On a property of neighborhood hypergraphs

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Abstract. The aim of the paper is to show that no simple graph has a proper subgraph with the same neighborhood hypergraph. As a simple consequence of this result we infer that if a clique hypergraph $\mathcal{G}$ and a hypergraph $\mathcal{H}$ have the same neighborhood hypergraph and the neighborhood relation in $\mathcal{G}$ is a subrelation of such a relation in $\mathcal{H}$, then $\mathcal{H}$ is inscribed into $\mathcal{G}$ (both seen as coverings). In particular, if $\mathcal{H}$ is also a clique hypergraph, then $\mathcal{H} = \mathcal{G}$.

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Recall (see e.g. [1]) that a hypergraph $\mathcal{G} = (V, \mathcal{E})$ consists of a finite set V of vertices and a finite sequence $\mathcal{E}$ of hyperedges, where each hyperedge is a non-empty subset of V, and the union of all hyperedges is V (note that a hypergraph may have multiple hyperedges). A hypergraph is *simple*, if no hyperedge is contained in another hyperedge.

With an ordinary graph G at least two hypergraphs can be associated. The first consists of all maximal cliques of G and is called the clique hypergraph. These hypergraphs form an important subclass of hypergraphs. For example, they are related with the Helly property (see [1]), and they also appear in the clique-transversal problem (see [4]), and consequently in graph coloring problems (see e.g. [5]).

The second hypergraph associated with G is formed by neighborhoods of vertices. Recall (see [1]) that two vertices of G are neighbors if they are adjacent or equal. The set of all neighbors of a vertex v is denoted by $N_G(v)$ and called the neighborhood of v. Next, we take all pairwise different neighborhoods in G to obtain a new hypergraph $\mathcal{N}(G)$ on the vertex set of G, called the neighborhood hypergraph of G. $\mathcal{N}(G)$ has no multiple hyperedges, but, in general, $\mathcal{N}(G)$ is not simple. Of course, hypergraphs $\mathcal{N}(H)_{\max}$ (consisting of all maximal hyperedges of $\mathcal{N}(G)$ with respect to inclusion) and $\mathcal{N}(G)_{\min}$ (consisting of all minimal hyperedges of $\mathcal{N}(G)$) are simple, but they play no important role here.

Theorem 1. *Let G be a simple graph and H its subgraph. If $\mathcal{N}(H) = \mathcal{N}(G)$, then $H = G$.*

Proof: Take a vertex v of H such that $N_H(v)$ is maximal up to inclusion. Let X_H be the set of all vertices w of H such that $N_H(w) = N_H(v)$, and X_G be the set of all vertices w of G such that $N_G(w) = N_H(v)$.

Since $N_H(v)$ corresponds to a maximal (up to inclusion) hyperedge of $\mathcal{N}(H) = \mathcal{N}(G)$ and H is a subgraph of G, we infer that

$$X_H \subseteq X_G,$$

in particular $N_H(v) = N_G(v)$.

Assume that there is a vertex $w \in X_G \setminus X_H$. Since $N_G(w) = N_H(v)$ and $w \in N_G(w)$ (by the definition), the vertices v and w are adjacent in H, thus also in G.

Since $N_H(w) \subseteq N_G(w) = N_H(v)$ and $N_H(w) \neq N_H(v)$ (by the assumption), there exists a vertex u such that

$$u \in N_H(v) = N_G(w) \quad \text{and} \quad u \notin N_H(w).$$

Then

$$N_H(u) \neq N_G(u).$$

Hence and by the equality $\mathcal{N}(\mathcal{H}) = \mathcal{N}(\mathcal{G})$, there is a vertex u' such that

$$N_H(u) = N_G(u').$$

Then $v \in N_H(u) = N_G(u')$, i.e. the vertices v and u' are adjacent in G.

On the other hand,

$$w \notin N_H(u) = N_G(u'),$$

so

$$u' \notin N_G(w) = N_H(v) = N_G(v),$$

i.e. the vertices u' and v are not adjacent in G. This contradiction implies

$$X_H = X_G.$$

Observe now that the vertices of $X = X_H = X_G$ form a clique in both the graphs H and G and the sets of neighbors of every vertex of X in the rest of the graphs H and G are the same. Thus to end the proof it is sufficient to apply the induction (on the order of graph) to the pair of graphs $H \setminus X$ and $G \setminus X$ (note that they may have isolated vertices, but it is not a problem). $\square$

Recall that a simple hypergraph $\mathcal{G}$ is said to be a *clique hypergraph*, if it is the clique hypergraph of some graph G. Observe that neighborhoods of each vertex v in $\mathcal{G}$ and G are the same, in particular $\mathcal{N}(\mathcal{G}) = \mathcal{N}(G)$ (where neighbors in a hypergraph are defined analogously as for a graph). Moreover, G is uniquely determined (therefore it will be sometimes denoted by $G_{\mathcal{G}}$). Because two different vertices of G are adjacent if and only if they are both contained in a hyperedge of $\mathcal{G}$. Hence it also easy follows (see [1]) that a simple hypergraph $\mathcal{G} = (V, \mathcal{E})$ is a clique hypergraph if and only if for each subset A of V, the following condition holds:

(C_2) if every pair of vertices of A belongs to some hyperedge of $\mathcal{G}$, then A is contained in a hyperedge of $\mathcal{G}$.

(Hypergraphs satisfying (C_2), not necessarily simple, were called conformal by Berge in [1]. However, today the concept of conformality has a slightly different meaning (see e.g. [6]).)

(C_2) is related with the Helly property (see [1]). More precisely, a hypergraph $\mathcal{G} = (V, \mathcal{E})$ has the Helly property (i.e. for any $\mathcal{F} \subseteq \mathcal{E}$, if any two hyperedges in $\mathcal{F}$ have a non-empty intersection, then the intersection of $\mathcal{F}$ is also non-empty) if and only if its dual $\mathcal{G}^*$ satisfies (C_2). $\mathcal{G}^* = (\mathcal{E}, V^*)$ is the hypergraph whose vertices are hyperedges of $\mathcal{G}$ and the set of hyperedges is $V^* = \{\mathcal{G}(v) : v \in V\}$, where $\mathcal{G}(v) = \{E \in \mathcal{E} : v \in E\}$. Gilmore's Theorem (see Chapter 1, §7 in [1]) gives the following necessary and sufficient condition for a hypergraph $\mathcal{G}$ to satisfy (C_2): for every three hyperedges E_1, E_2, E_3 of $\mathcal{G}$, there is a hyperedge of $\mathcal{G}$ containing the set $(E_1 \cap E_2) \cup (E_2 \cap E_3) \cup (E_3 \cap E_1)$. The condition can be easily translated into the Helly property (see [1]). This result have been generalized by Berge and Duchet in [3] (see also [1]) to hypergraphs with the k-Helly property (i.e. for any family $\mathcal{F}$ of hyperedges of $\mathcal{G}$, if every subfamily of $\mathcal{F}$ with at most k elements has a non-empty intersection, then $\mathcal{F}$ also has a non-empty intersection). The k-Helly property corresponds with the condition (C_k) obtained from (C_2) by replacing "every pair" with "every subset with at most k vertices".

We say that a hypergraph $\mathcal{H}$ is *inscribed into* a hypergraph $\mathcal{G}$ if for any hyperedge F of $\mathcal{H}$ there is a hyperedge E of $\mathcal{G}$ such that $F \subseteq E$. It is just a reformulation of the well-known notion for covering in the case of hypergraphs.

Theorem 2. *Let $\mathcal{G}$ be a clique hypergraph and $\mathcal{H}$ be an arbitrary hypergraph with the same vertex set such that*

($*$) $N_{\mathcal{G}}(v) \subseteq N_{\mathcal{H}}(v)$ *for each vertex* v,

($**$) $\mathcal{N}(\mathcal{G}) = \mathcal{N}(\mathcal{H})$.

Then $\mathcal{H}$ is inscribed into $\mathcal{G}$.

Proof: Take an auxiliary graph H with the same vertex set as $\mathcal{H}$ such that two different vertices of H are adjacent if and only if they are contained in a common hyperedge of $\mathcal{H}$. Then $N_H(v) = N_{\mathcal{H}}(v)$ for any vertex v. Hence and by ($*$) we first infer that the graph $G_{\mathcal{G}}$ is a subgraph of H. Secondly, $\mathcal{N}(G_{\mathcal{G}}) = \mathcal{N}(H)$ by ($**$). Thus by Theorem 1 we obtain $G_{\mathcal{G}} = H$, i.e. $\mathcal{G}$ is the clique hypergraph of H. It easily implies that $\mathcal{H}$ is inscribed into $\mathcal{G}$. □

By the above proof we obtain in particular that for any hypergraph $\mathcal{H}$ there exists exactly one clique hypergraph $\mathcal{H}'$ with the same vertex set such that $\mathcal{H}$ is inscribed into $\mathcal{H}'$ and $N_{\mathcal{H}'}(v) = N_{\mathcal{H}}(v)$ for each vertex v (it is sufficient to take the graph H for $\mathcal{H}$ as above and its clique hypergraph).

This fact and Theorem 2 (because the relation "to be inscribed into" is a partial order for simple hypergraphs) imply that $\mathcal{G}$ is a clique hypergraph if and only if for each simple hypergraph $\mathcal{H}$ with the same vertex set, if $\mathcal{G}$ is inscribed into $\mathcal{H}$ and $\mathcal{N}(\mathcal{H}) = \mathcal{N}(\mathcal{G})$, then $\mathcal{H} = \mathcal{G}$. In particular

Corollary 3. *Let $\mathcal{G}$ and $\mathcal{H}$ be clique hypergraphs with the same vertex set satisfying $(*)$ and $(**)$. Then $\mathcal{G} = \mathcal{H}$.*

By Theorem 2 we obtain also that if a clique hypergraph $\mathcal{G}$ is a subhypergraph of a hypergraph $\mathcal{H}$ and $\mathcal{N}(\mathcal{G}) = \mathcal{N}(\mathcal{H})$, then $\mathcal{H}$ is inscribed into $\mathcal{G}$. In particular, if $\mathcal{H}$ is simple, then $\mathcal{G} = \mathcal{H}$.

Now we translate the above results for hypergraphs having the Helly property. Observe that Theorem 2 holds also for hypergraphs satisfying (C_2). Because if $\mathcal{G}$ is such a hypergraph, then $\mathcal{G}_{\max}$ is a clique hypergraph, and also $N_{\mathcal{G}_{\max}}(v) = N_{\mathcal{G}}(v)$ for any vertex v.

For hypergraphs $\mathcal{G} = (V, (E_1, \dots, E_n))$ and $\mathcal{H} = (W, (E'_1, \dots, E'_n))$ we will "assume" in the results below that $\mathcal{G}^*$ and $\mathcal{H}^*$ (and also $\mathcal{N}(\mathcal{G}^*)$ and $\mathcal{N}(\mathcal{H}^*)$) have the same vertex set $\{E_1, \dots, E_n\}$. Say more formally, we identify hyperedges E_i and E'_i, i.e. the equality $\mathcal{G}^* = \mathcal{H}^*$ denotes that the natural correspondence $E_i \longmapsto E'_i$ forms an isomorphism between these hypergraphs.

Corollary 4. *Let $\mathcal{G} = (V, (E_1, \dots, E_n))$ be a hypergraph with the Helly property. Let $\mathcal{H} = (W, (E'_1, \dots, E'_n))$ be a hypergraph satisfying*

$(*)$ *for any $1 \le i, j \le n$, $E_i \cap E_j \neq \emptyset \Longrightarrow E'_i \cap E'_j \neq \emptyset$,*

$(**)$ $\mathcal{N}(\mathcal{H}^*) = \mathcal{N}(\mathcal{G}^*)$.

Then for each $w \in W$, there is $v \in V$ such that for any $1 \le i \le n$,

$$w \in E'_i \Longrightarrow v \in E_i.$$

Proof: $(*)$ implies $N_{\mathcal{G}^*}(E_i) \subseteq N_{\mathcal{H}^*}(E'_i)$ for each $i = 1, 2, \dots, n$. Hence, $\mathcal{H}^*$ is inscribed into $\mathcal{G}^*$. This implies the thesis. □

The implication in the above result cannot be replaced by the equivalence. Take the following two hypergraphs $\mathcal{G} = \big(\{1,2,3,4\}, (\{1,2\}, \{2,3\}, \{3,4\})\big)$ and $\mathcal{H} = \big(\{1,2,3,4\}, (\{1,2\}, \{2,3,5\}, \{3,4\})\big)$. Then $\mathcal{G}$ and $\mathcal{H}$ satisfy the conditions $(*)$ and $(**)$, and $\mathcal{G}$ has the Helly property. On the other hand, $\mathcal{H}(5) = \big\{\{2,3,5\}\big\}$, and $\mathcal{G}(2) = \big\{\{1,2\}, \{2,3\}\big\}$, $\mathcal{G}(3) = \big\{\{2,3\}, \{3,4\}\big\}$.

Take a hypergraph $\mathcal{G} = (V, (E_1, \dots, E_n))$ and note that $\mathcal{G}^*$ is simple if and only if for each vertices $v, w \in V$, the following condition holds:

$$(DS) \qquad \{E_i \colon v \in E_i\} \subseteq \{E_j \colon w \in E_j\} \Longrightarrow v = w.$$

Thus by Corollary 3 we obtain (because $(\mathcal{G}^*)^* = \mathcal{G}$):

Corollary 5. *Let hypergraphs with the Helly property* $\mathcal{G} = (V, (E_1, \dots, E_n))$ *and* $\mathcal{H} = (W, (E'_1, \dots, E'_n))$ *satisfy* (DS) *and* $(*)$, $(**)$ *of Corollary* 4. *Then* $\mathcal{G} = \mathcal{H}$ *(strictly formally,* $\mathcal{G}$ *and* $\mathcal{H}$ *are isomorphic).*

Using the last corollary of Theorem 1 (i.e. its modified version in which we assume that $\mathcal{G}$ satisfies (C_2)) we can also show that if a hypergraph $\mathcal{G}$ having the Helly property is a subhypergraph of a hypergraph $\mathcal{H}$ and $\mathcal{N}(\mathcal{G}^*) = \mathcal{N}(\mathcal{H}^*)$, then $\mathcal{H}$ has also the Helly property. If $\mathcal{H}$ satisfies additionally (DS), then $\mathcal{H} = \mathcal{G}$.

Observe that to a given hypergraph $\mathcal{G} = (V, (E_1, \dots, E_n))$ new vertices can be added in such a way that the obtained hypergraph has the Helly property. More precisely, there is a hypergraph $\mathcal{G}' = (V', (E'_1, \dots, E'_n))$ such that

(i) $E_i \subseteq E'_i$ for $i = 1, \dots, n$,
(ii) for each $1 \le i, j \le n$, $\quad E'_i \cap E'_j \ne \emptyset \Longleftrightarrow E_i \cap E_j \ne \emptyset$,
(iii) $\mathcal{G}'$ has the Helly property.

Take the dual hypergraph $\mathcal{G}^*$, and the graph G with vertices $E_1, \dots, E_n$ such that E_i and E_j $(i \ne j)$ are adjacent if and only if they both belong to a hyperedge of $\mathcal{G}^*$. Next, take the hypergraph $\mathcal{H}$ consisting of all maximal cliques of G and all hyperedges of $\mathcal{G}^*$. Then $\mathcal{G}^*$ is inscribed into $\mathcal{H}$, so $\mathcal{H}_{\max}$ is a clique hypergraph, which implies that $\mathcal{H}$ satisfies (C_2). Moreover, $N_{\mathcal{H}}(E_i) = N_G(E_i) = N_{\mathcal{G}^*}(E_i)$ for each $i = 1, \dots, n$. Thus it is sufficient to take $\mathcal{G}' = \mathcal{H}^*$.

Now we show that the assumptions of Theorems 1 and 2 (thus also their corollaries) are necessary. First, the following graphs $G = \big(\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\big)$ and $H = \big(\{1,3\}, \{3,4\}, \{2,4\}, \{1,2\}\big)$ are different, but they have the same neighborhood hypergraph (because $\mathcal{N}(G)$ and $\mathcal{N}(H)$ consist of all three-element subsets of $\{1,2,3,4\}$). Further, the clique hypergraphs of G and H are equal to G and H, respectively.

Secondly, take the following hypergraphs $\mathcal{G} = (\{1,5,6,7\}, \{1,4,5,7\}, \{2,3,4,7\})$ and $\mathcal{H} = (\{1,5,6,7\}, \{1,2,4,5,7\}, \{2,3,4,7\})$. It is easy to see that they are clique hypergraphs. $\mathcal{G}$ and $\mathcal{H}$ satisfy $(*)$ of Theorem 2, and $(**)$ does not hold, since $N_{\mathcal{G}}(1) = \{1,4,5,6,7\} \notin \mathcal{N}(\mathcal{H})$. On the other hand, $\mathcal{N}(\mathcal{G})_{\max} = \mathcal{N}(\mathcal{H})_{\max}$ (because they have exactly one hyperedge $N_{\mathcal{G}}(7) = N_{\mathcal{H}}(7) = \{1, 2, \dots, 7\}$) and $\mathcal{N}(\mathcal{G})_{\min} = \mathcal{N}(\mathcal{H})_{\min}$ (because they have exactly two hyperedges $N_{\mathcal{G}}(3) = N_{\mathcal{H}}(3) = \{2,3,4,7\}$ and $N_{\mathcal{G}}(6) = N_{\mathcal{H}}(6) = \{1,5,6,7\}$). Observe also that $G_{\mathcal{G}}$ is a proper subgraph of $G_{\mathcal{H}}$ (where $G_{\mathcal{G}}$ and $G_{\mathcal{H}}$ are the graphs corresponding to $\mathcal{G}$ and $\mathcal{H}$), although $\mathcal{N}(G_{\mathcal{H}})_{\max} = \mathcal{N}(G_{\mathcal{G}})_{\max}$ and $\mathcal{N}(G_{\mathcal{H}})_{\min} = \mathcal{N}(G_{\mathcal{G}})_{\min}$.

Finally observe that our results are not true for infinite graphs and hypergraphs. Let $A = \{a_i \colon i \in \mathbb{Z}\}$ and $B = \{b_i \colon i \in \mathbb{Z}\}$ be two infinite disjoint sets (where $\mathbb{Z}$ is the set of all integers), and take

$$G_1 = \big\{\{a_i, a_j\} \colon i \ne j\big\} \cup \big\{\{b_i, b_j\} \colon i \ne j\big\} \cup \big\{\{a_i, b_j\} \colon j \le i\big\},$$
$$G_2 = \big\{\{a_i, a_j\} \colon i \ne j\big\} \cup \big\{\{b_i, b_j\} \colon i \ne j\big\} \cup \big\{\{a_i, b_j\} \colon j \le i - 1\big\}.$$

Then first G_2 is a proper subgraph of G_1. Secondly, for each $i \in \mathbb{Z}$,

$$\begin{aligned} N_{G_1}(a_i) &= A \cup \{b_j\colon\ j \leq i\}, & N_{G_1}(b_i) &= B \cup \{a_j\colon\ j \geq i\}, \\ N_{G_2}(a_i) &= A \cup \{b_j\colon\ j \leq i-1\}, & N_{G_2}(b_i) &= B \cup \{a_j\colon\ j \geq i+1\}. \end{aligned}$$

Hence, $N_{G_2}(a_i) = N_{G_1}(a_{i-1}) \subseteq N_{G_1}(a_i)$ and $N_{G_2}(b_i) = N_{G_1}(b_{i+1}) \subseteq N_{G_1}(b_i)$. In particular, $\mathcal{N}(G_1) = \mathcal{N}(G_2)$.

By the above facts we have also that the clique hypergraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of the graphs G_1 and G_2 satisfy assumptions of Theorem 2. But they are not equal, because $G_1 \neq G_2$.

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