant $C > 0$ such that*

$$\int_{-\infty}^{+\infty} |I_\alpha^-(f)(x)|^{p'} \ [M_{\alpha p'}^+(M^{[p']}w)(x)]\,dx \le C\int_{-\infty}^{+\infty} |f(x)|^{p'} \ w(x)\,dx$$

for every measurable function f and every weight w where $M^{[p']}$ is the maximal Hardy-Littlewood operator iterated $[p']$ times.

Proof: The assertion is an immediate consequence of Theorem 1.4. □

References

[1] Bennett C., Sharpley R., *Interpolation of Operators*, Academic Press, New York, 1988.

[2] Cruz-Uribe D., *New proofs of two-weight norm inequalities for the maximal operator*, Georgian Math. J. **7** (2000), no. 1, 33–42.

[3] Fefferman C., Stein E.M., *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[4] García-Cuerva J., Rubio de Francia J.L., *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[5] Martín–Reyes F.J., *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 691–698.

[6] Martín-Reyes F.J., Pick L., de la Torre A., A_∞^+ *condition*, Canad. J. Math. **45** (1993), no. 6, 1231–1244.

[7] Martín-Reyes F.J., de la Torre A., *Two weight norm inequalities for fractional one-sided maximal operators,*, Proc. Amer. Math. Soc. **117** (1993), no. 2, 483–489.

[8] Pérez C., *Banach function spaces and the two-weight problem for maximal functions*, Proceedings of the Conference on Function Spaces, Differential Operators and Nonlinear Analysis, Paseky nad Jizerou, (1995), pp. 141–158.

[9] Riveros M.S., de Rosa L., de la Torre A., *Sufficient conditions for one-sided operators*, J. Fourier Anal. Appl. **6** (2000), no. 6, 607–621.

[10] Sawyer E., *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), no. 1, 53–61.

Juan José Olleros 2969 8°B, 1426 Ciudad de Buenos Aires, Argentina

E-mail: liliana_ de_ rosa@yahoo.com.ar

(*Received* November 11, 2004, *revised* September 27, 2005)