How non-symmetric can a copula be?

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Abstract. A two-place function measuring the degree of non-symmetry for (quasi-)copulas is considered. We construct copulas which are maximally non-symmetric on certain subsets of the unit square. It is shown that there is no copula (and no quasi-copula) which is maximally non-symmetric on the whole unit square.

Keywords: copula, quasi-copula, symmetry, opposite diagonal

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1. Introduction

Copulas (first mentioned in [11], for an excellent survey see [9]) and quasi-copulas (introduced in [1] and conveniently characterized in [4]) play a key role in the analysis of bivariate distribution functions with given marginals. The basic result in this context is Sklar's Theorem ([11], [12]) showing that the joint distribution of a random vector and the corresponding marginal distributions are linked by some copula: if (X, Y) is a random vector, $F_X, F_Y: [-\infty, \infty] \to [0, 1]$ are its marginal distribution functions, then $H_{XY}: [-\infty, \infty]^2 \to [0, 1]$ is a joint distribution of (X, Y) if and only if there is a two-dimensional copula C_{XY} such that for all $(x, y) \in [-\infty, \infty]^2$ we have

$$H_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y)).$$

Moreover, if F_X and F_Y are continuous then C_{XY} is unique, otherwise C_{XY} is uniquely determined only on $\mathrm{Ran}(F_X) \times \mathrm{Ran}(F_Y)$.

Recall that a (*two-dimensional*) *copula* is a function $C: [0, 1]^2 \to [0, 1]$ such that $C(0, x) = C(x, 0) = 0$ and $C(1, x) = C(x, 1) = x$ for all $x \in [0, 1]$, and C is 2-increasing, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \le x_2$ and $y_1 \le y_2$ we have

$$C(x_1, y_1) + C(x_2, y_2) \ge C(x_1, y_2) + C(x_2, y_1).$$

A *quasi-copula* is a function $Q: [0, 1]^2 \to [0, 1]$ such that $Q(0, x) = Q(x, 0) = 0$ and $Q(1, x) = Q(x, 1) = x$ for all $x \in [0, 1]$, Q is non-decreasing (in each component), and Q is 1-Lipschitz, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$

$$|Q(x_1, y_1) - Q(x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|.$$

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Obviously, each copula is a quasi-copula, but not vice versa. Each copula C satisfies

$$W \leq C \leq M, \tag{1.1}$$

where the Fréchet-Hoeffding lower and upper bounds W and M are given by $W(x,y) = \max(x+y-1,0)$ and $M(x,y) = \min(x,y)$, respectively, and the same holds for quasi-copulas.

In general, a copula is neither symmetric (commutative) nor associative (see [8]), and it is well-known that each associative copula is also symmetric and, consequently, a (continuous) triangular norm [6], [10](again the converse does not necessarily hold).

There is a close relationship between symmetric copulas and interchangeable random variables X and Y (where the random vectors (X,Y) and (Y,X) are identically distributed). Clearly, two interchangeable random variables X and Y must be identically distributed, i.e., have a common univariate distribution function, and for identically distributed random variables X and Y their interchangeability is equivalent to the symmetry of their copula C_{XY} (see [9, Theorem 2.7.4]).

As a consequence, for exchangeable random variables X and Y with copula C, the symmetry of C implies $C(y,x) = C(x,y)$. In general (i.e., for non-exchangeable random variables X and Y) this is no more true, but any estimate of the value $C(y,x)$ by means of $C(x,y)$ will be helpful when modelling bivariate statistical data, especially in order to exclude irrelevant models.

Therefore, we are interested in "how non-symmetric" a copula can be, and we construct copulas which are "maximally" non-symmetric on certain distinguished subsets of the unit square. Finally we show that no copula (and no quasi-copula) can be "maximally" non-symmetric on the whole unit square.

2. Degree of non-symmetry

Given a copula C, the function $d_C\colon [0,1]^2 \to [0,1]$ defined by

$$d_C(x,y) = |C(x,y) - C(y,x)|$$

provides a "measure" of its non-symmetry at each point of the unit square $[0,1]^2$, and its Chebyshev norm $\|d_C\|_\infty$ given by

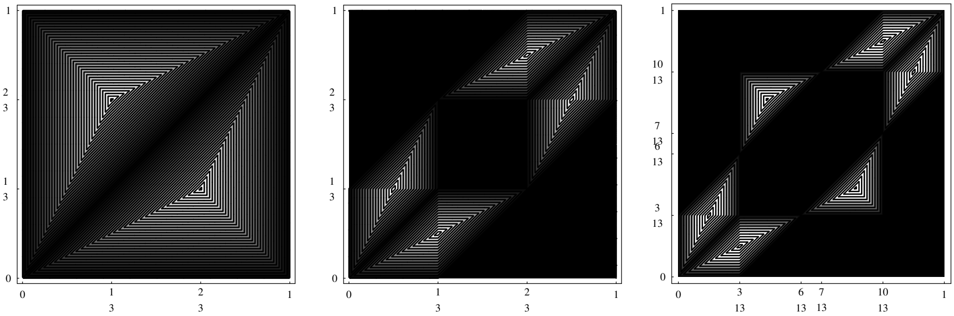
$$\|d_C\|_\infty = \sup\{d_C(x,y) \mid (x,y) \in [0,1]^2\}$$

can be viewed as the *degree of non-symmetry* of C. Obviously, for each copula C the function d_C vanishes on the boundary as well as on the diagonal $\{(x,x) \mid x \in [0,1]\}$ of $[0,1]^2$. Also, a copula C is symmetric if and only if $\|d_C\|_\infty = 0$.

Example 2.1. The copula C given by $C(x,y) = xy - x^3y(1-x)(1-y)$ is non-symmetric, and we obtain $d_C(x,y) = xy(1-x)(1-y)|x^2-y^2|$. A simple computation then yields $\|d_C\|_\infty = d_C(0.3418922, 0.7768102) = 0.0189801$.

In order to find out the maximal degree of non-symmetry of copulas consider the function $d^*\colon [0,1]^2 \to [0,1]$ defined by

$$d^* = \sup\{d_C \mid C \text{ is a copula}\}.$$



d^* $\qquad$ $d^* - d_{C_{1/3}}$ $\qquad$ $d^* - d_{C_{3/13}}$

FIGURE 1: Maximal non-symmetry

We now compute the function d^* (see Figure 1 left) and show that for each point $(x_0, y_0) \in [0,1]^2$ we can find a copula C such that d_C and d^* coincide on two straight line segments containing the points (x_0, y_0) and (y_0, x_0).

Proposition 2.2.

(i) *For all* $(x,y) \in [0,1]^2$ *we have* $d^*(x,y) = \min(|x-y|, x, y, 1-x, 1-y)$.

(ii) *For each* $\lambda \in [0,1]$ *the function* $C_\lambda\colon [0,1]^2 \to [0,1]$ *given by*

$$C_\lambda(x,y) = \max(M(x, y-\lambda), W(x,y))$$

is a copula such that we have $d_{C_\lambda}(x,y) = d^*(x,y)$ *for all* $(x,y) \in [0,1]^2$ *with* $|x-y| = \lambda$.

PROOF: Let C be a copula and assume, without loss of generality, $x \le y$ and $C(x,y) \le C(y,x)$. Then the monotonicity of C yields $C(x,y) \le C(y,x) \le C(y,y)$ which, together with (1.1) and the fact that C is 1-Lipschitz, implies $d_C(x,y) \le \min(|x-y|, M(x,y) - W(x,y))$. A simple computation shows that the latter expression coincides with $\min(|x-y|, x, y, 1-x, 1-y)$, i.e., for all $(x,y) \in [0,1]^2$

$$d_C(x,y) \le \min(|x-y|, x, y, 1-x, 1-y).$$

Now fix an arbitrary point $(x_0, y_0) \in [0,1]^2$ and put $\lambda = |x_0 - y_0|$. If we can show that C_λ in (ii) is a copula satisfying

$$d_{C_\lambda}(x_0, y_0) = \min(\lambda, x_0, y_0, 1 - x_0, 1 - y_0) \tag{2.1}$$

this will complete the proof of (i).

Since C_λ is a shuffle of M it is a copula (see [9]). Note that for each $(x,y) \in [0,1]^2$

$$\begin{aligned} d_{C_\lambda}(x,y) = \min(\max(&\min(x - \lambda, y, 1 - x, 1 - \lambda - y), \\ &\min(y - \lambda, x, 1 - y, 1 - \lambda - x), 0), |x - y|, \lambda). \end{aligned}$$

Then the verification of (2.1) is a matter of simple but tedious checking of all possible cases. Since λ only depends on $|x_0 - y_0|$, the proof of (ii) is complete, too. □

An immediate consequence of Proposition 2.2 is the following:

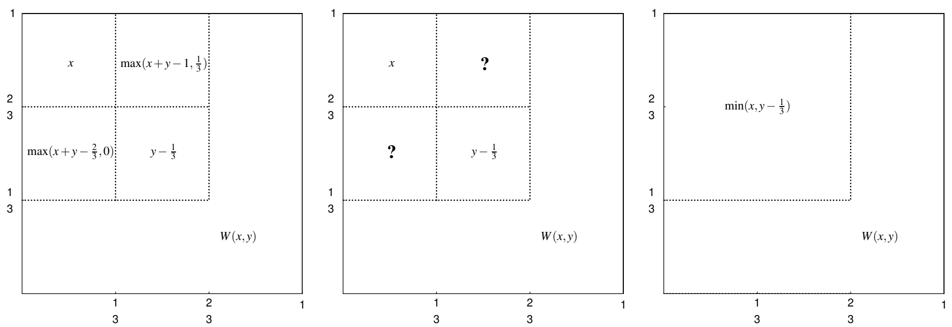
Corollary 2.3. *For each copula C and each $(x,y) \in [0,1]^2$ we have:*

$$C(y,x) \in [\max(W(y,x), C(x,y) - |x-y|), \min(M(y,x), C(x,y) + |x-y|)]\,.$$

Observe that the estimate for $C(y,x)$ in Corollary 2.3 is better than the estimate derived from the Fréchet-Hoeffding bounds W and M: if for a copula C we have $C(0.5, 0.6) = 0.3$ then the Fréchet-Hoeffding bounds imply $C(0.6, 0.5) \in [0.1, 0.5]$, whereas Corollary 2.3 tells us $C(0.6, 0.5) \in [0.2, 0.4]$.

Although copulas form a proper subclass of the class of quasi-copulas, the fact that we did not need the 2-increasingness of copulas implies:

Corollary 2.4. *We also have $d^* = \sup\{d_Q \mid Q$ is a quasi-copula$\}$.*



smallest: $\underline{Q}_{\omega^*}$ arbitrary greatest: $C_{1/3}$

FIGURE 2: Copulas with opposite diagonal ω^*

Some straightforward calculations show that the maximal value of d^* equals $\frac{1}{3}$ and that there is indeed a copula, namely, $C_{1/3}$ (see Figure 2 right) such that $d_{C_{1/3}}$ attains this maximal value in the points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ (see Figure 1 center):

Corollary 2.5.

(i) *For each* $\lambda \in \left[0, \frac{1}{3}\right]$ *we have*

$$\|d_{C_\lambda}\|_\infty = d_{C_\lambda}(\lambda, 1-\lambda) = \lambda.$$

(ii) *In particular, we have*

$$\|d^*\|_\infty = d^*(\tfrac{1}{3}, \tfrac{2}{3}) = \tfrac{1}{3} = d_{C_{1/3}}(\tfrac{1}{3}, \tfrac{2}{3}) = \|d_{C_{1/3}}\|_\infty.$$

Example 2.6. From the proof of Proposition 2.2 it follows that in the class $\mathcal{S}_M$ of shuffles of M (see [9]) for each $(x_0, y_0) \in [0,1]^2$ we can find a copula $C \in \mathcal{S}_M$ such that $d_C(x_0, y_0) = d^*(x_0, y_0)$. For other well-known classes of copulas this does not hold:

(i) Evidently, for each associative (and, consequently, for each Archimedean) copula C the function d_C vanishes on the whole unit square $[0,1]^2$.

(ii) In the class $\mathcal{A}$ of maximum attractors [2] (compare also [3], [7]) we obtain

$$\sup\{\|d_C\|_\infty \mid C \in \mathcal{A}\} = \tfrac{1}{5} \cdot (\tfrac{4}{5})^4.$$

This extremal value is attained in the points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ by the function d_{C_A}, where the maximum attractor C_A is given by

$$C_A(x,y) = (xy)^{A\left(\frac{\log x}{\log(xy)}\right)}$$

and the dependence function $A\colon [0,1] \to [0,1]$ by

$$A(x) = \begin{cases} 1-x & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{x+1}{2} & \text{otherwise.} \end{cases}$$

Example 2.7. Although for the copula $C_{1/3}$ we know that $d_{C_{1/3}}$ attains the maximal value of d^* in $(\frac{1}{3}, \frac{2}{3})$, there are other members of the family $(C_\lambda)_{\lambda \in [0,1]}$ such that the area of the subset of $[0,1]^2$ on which d_{C_λ} and d^* coincide is greater. In general, for $\lambda \in [0,1]$ the area of the subset of $[0,1]^2$ on which d_{C_λ} and d^* coincide equals $(1-\lambda)^2 + (\max(1-2\lambda, 0))^2 - 2(\max(1-3\lambda, 0))^2$, assuming its maximal value $\frac{9}{13}$ for $\lambda = \frac{3}{13}$ (see Figure 1 right).

3. Non-symmetry and opposite diagonal section

A closer look at the copula $C_{1/3}$ shows that the functions $d_{C_{1/3}}$ and d^* coincide on $\left[\frac{1}{3}, \frac{2}{3}\right]^2 \cup \{(x, y) \in [0, 1]^2 \mid |x - y| \geq \frac{1}{3}\}$ (see Figure 1 center). This means, in particular, that we have $d_{C_{1/3}}(x, 1-x) = d^*(x, 1-x)$ for all $x \in [0, 1]$, i.e., $C_{1/3}$ is "maximally non-symmetric" on the whole opposite diagonal $\{(x, 1-x) \mid x \in [0, 1]\}$ of the unit square $[0, 1]^2$ (note that $C_{1/3}$ is the only copula in the family $(C_\lambda)_{\lambda \in [0,1]}$ with this property).

From [5] we know that, for a given copula C, its opposite diagonal section $\omega_C\colon [0, 1] \to [0, 1]$ defined by $\omega_C(x) = C(x, 1 - x)$ must be a 1-Lipschitz function satisfying $\omega_C(0) = \omega_C(1) = 0$.

Therefore, if for some copula C we require $d_C(x, 1 - x) = d^*(x, 1 - x)$ for all $x \in [0, 1]$, the only possibilities are either $\omega_C = \omega^*$ or $\omega_C = \omega_1$, where the functions $\omega^*, \omega_1\colon [0, 1] \to [0, 1]$ are given by

$$\begin{aligned} \omega^*(x) &= \max(\min(x, \tfrac{2}{3} - x), 0), \\ \omega_1(x) &= \max(\min(1 - x, x - \tfrac{1}{3}), 0). \end{aligned}$$

However, if for some (necessarily non-symmetric) copula C we have $\omega_C = \omega^*$ then for the copula C_1 defined by $C_1(x, y) = C(y, x)$ we have $\omega_{C_1} = \omega_1$. This means that we can restrict our considerations to copulas C satisfying $\omega_C = \omega^*$.

From [5, Proposition 7.3] it follows that $C_{1/3}$ is just the greatest copula with opposite diagonal section ω^*. Moreover, because of [5, Proposition 6.5(ii)] the smallest quasi-copula $\underline{Q}_{\omega^*}$ with opposite diagonal section ω^* is given by

$$\underline{Q}_{\omega^*}(x, y) = \begin{cases} x & \text{if } (x, y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right], \\ \max(x + y - \frac{2}{3}, 0) & \text{if } (x, y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right[, \\ \max(x + y - 1, \frac{1}{3}) & \text{if } (x, y) \in \left]\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right], \\ y - \frac{1}{3} & \text{if } (x, y) \in \left]\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right[, \\ W(x, y) & \text{otherwise.} \end{cases}$$

For our special opposite diagonal section ω^*, the quasi-copula $\underline{Q}_{\omega^*}$ turns out to be a copula since it is again a shuffle of M (see Figure 2 left).

With these preliminary considerations, we are able to show:

Proposition 3.1. *There is no copula C such that $d_C = d^*$.*

PROOF: Suppose that C is a copula such that $d_C = d^*$. Then, in particular, d_C and d^* must coincide on the opposite diagonal, i.e., we must have either $\omega_C = \omega^*$ or $\omega_C = \omega_1$. Assume without loss of generality that $\omega_C = \omega^*$. Since $\underline{Q}_{\omega^*}$ and $C_{1/3}$ are the smallest and greatest copula with opposite diagonal section ω^*, it follows immediately that each copula C with $\omega_C = \omega^*$ coincides with $\underline{Q}_{\omega^*}$ and $C_{1/3}$ on

$$[0,1]^2 \setminus \Big(\big( ]0, \tfrac{1}{3}[\times]\tfrac{1}{3}, \tfrac{2}{3}[\big) \cup \big( ]\tfrac{1}{3}, \tfrac{2}{3}[\times]\tfrac{2}{3}, 1[\big) \Big)$$

(see Figure 2 center — the question marks indicate the regions where C is not uniquely determined by the lower and upper bounds $\underline{Q}_{\omega^*}$ and $C_{1/3}$). As a consequence, C coincides with the symmetric copula W on the set $\left[0, \frac{1}{3}\right]^2 \cup \left[\frac{2}{3}, 1\right]^2$, implying that d_C vanishes on this set. Since d^* vanishes only on the boundary and the diagonal of $[0,1]^2$ this shows that for no copula C the equality $d_C = d^*$ can hold. □

Since again the 2-increasingness of copulas was not used in our argument, we also have shown:

Corollary 3.2. *There is no quasi-copula Q such that $d_Q = d^*$.*

Example 3.3. Clearly, for each symmetric (quasi-)copula C the value $\|d^* - d_C\|_\infty$ attains its maximum $\frac{1}{3}$. For the family $(C_\lambda)_{\lambda \in [0,1]}$ of copulas considered in Proposition 2.2(ii) we obtain $\|d^* - d_{C_\lambda}\|_\infty = \min(\max(\frac{1}{3} - \lambda, \frac{\lambda}{2}), \frac{1}{3})$. This value is minimal for $\lambda = \frac{2}{9}$, and we get $\|d^* - d_{C_{2/9}}\|_\infty = \frac{1}{9}$. Observe, however, that also for $\lambda \in \left[\frac{2}{3}, 1\right[$ we get the maximal value $\|d^* - d_{C_\lambda}\|_\infty = \frac{1}{3}$, although the corresponding copulas C_λ are non-symmetric.

Note added in proof: Similar results were obtained independently by R.B. Nelsen (Extremes of nonexchangeability, Statist. Papers, to appear).

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