Comment.Math.Univ.Carolin. 47,1 (2006)103–111

Baireness of $C_k(X)$ for ordered X

Michael Granado, Gary Gruenhage

Abstract. We show that if X is a subspace of a linearly ordered space, then $C_k(X)$ is a Baire space if and only if $C_k(X)$ is Choquet iff X has the Moving Off Property.

Keywords: Baire, linearly ordered space, compact-open topology, Choquet, Moving Off Property

Classification: Primary 54F05; Secondary 54C35, 54E52

1. Introduction

Let $C_k(X)$ denote the space of continuous real-valued functions on X endowed with the compact-open topology. G. Gruenhage and D. Ma [GM] defined the Moving Off Property (MOP), and showed that, for locally compact spaces X, $C_k(X)$ is Baire if and only if X has MOP. This result holds more generally for the class of q-spaces, which includes all locally compact and all first-countable spaces.

It is an open question whether the Gruenhage-Ma result holds for all completely regular X. We provide some evidence of an affirmative answer to the question by showing that it holds whenever X is a GO-space (i.e., a subspace of a linearly ordered space).

It is also an unsolved problem to find any internal property P of topological spaces such that X has P iff $C_k(X)$ is Baire. A key to our result that P=MOP works for GO-spaces is to first obtain a structural result which characterizes when a GO-space has the MOP. We then use this result to obtain our main theorem. In the final section, we apply our results to some special cases.

All spaces are assumed to be completely regular.

2. Definitions and background results

Recall that a collection $\mathcal{J}$ of subsets of a space X is *discrete* if every point of X has a neighborhood meeting at most one member of $\mathcal{J}$. We say $\mathcal{J}$ has a *discrete open expansion* if for every $J \in \mathcal{J}$, there is an open superset U_J of J such that $\{U_J : J \in \mathcal{J}\}$ is a discrete collection.

The second author acknowledges support from National Science Foundation grant DMS-0405216.

A collection $\mathcal{K}$ of nonempty compact subsets of a space X is said to be a *moving off collection* if for each compact subset M of X there exists a $K \in \mathcal{K}$ with $M \cap K = \emptyset$. A space X is said to have the *Moving Off Property* (MOP) if every moving off collection $\mathcal{K}$ in X contains an infinite subcollection $\mathcal{K}'$ which has a discrete open expansion.

A space X is said to have the *Weak Moving Off Property* (WMOP) if every moving off collection in X contains an infinite discrete subcollection. This property, which we mention here primarily for completeness, was considered by A. Bouziad [B]. The WMOP is equivalent to the MOP in locally compact or normal spaces; in particular, the concepts coincide in the class of GO-spaces. While the WMOP seems more elegant than the MOP, it cannot serve to characterize Baireness of $C_k(X)$. Example 4.8 of [G_2] gives a completely regular space X with the WMOP but not the MOP, hence by the next theorem, $C_k(X)$ is not Baire.

Theorem 2.1 ([GM]). *If $C_k(X)$ is Baire, then X has the MOP.*

We will also make use of the following results.

Theorem 2.2 ([GM]). *Suppose X has the MOP. Then:*

(a) *if X has a countable local base at p, then X is locally compact at p;*
(b) *if X is countably compact, then X is compact.*

Theorem 2.3 ([G_1]). *If a space X is paracompact and locally compact, then X has the MOP.*

Let X be a nonempty topological space. The *Choquet game G_X* of X is defined as follows: Players Empty (E) and Nonempty (NE) take turns in choosing nonempty open subsets of X. Player E starts by choosing $U_0 \subset X$ and NE responds with $V_0 \subset U_0$. In the n^{th} round, $n \geq 1$, E and NE choose in turn non-empty open sets U_n and V_n, with $V_n \subset U_n \subset V_{n-1}$. We say that E *wins* the game if $\bigcap_n U_n = \emptyset$; otherwise NE wins.

It is well-known that a space X is a Baire space iff E has no winning strategy in the Choquet game. If NE has a winning strategy, then X is said to be a *Choquet space*[1]. Choquet spaces are also called *weakly α-favorable spaces*.

Ma [Ma] proved the following characterization of Choquetness of $C_k(X)$ for locally compact X:

Theorem 2.4. *Suppose X is locally compact. Then $C_k(X)$ is Choquet iff X is paracompact.*

We will use the following characterization of paracompactness in GO-spaces ([EL]):

[1] We are following the terminology of Kechris [Ke].

Theorem 2.5. *Let X be a GO-space. Then X is not paracompact if and only for some regular uncountable cardinal κ, there exists a closed subspace T of X which is homeomorphic to a stationary subset S of κ; furthermore, when such T and S exist, one may assume that there is a homeomorphism $h : S \to T$ that is either order-preserving or order-reversing.*

We follow Kunen [Ku] for set-theoretic terminology. A subset A of an ordered set X is *cofinal* (resp., *coinitial*) in X if for every $x \in X$, there is $a \in A$ with $x \leq a$ (resp., $a \leq x$). A cardinal κ is *regular* if there is no cofinal subset A of κ with $|A| < \kappa$. A subset C of an uncountable regular cardinal κ is *closed unbounded* (*c.u.b.*) in κ if it is cofinal in κ and closed in the order topology, and a subset S of κ is *stationary* in κ if $S \cap C \neq \emptyset$ whenever C is c.u.b. in κ. The main set-theoretic fact that we will use about stationary sets is the so-called Pressing Down Lemma:

Theorem 2.6. *Let S be a stationary subset of a regular uncountable cardinal κ. Suppose $f : S \to \kappa$ is such that $f(\alpha) < \alpha$ for every $\alpha \in S$, $\alpha > 0$. Then there is some $\beta \in \kappa$ and a stationary (hence unbounded) subset T of S such that $f(\alpha) = \beta$ for every $\alpha \in T$.*

We also use the more elementary fact that for any stationary set S, the set S' of limit points in S is stationary as well.

3. Characterization of the MOP for ordered spaces

Let X be a *GO*-space. Then there exists a compact ordered space X^* containing X as a dense subspace ([L]). Elements of $X^* \backslash X$ are called *gaps* in X. For $A \subset X$, we will denote by $\sup A$ and $\inf A$ the obvious elements of X^* (which may or may not be in X).

A subset C of X is called *convex in* X, if for all $a, b \in C$, $\{x \in X : a < x < b\} \subseteq C$.

Let X be a GO space, and let $p \in X$. Then X is said to be *left first-countable* (resp., *left locally compact*) at p if p is a point of first-countability (resp., local compactness) in $(\leftarrow, p]$. The terms *right first-countable* and *right locally compact* are defined analogously.

Define an equivalence relation on a GO-space X by $a \sim b$ iff $[a, b]$ or $[b, a]$ is compact, and let $\mathcal{J}$ be the collection of equivalence classes. Note that $\mathcal{J}$ is a pairwise-disjoint collection of convex subsets of X. It is also easy to see that each $J \in \mathcal{J}$ is a locally compact subspace of X. The following theorem is the main result of this section.

Theorem 3.1. *Let X be a GO-space. Then X has the MOP iff the following two properties hold:*

(I) *every $J \in \mathcal{J}$ is σ-compact;*

(II) *for any point $p \in X$, if X is left first-countable at p, then X is left locally compact at p, and if X is right first-countable at p, then X is right locally compact at p.*

Proof: We first show the reverse direction. Assume (I) and (II) hold; we will show that X has the MOP.

Claim. If K is a compact subset of X, then $\mathcal{J}(K) = \{J \in \mathcal{J} : J \cap K \neq \emptyset\}$ is finite.

To see this, suppose by way of contradiction that there is a compact set K and a countably infinite $\mathcal{J}' \subset \mathcal{J}$ such that, for every $J \in \mathcal{J}'$, $J \cap K \neq \emptyset$. Then there is a point p in K such that every neighborhood of p meets infinitely many $J \in \mathcal{J}'$. W.l.o.g., each $J \in \mathcal{J}'$ falls to the left of p. It follows that X is left first-countable at p, hence by (II), is left locally compact at p. But then some point $y < p$ is in the same equivalence class as p, yet $[y, p]$ meets infinitely many distinct equivalence classes; this is a contradiction which proves the claim.

Let $\mathcal{K}$ be a moving off collection of compact sets. Each $J \in \mathcal{J}$ is σ-compact and locally compact, so we can write $J = \bigcup_{n\in\omega} J_n$, where each J_n is compact and every compact subset of J is contained in some J_n.

Now choose $K_0 \in \mathcal{K}$. If $K_i \in \mathcal{K}$ has been chosen for each $i < n$, choose $K_n \in \mathcal{K}$ disjoint from

$$(\bigcup_{i<n} K_i) \cup \bigcup\{J_i : i \leq n, J \in \bigcup_{i<n} \mathcal{J}(K_i)\}.$$

We show that $\{K_i\}_{i\in\omega}$ is a discrete subcollection of $\mathcal{K}$. Suppose p is a limit point. W.l.o.g., p is a limit from the left. In the same way as in the proof of the Claim, there is $y < p$ such that $[y, p]$ is compact. Then $[y, p] \subset J$ for some $J \in \mathcal{J}$, and $[y, p] \cap K_n \neq \emptyset$ for some n. So $J \in \mathcal{J}(K_n)$, and $[y, p] \subset J_m$ for some m. Then if $l > \max\{m, n\}$, by the construction $K_l \cap J_m = \emptyset$. Thus $[y, p]$ meets only finitely many K_i, a contradiction. This completes the proof of the reverse direction.

Now we prove the forward direction. Suppose X has the MOP. Then so does any closed subset of X, in particular, closed intervals. By Theorem 2.2(a), points of first-countability must be points of local compactness. It follows that left (resp., right) first-countable implies left (resp., right) compact at any point, so (II) holds.

To see that (I) holds, let J be a $\sim$ equivalence class, and suppose J is not σ-compact. Then J either has no countable cofinal subset or no countable coinitial subset. Suppose w.l.o.g. that J has no countable cofinal subset. Then $\sup J \notin J$, hence $\sup J \notin X$, so J is closed (on the right) in X. Let κ be the minimal cardinal of a cofinal subset of J. Note that κ is regular. Since $[a, b]$ is compact for every $a, b \in J$, one sees that $\sup A \in J$ for any subset of J of cardinality less than κ. It follows that one may construct by induction a continuous increasing mapping $f : \kappa \to J$ with $\sup J = \sup \operatorname{ran}(f)$. But then $\operatorname{ran}(f)$ is a closed in X copy of the ordinal space κ. Since κ is countably compact but not compact, this contradicts Theorem 2.2(b) and completes the proof of the theorem. □

4. Baireness of $C_k(X)$ for ordered X

In this section, we use the characterization of the MOP for GO-spaces obtained in the last section to prove the following theorem:

Theorem 4.1. *Let X be GO-space. The following are equivalent:*

(a) *$C_k(X)$ is Baire;*
(b) *X has the MOP;*
(c) *$C_k(X)$ is Choquet.*

For the proof of the above result and for results in the next section, it will be handy to have the following lemma.

Lemma 4.2. *Suppose X is a GO-space, $p \in X$, and that S is a cofinal subset of $(\leftarrow, p)$ which is homeomorphic to a stationary subset of a regular uncountable cardinal. Suppose also that X is left locally compact at every point of S, and that p is a limit point of S. Then X is left locally compact at p.*

Proof: Let S' be the set of limit points of S inside S. Then for each α in S', there is some $\beta_\alpha \in S$ with $\beta_\alpha < \alpha$ such that the closed interval $[\beta_\alpha, \alpha]$ is compact. Since S' is stationary, by the Pressing Down Lemma there is $\beta \in S$ and an unbounded subset T of S' such that $[\beta, \alpha]$ is compact for every $\beta \in T$. It follows that $[\beta, p]$ is compact, and the lemma is proved. □

Proof of Theorem 4.1: By Theorem 2.1, (a) implies (b) is true for any space X. That (c) implies (a) is immediate from the definitions. It remains to prove (b) implies (c). To this end, suppose X has the MOP.

We need to define a winning strategy for NE in the Choquet game on $C_k(X)$. W.l.o.g., we may assume both players restrict their choices to basic open sets of the form

$$B(f, K, \epsilon) = \{g \in C(X) : \forall\, x \in K(|f(x) - g(x)| < \epsilon)\}$$

where $f \in C(X)$, K is compact, and $\epsilon > 0$. Some ideas in the proof below are similar to those in Theorem 8.3 of [MN]. Indeed, it is possible to prove in our case that II has a winning strategy in the game $\Gamma_2(X)$ defined in [MN], and quote their Theorem 8.3 to conclude that $C_k(X)$ is Choquet. However, there is a gap in their proof of Theorem 8.3; although that gap can be fixed, we choose here to give instead a direct proof of (b) implies (c).

As in the proof of Theorem 3.1, any compact set K meets only a finite collection $\mathcal{J}(K)$ of members of $\mathcal{J}$, the family of $\sim$ equivalence classes. And since each member J of $\mathcal{J}$ is σ-compact and locally compact , we can write $J = \bigcup_{n\in\omega} J_n$, where $J_0, J_1, \dots$ is an increasing sequence of compact subsets of J such that every compact subset of J is contained in J_n for some n.

Now suppose $B(f_n, K_n, \epsilon_n)$ is E's move in the n^{th} round. Let NE respond with $B(f_n, L_n, \epsilon'_n)$, where

$$L_n = L_{n-1} \cup K_n \cup \bigcup\{J_n : J \in \mathcal{J}(L_{n-1} \cup K_n)\},$$

and $\epsilon'_n = \min\{\epsilon_n/2, 1/2^n\}$.

Note that by induction, the L_n's are increasing, and $L_n \supset \bigcup_{i\leq n} K_i$. Also, if $m < n < l$, then $f_l \in B(f_n, L_n, \epsilon'_n)$, so $|f_l(x) - f_n(x)| < 1/2^n$ for all x in L_n, hence for all $x \in L_m$. It follows that, for each fixed m, $\{f_n \restriction L_m : n \in \omega\}$ is a Cauchy sequence in the topology of uniform convergence, hence converges to a unique $g_m : L_m \to \mathbb{R}$. Note that $g_n \restriction L_m = g_m$ for $n \geq m$; thus if $L = \bigcup_{n\in\omega} L_n$, then $g = \bigcup_{n\in\omega} g_n$ is a function from L to $\mathbb{R}$.

We plan to show that L is closed in X and that g is continuous on L. To this end, we will show that if $L'_0 = L_0$ and $L'_n = \overline{L_n \backslash L_{n-1}}$ for $n \geq 1$, then $\{L'_n\}_{n\in\omega}$ is a locally finite collection. Suppose by way of contradiction that every neighborhood of a point p meets L'_n for infinitely many n. W.l.o.g., p is a limit from the left of the L'_n's.

Claim. X is left locally compact at p. If p is not a limit point from the left of L'_n for any n, then $(\leftarrow, p)$ has countable cofinality, so X must be left first-countable at p and hence by Theorem 3.1, X is left locally compact at p; thus the claim holds in this case. Now assume p is a limit point from the left of L'_n for some fixed n. If there is a countable subset of L'_n cofinal in $(\leftarrow, p)$, then again X is left first-countable at p and the claim holds as before. So suppose the cofinality of $L'_n \cap (\leftarrow, p)$ is uncountable. Then for some uncountable regular cardinal κ, there is a continuous increasing function $\theta : \kappa \to L'_n \cap (\leftarrow, p)$ whose range is cofinal in $L'_n \cap (\leftarrow, p)$. Let S be the subset of κ consisting of the limits of countable cofinality in κ. Then S is stationary in κ, and X is left first-countable, hence left locally compact, at each point of $\theta(S)$. Now the claim follows by applying Lemma 4.2 to $\theta(S)$.

From the claim, we easily get a contradiction. Let $y < p$ such that $[y, p]$ is compact. Then $[y, p] \subset J_i$ some $J \in \mathcal{J}$ and $i \in \omega$. Since p is a limit from the left of $\{L'_n\}_{n\in\omega}$, we have that $L_m \cap [y, p] \neq \emptyset$ for infinitely many m, and it follows from the construction that for sufficiently large n, $L_n \supset J_i$. This is easily seen to be a contradiction to the assumption that p is a limit from the left of the L'_n's.

Now, since we have shown that $\{L'_n\}_{n\in\omega}$ is a locally finite collection of closed sets, we have that $L = \bigcup_{n\in\omega} L'_n$ is closed in X, and furthermore, since $g \restriction L'_n$ is continuous for each n, we also have that g is continuous on L. Hence g extends to a continuous $g^* : X \to \mathbb{R}$ and it is straightforward to show that $g^* \in B(f_m, K_m, \epsilon_m)$ for every $m \in \omega$. This completes the proof. □

5. Applications

In this section, we apply our main result to get further results in some special cases.

Lemma 5.1. *Let κ be a regular uncountable cardinal.*

(a) *Suppose S is a stationary co-stationary subset of κ. Then there is a c.u.b. C in κ such that S is not locally compact at any point of $C \cap S$;*

(b) *If N is a non-stationary subset of κ, then $\kappa\backslash N$ does not have the MOP.*

Proof: For (a), let $D = \overline{S} \cap \overline{\kappa\backslash S}$, and let C be the set of non-isolated points of the subspace D. Then C is c.u.b., and it is easy to check that no point of $C \cap S$ is a point of local compactness in S.

For (b), consider a c.u.b. $C \subset \kappa\backslash N$. Then C is countably compact but not compact, hence cannot have the MOP □

We now get the following characterizations for GO-spaces which are locally compact or first-countable:

Theorem 5.2. *Let X be a locally compact GO-space. Then the following are equivalent:*

(a) *X has the MOP;*
(b) *X is paracompact;*
(c) *$C_k(X)$ is Baire;*
(d) *$C_k(X)$ is Choquet.*

Proof: By Theorem 2.4, $C_k(X)$ is Choquet iff X is paracompact, and by Theorem 2.1, Baireness of $C_k(X)$ implies X has the MOP.

Thus it remains to show that for a locally compact GO-space X, if X has the MOP, then X is paracompact. Suppose X is not paracompact. Then X contains a closed subset S homeomorphic to a stationary subset of a regular uncountable cardinal κ. Since S is locally compact, by Lemma 5.1(a), S cannot be co-stationary, hence must contain a copy of a club C in κ. But by 5.1(b), C does not have the MOP, contradiction. □

Corollary 5.3. *Let X be a first-countable GO-space. Then the following are equivalent:*

(a) *X has the MOP;*
(b) *X is paracompact and locally compact;*
(c) *$C_k(X)$ is Baire;*
(d) *$C_k(X)$ is Choquet.*

Proof: Recall that first-countable implies locally compact for spaces having the MOP. Hence this corollary is an immediate consequence of the previous theorem.
□

Now we apply our results to obtain a characterization for GO-spaces with a well-order, or, equivalently, subspaces of an ordinal. For a space X, we denote by LC(X) the points of local compactness. Note that LC(X) is an open locally compact subspace of X.

For X a subset of an ordinal, we say a point $x \in X$ has *countable cofinality relative to* X if there is a countable subset of X which is cofinal in $X \cap (\leftarrow, x)$.

Theorem 5.4. *Let X be a subspace of an ordinal. Then X has the MOP iff $LC(X)$ is paracompact and contains all points of countable cofinality relative to X.*

Proof: We first prove the forward direction. Suppose X has the MOP. That $\mathrm{LC}(X)$ contains all points of countable cofinality relative to X is immediate from Theorem 3.1. Suppose $\mathrm{LC}(X)$ is not paracompact. Then there is a closed subset Y of $\mathrm{LC}(X)$ such that Y is homeomorphic to a stationary subset S of a regular uncountable cardinal κ. Since X is well-ordered, we may assume that there is an order-preserving homeomorphism $h : S \to Y$. By Lemma 5.1, since each point of Y is a point of local compactness, S cannot be co-stationary, i.e., S contains some c.u.b. C in κ. Since $h(C)$ cannot have the MOP, $h(C)$ cannot be closed in X. It follows that $p = \sup(h(C)) = \sup(Y)$ is a point of $X\backslash\mathrm{LC}(X)$ and is a limit point of $h(C)$. But since each point of Y is a point of local compactness, by Lemma 4.2, $p \in \mathrm{LC}(X)$, which is a contradiction.

For the reverse direction, suppose $\mathrm{LC}(X)$ is paracompact and contains all points of countable cofinality relative to X. Then condition (II) of Theorem 3.1 holds, so we need to show (I) holds. Let J be a $\sim$ equivalence class. Then $J \subset \mathrm{LC}(X)$. Note that J is σ-compact iff J has a countable cofinal subset. Suppose J has no countable cofinal subset. Then $p = \sup J \notin J$ and there is a copy K of a regular uncountable cardinal in J such that $\sup(K) = \sup(J)$. Since $\mathrm{LC}(X)$ is paracompact, K cannot be closed in $\mathrm{LC}(X)$ by Theorem 2.5 , so $p \in \mathrm{LC}(X)$ and hence $p \in J$ by Lemma 4.2, which is a contradiction. □

Examples.

(a) A subspace X of the space ω_1 of countable ordinals has the MOP iff X is locally compact and non-stationary.
(b) Let Y be the set of ordinals in ω_2 of uncountable cofinality. Then Y has the MOP.
(c) Let Y^* be Y above but with the reverse ordering, and let X be the linearly ordered space consisting of the ordinal space $\omega_1 + 1$ followed a copy of Y^*. Then X has the MOP, but $\mathrm{LC}(X)$ is not paracompact.

Proof: To see (a), note that in this situation we have $\mathrm{LC}(X) = X$, and recall that a subset X of ω_1 is paracompact iff X is non-stationary. To see (b), note that $\mathrm{LC}(Y)$ is the set of isolated points of Y, hence $\mathrm{LC}(Y)$ is paracompact. Finally, for (c), note that the $\sim$ equivalence classes consist of $\omega_1 + 1$ and singletons $y \in Y^*$. Thus X satisfies condition (I) of Theorem 3.1. It is easy to see that X also satisfies condition (II), so X has the MOP. But $\mathrm{LC}(X)$ contains the space ω_1 of countable ordinals as a relatively closed subspace, hence $\mathrm{LC}(X)$ is not paracompact. □

It follows from the last example that the characterization of the MOP in well-ordered GO-spaces given by Theorem 5.4 does not hold for general GO-spaces.

References

[B] Bouziad A., *Coincidence of the upper Kuratowski topology with the co-compact topology on compact sets, and the Prohorov property*, Topology Appl. **120** (2002), 283–299.

[EL] Engelking R., Lutzer D., *Paracompactness in ordered spaces*, Fund. Math. **94** (1977), 49–58.

[G_1] Gruenhage G., *Games, covering properties and Eberlein compacts*, Topology Appl. **23** (1986), 291–297.

[G_2] Gruenhage G., *The story of a topological game*, Rocky Mountain J. Math., to appear.

[GM] Gruenhage G., Ma D.K., *Baireness of $C_k(X)$ for locally compact X*, Topology Appl. **80** (1997), 131–139.

[Ke] Kechris A.S., *Classical Descriptive Set Theory*, Springer, New York, 1995.

[Ku] Kunen K., *Set Theory*, North-Holland, Amsterdam, 1980.

[L] Lutzer D.J., *On generalized ordered spaces*, Dissertationes Math. **89** (1971).

[Ma] Ma D.K., *The Cantor tree, the γ-property, and Baire function spaces*, Proc. Amer. Math. Soc. **119** (1993), 903–913.

[MN] McCoy R.A., Ntantu I., *Completeness properties of function spaces*, Topology Appl. **22** (1986), 191–206.

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

E-mail: granamf@auburn.edu
garyg@auburn.edu

(*Received* May 26, 2005, *revised* December 12, 2005)