# Direct limit of matricially Riesz normed spaces

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*Abstract.* In this paper, the $\mathcal{F}$-Riesz norm for ordered $\mathcal{F}$-bimodules is introduced and characterized in terms of order theoretic and geometric concepts. Using this notion, $\mathcal{F}$-Riesz normed bimodules are introduced and characterized as the inductive limits of matricially Riesz normed spaces.

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## 1. Introduction

Effros and Ruan, as suggested by B.E. Johnson, initiated a study of normed $\mathcal{F}$-bimodules as direct limits of matrix normed spaces [2]. In [6] the authors studied the direct limit of matrix ordered spaces. Continuing this line, in this paper we discuss the direct limits of matricially Riesz normed spaces (studied by [4], [5]). As a consequence we introduce the notion of $\mathcal{F}$-Riesz normed bimodules.

We recall the following notions discussed in [6] (see also [2]).

**Matricial notions.**
Let $V$ be a complex vector space. Let $M_n(V)$ denote the set of all $n \times n$ matrices with entries from $V$. For $V = \mathcal{C}$, we denote $M_n(\mathcal{C})$ by $M_n$. For $\alpha = [\alpha_{ij}] \in M_n$ and $v = [v_{ij}] \in M_n(V)$ we define

$$\alpha v = \left[\sum_{j=1}^{n} \alpha_{ij} v_{jk}\right], \quad v\alpha = \left[\sum_{j=1}^{n} v_{ij} \alpha_{jk}\right].$$

Then $M_n(V)$ is a $M_n$-bimodule for all $n \in \mathbb{N}$. In particular $M_n(V)$ is a complex vector space for all $n \in \mathbb{N}$. For $v \in M_n(V)$, $w \in M_m(V)$, we define

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+m}(V).$$

Next, we consider the family $\{M_n\}$. For each $n, m \in \mathbb{N}$ define $\sigma_{n,n+m} : M_n \longrightarrow M_{n+m}$ by $\sigma_{n,n+m}(\alpha) = \alpha \oplus 0_m$. Then $\sigma_{n,n+m}$ is a vector space isomorphism with

$$\sigma_{n,n+m}(\alpha\beta) = \sigma_{n,n+m}(\alpha)\sigma_{n,n+m}(\beta).$$

Thus we may "identify" $M_n$ in $M_{n+m}$ as a subalgebra for every $m \in \mathbb{N}$. More generally, we may identify $M_n$ in the set $\mathcal{F}$ of $\infty \times \infty$ complex matrices, having entries zero after first $n$ rows and first $n$ columns. Then $\mathcal{F}$ may be considered as the direct or inductive limit of the family $\{M_n\}$. In this sense

$$\mathcal{F} = \bigcup_{n=1}^{\infty} M_n.$$

Let $e_{ij}$ denote the $\infty \times \infty$ matrix with 1 at the $(i,j)$th entry and 0 elsewhere. Then the collection $\{e_{ij}\}$ is called the set of matrix units in $\mathcal{F}$. We write $1_n$ for $\sum_{i=1}^{n} e_{ii}$.

For $i, j, k, l \in \mathbb{N}$, we have $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Note that for any $\alpha \in \mathcal{F}$, there exist complex numbers $\alpha_{ij}$ such that

$$\alpha = \sum_{i,j} \alpha_{ij}e_{ij} \quad \text{(a finite sum)}.$$

Thus $\mathcal{F}$ is an algebra.

For $\alpha = \sum_{i,j} \alpha_{ij}e_{ij} \in \mathcal{F}$, we define $\alpha^* = \sum_{i,j} \bar{\alpha}_{ji}e_{ij} \in \mathcal{F}$. Then $\alpha \longmapsto \alpha^*$ is an involution. In other words, $\mathcal{F}$ is a $*$-algebra.

**Definition 1.1.** Let $V$ be a complex vector space. Consider the family $\{M_n(V)\}$. For each $n, m \in \mathbb{N}$, define $T_{n,n+m} : M_n(V) \longrightarrow M_{n+m}(V)$ by $T_{n,n+m}(v) = v \oplus 0_m$, $0_m \in M_m(V)$. Then $T_{n,n+m}$ is an injective homomorphism. Let $\mathcal{V}$ be the inductive limit of the directed family $\{M_n(V), T_{n,n+m}\}$. We shall call $\mathcal{V}$ the *matricial inductive limit* or *direct limit* of $V$.

The matricial inductive limit of a complex vector space $V$ may be characterized in the following sense:

**Theorem 1.2.** *Let $\mathcal{W}$ be a non-degenerate $\mathcal{F}$-bimodule. Put $W = e_{11}\mathcal{W}e_{11}$. Then $W$ is a complex vector space and $\mathcal{W}$ is its matricial inductive limit* ([2]).

**Definition 1.3** (Matrix normed space)**.** Let $V$ be a complex vector space. Then $M_n(V)$, the space of $n \times n$ matrices with entries from $V$, is an $M_n$-bimodule for all $n \in \mathbb{N}$. A *matrix norm* on $V$ is a sequence $\{\|\cdot\|_n\}$ such that $\|\cdot\|_n$ is a norm on $M_n(V)$ for all $n \in \mathbb{N}$. We say that $(V, \{\|\cdot\|_n\})$ is a *matrix normed space* if $\|v \oplus 0_m\|_{n+m} = \|v\|_n$ and $\|\alpha v \beta\|_n \leq \|\alpha\| \, \|v\|_n \, \|\beta\|$ for all $v \in M_n(V)$, $\alpha, \beta \in M_n$ and $n, m \in \mathbb{N}$ ([7]).

**Definition 1.4** ($\mathcal{F}$-bimodule norm)**.** Let $\mathcal{V}$ be a non-degenerate $\mathcal{F}$-bimodule. Let $\|\cdot\|$ be a norm on $\mathcal{V}$. Then we say $\|\cdot\|$ is an $\mathcal{F}$-bimodule norm on $\mathcal{V}$ if $\|\alpha v \beta\| \leq \|\alpha\| \, \|v\| \, \|\beta\|$, for any $\alpha, \beta \in \mathcal{F}$, $v \in \mathcal{V}$. In this case we say that $\mathcal{V}$ is a non-degenerate *normed $\mathcal{F}$-bimodule*.

**Theorem 1.5.** *Let $(V, \{\|\cdot\|_n\})$ be a matrix normed space. Let $\mathcal{V}$ be the matricial inductive limit of $V$. For each $v \in \mathcal{V}$, we define $\|v\|$ as follows: let $n \in \mathbb{N}$ be such that $1_n v 1_n = v$. Write $\|v\| = \|v\|_n$. Then this definition is independent of the choice of $n$ and introduces an $\mathcal{F}$-bimodule norm on $\mathcal{V}$ such that $(\mathcal{V}, \|\cdot\|)$ is a non-degenerate normed $\mathcal{F}$-bimodule.*

*Conversely, let $(\mathcal{W}, \|\cdot\|)$ be a non-degenerate normed $\mathcal{F}$-bimodule and let $W = 1_1 \mathcal{W} 1_1$ and $\|\cdot\|_n = \|\cdot\| |_{M_n(W)}$ for all $n \in \mathbb{N}$. Then $(W, \{\| \|_n\})$ is a matrix normed space whose matricial inductive limit is $(\mathcal{W}, \|\cdot\|)$.*

**Remark.** This characterization can be extended to $*$ vector spaces as follows: Let $V$ be a $*$ vector space and let $\mathcal{V}$ be the matricial inductive limit of $V$, so that $\mathcal{V}$ is a non-degenerate $\mathcal{F}$-bimodule ([6]). Let $(V, \{\|\cdot\|_n\})$ be a matrix normed space such that for every $n \in \mathbb{N}$ and $v \in M_n(V)$, $\|v^*\|_n = \|v\|_n$. Let $(\mathcal{V}, \|\cdot\|)$ be the matricial inductive limit of the matrix normed space $(V, \{\|\cdot\|_n\})$. Then $\|v^*\| = \|v\|$ for all $v \in \mathcal{V}$.

Next, we recall the definition of an ordered $\mathcal{F}$-bimodule and its characterization as a matricial inductive limit space from [6]:

**Definition 1.6** (Ordered $\mathcal{F}$-bimodule)**.** Let $\mathcal{V}$ be a $*$-$\mathcal{F}$-bimodule. Let $\mathcal{V}^+$ be a bimodule cone in $\mathcal{V}_{sa}$. That is

1. $v_1, v_2 \in \mathcal{V}^+ \Rightarrow v_1 + v_2 \in \mathcal{V}^+$,
2. $v \in \mathcal{V}^+, \alpha \in \mathcal{F} \Rightarrow \alpha^* v \alpha \in \mathcal{V}^+$.

Then $(\mathcal{V}, \mathcal{V}^+)$ will be called an *ordered $\mathcal{F}$-bimodule*.

The following result is obtained from [6].

**Theorem 1.7.** *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Let $\mathcal{V}$ be the matricial inductive limit of $V$. Then $(\mathcal{V}, \mathcal{V}^+)$ is a non-degenerate ordered $\mathcal{F}$-bimodule, where $\mathcal{V}^+ = \bigcup_{n=1}^{\infty} M_n(V)^+$. Conversely, let $(\mathcal{W}, \mathcal{W}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Put $W = 1_1 \mathcal{W} 1_1$ and $M_n(W)^+ = 1_n \mathcal{W}^+ 1_n$ for all $n \in \mathbb{N}$. Then $(W, \{M_n(W)^+\})$ is a matrix ordered space with $\mathcal{W}^+ = \bigcup_{n=1}^{\infty} M_n(W)^+$.*

## 2. $\mathcal{F}$-Riesz norm

We now characterize $\mathcal{F}$-bimodule norms.

**Definition 2.1.** Let $\mathcal{V}$ be a non-degenerate $\mathcal{F}$-bimodule. Let $\mathcal{U} \subset \mathcal{V}$. We say $\mathcal{U}$ is *absolutely $\mathcal{F}$-convex* if $\sum_{i=1}^{k} \alpha_i u_i \beta_i \in \mathcal{U}$ whenever $u_1, u_2, \ldots, u_k \in \mathcal{U}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k \in \mathcal{F}$ with $\sum_{i=1}^{k} \|\alpha_i\|^2 \leq 1$ and $\sum_{i=1}^{k} \|\beta_i\|^2 \leq 1$. If the property holds true only for $k = 1$ then we say $\mathcal{U}$ is *$\mathcal{F}$-circled*.

**Theorem 2.2.** *The open unit ball of a non-degenerate normed $\mathcal{F}$-bimodule $(\mathcal{V}, \|\cdot\|)$ is absolutely $\mathcal{F}$-convex and absorbing.*

Proof: Let $\mathcal{U}$ denote the open unit ball of $(\mathcal{V}, \|\cdot\|)$. Let $u_1, u_2, \dots, u_k \in \mathcal{U}$ and $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$ with $\sum_{i=1}^k \|\alpha_i\|^2 \le 1$ and $\sum_{i=1}^k \|\beta_i\|^2 \le 1$. Consider $u = \sum_{i=1}^k \alpha_i u_i \beta_i$. Then

$$\begin{aligned}\|u\| = \left\|\sum_{i=1}^k \alpha_i u_i \beta_i\right\| &\le \sum_{i=1}^k \|\alpha_i\| \, \|u_i\| \, \|\beta_i\| < \sum_{i=1}^k \|\alpha_i\| \, \|\beta_i\| \\ &\le \left(\sum_{i=1}^k \|\alpha_i\|^2\right)^{1/2} \left(\sum_{i=1}^k \|\beta_i\|^2\right)^{1/2} \le 1.\end{aligned}$$

Therefore $u \in \mathcal{U}$. Thus $\mathcal{U}$ is absolutely $\mathcal{F}$-convex. To show that $\mathcal{U}$ is absorbing consider a $v \in \mathcal{V}$ and $\epsilon > 0$. Put $v_1 = \frac{v}{(\|v\|+\epsilon)}$. Then $v_1 \in \mathcal{U}$ and $v = v_1 (\|v\| + \epsilon)$. Therefore $\mathcal{U}$ is absorbing. $\square$

The following theorem completes the characterization of $\mathcal{F}$-bimodule norms among norms on $\mathcal{V}$.

**Theorem 2.3.** *Let $\mathcal{A} \subset \mathcal{V}$ be absolutely $\mathcal{F}$-convex and absorbing. Then the gauge of $\mathcal{A}$,*

$$p(v) = \inf\{k > 0 \mid v \in k\mathcal{A}\}$$

*determines an $\mathcal{F}$-bimodule semi-norm on $\mathcal{V}$.*

Proof: First we note that $p(v) \ge 0$ for all $v \in \mathcal{V}$. From the definition, we get that $p(kv) = |k| p(v)$ for all $k \in \mathcal{C}$. We now show that $p(v + w) \le p(v) + p(w)$ for all $v, w \in \mathcal{V}$. Let $v, w \in \mathcal{V}$ and $S\epsilon > 0$. Then there exist $k_1, k_2 > 0$ such that $k_1 < p(v) + \frac{\epsilon}{2}$ with $v \in k_1\mathcal{A}$ and $k_2 < p(w) + \frac{\epsilon}{2}$ with $w \in k_2\mathcal{A}$. We show that $v + w \in (k_1 + k_2)\mathcal{A}$. We set $\alpha = \frac{k_1}{k_1+k_2}$, $\beta = \frac{k_2}{k_1+k_2}$. Then $\alpha + \beta = 1$. Also $\frac{\alpha v}{k_1} = \frac{v}{k_1+k_2}$, $\frac{\beta w}{k_2} = \frac{w}{k_1+k_2}$. Thus we get $\frac{\alpha v}{k_1} + \frac{\beta w}{k_2} = \frac{v+w}{k_1+k_2}$. As $\mathcal{A}$ is absolutely $\mathcal{F}$-convex, it is convex. Thus $v + w \in (k_1 + k_2)\mathcal{A}$. It follows that

$$p(v + w) \le k_1 + k_2 < p(v) + p(w) + \epsilon.$$

As $\epsilon > 0$ is arbitrary we get that $p(v + w) \le p(v) + p(w)$. Next, we show that $p(\alpha v \beta) \le \|\alpha\| \, p(v) \, \|\beta\|$ for all $\alpha, \beta \in \mathcal{F}$, $v \in \mathcal{V}$. First, let $v \in \mathcal{A}$. Then $p(v) \le 1$. Let $\alpha, \beta \in \mathcal{F}$ with $\|\alpha\| \le 1$, $\|\beta\| \le 1$. Since $\mathcal{A}$ is absolutely $\mathcal{F}$-convex, $\alpha v \beta \in \mathcal{A}$. Therefore $p(\alpha v \beta) \le 1$. Now let $v \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$, $\epsilon > 0$. Put $v_1 = \frac{v}{p(v)+\epsilon}$. Then $p(v_1) = \frac{p(v)}{p(v)+\epsilon} < 1$. That is $v_1 \in \mathcal{A}$. Without loss of generality we may take $\alpha \ne 0$, $\beta \ne 0$. Let $\alpha_1 = \frac{\alpha}{\|\alpha\|}$, $\beta_1 = \frac{\beta}{\|\beta\|}$. Then $p(\alpha_1 v_1 \beta_1) \le 1$ so that

$$p(\alpha v \beta) \le \|\alpha\| \left(p(v) + \epsilon\right) \|\beta\|.$$

As $\epsilon > 0$ is arbitrary we get

$$p(\alpha v\beta) \leq \|\alpha\| \left(p(v)\right) \|\beta\| .$$

Hence $p(\cdot)$ is a $\mathcal{F}$-semi-norm on $\mathcal{V}$. $\square$

In the rest of the paper we will be dealing with non-degenerate ordered $\mathcal{F}$-bimodules. We introduce some more notations.

We write $I_n = \sum_{i=1}^{n} e_{ii}$, $J_n = \sum_{i=1}^{n} e_{i,n+i}$ for any $n \in \mathbb{N}$. Note that $\|I_n\| = \|J_n\| = 1$ and $J_n I_n = 0$, $I_n J_n = J_n$, $J_n J_n = 0$, $J_n J_n^* = I_n$. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule ([6]). Let $u_1, u_2 \in \mathcal{V}^*$ and $n \in \mathbb{N}$ such that $1_n u_1 1_n = u_1$, $1_n u_2 1_n = u_2$. We denote $u_1 + J_n^* u_2 J_n$ by $(u_1, u_2)_n^+$. For any $v \in \mathcal{V}$ and an $n \in \mathbb{N}$ with $1_n v 1_n = v$ we denote $I_n v J_n + J_n^* v^* I_n$ by $sa_n(v)$.
Before we define $\mathcal{F}$-Riesz norm, we need the following reformulation of the concept that $\mathcal{V}^+$ is generating.

**Proposition 2.4.** *Let $\left(\mathcal{V}, \mathcal{V}^+\right)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Then $\mathcal{V}^+$ is generating if and only if for every $v \in \mathcal{V}$ there exist $u_1, u_2 \in \mathcal{V}^+$ such that $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$, for a suitable $n \in \mathbb{N}$.*

**Note.** In the notation $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$, we say that $n \in \mathbb{N}$ is "*suitable*" provided $1_n u_1 1_n = u_1$, $1_n u_2 1_n = u_2$ and $1_n v 1_n = v$. This terminology will be used throughout the paper without any further explanation.

Proof: First, let $\mathcal{V}^+$ be generating. Let $v \in \mathcal{V}_{sa}$. Then by [6, Theorem 3.10] there exist $v_1, v_2 \in \mathcal{V}^+$ such that $v = v_1 - v_2$. Put $u = v_1 + v_2$. Then $u \in \mathcal{V}^+$ and $u \pm v \in \mathcal{V}^+$. Next let $v \in \mathcal{V}$ be arbitrary. Find an $n \in \mathbb{N}$ such that $1_n v 1_n = v$. Consider $sa_n(v)$: $sa_n(v) = I_n v J_n + J_n^* v^* I_n \in \mathcal{V}_{sa}$. Then as above there exists a $u \in \mathcal{V}^+$ such that $u \pm sa_n(v) \in \mathcal{V}^+$. Let $u' = I_{2n} u I_{2n} \in \mathcal{V}^+$. Then $u' \pm sa_n(v) \in \mathcal{V}^+$ for $I_{2n} sa_n(v) I_{2n} = sa_n(v)$. Set $u_1 = I_n u' I_n$, $u_2 = J_n u' J_n^*$. Then $(u_1, u_2)_n^+ = I_n u' I_n + J_n^* \left(J_n u' J_n^*\right) J_n$. We show that $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$. Note that

$$(1) \quad \begin{aligned} &I_n u' I_n - I_n u' J_n^* J_n - J_n^* J_n u' I_n + J_n^* J_n u' J_n^* J_n \mp sa_n(v) \\ &\qquad = (I_n - J_n^* J_n)\left(u' \pm sa_n(v)\right)(I_n - J_n^* J_n) \in \mathcal{V}^+. \end{aligned}$$

Similarly

$$(2) \quad \begin{aligned} &I_n u' I_n + I_n u' J_n^* J_n + J_n^* J_n u' I_n + J_n^* J_n u' J_n^* J_n \pm sa_n(v) \\ &\qquad = (I_n + J_n^* J_n)\left(u' \pm sa_n(v)\right)(I_n + J_n^* J_n) \in \mathcal{V}^+. \end{aligned}$$

Adding (1) and (2) suitably, we get

$$(u_1, u_2)_n^+ \pm sa_n(v) = I_n u' I_n + J_n^* \left(J_n u' J_n^*\right) J_n \pm sa_n(v) \in \mathcal{V}^+.$$

Conversely assume that for every $v \in \mathcal{V}$ there exist $u_1, u_2 \in \mathcal{V}^+$ such that $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$, for a suitable $n \in \mathbb{N}$. We show that $\mathcal{V}^+$ is generating. Let $v \in \mathcal{V}$. Then there exist $u_1, u_2 \in \mathcal{V}^+$ such that $(u_1, u_2)_n^+ \pm sa_n(v) \in \mathcal{V}^+$, for a suitable $n \in \mathbb{N}$. Therefore

$$(I_n + J_n)\left((u_1, u_2)_n^+ \pm sa_n(v)\right)(I_n + J_n^*) \in \mathcal{V}^+.$$

This gives $u_1 + u_2 \pm (v + v^*) \in \mathcal{V}^+$. Similarly

$$(I_n + iJ_n)\left((u_1, u_2)_n^+ \pm sa_n(v)\right)(I_n - iJ_n^*) \in \mathcal{V}^+$$

which gives $u_1 + u_2 \pm i\,(v - v^*) \in \mathcal{V}^+$. Put

$$\begin{aligned} v_0 &= \frac{1}{4}\left(u_1 + u_2 + v + v^*\right), \\ v_1 &= \frac{1}{4}\left(u_1 + u_2 - i(v - v^*)\right), \\ v_2 &= \frac{1}{4}\left(u_1 + u_2 - v - v^*\right), \\ v_3 &= \frac{1}{4}\left(u_1 + u_2 + i(v - v^*)\right). \end{aligned}$$

Then $v_0, v_1, v_2, v_3 \in \mathcal{V}^+$ and we have

$$v_0 + iv_1 - v_2 - iv_3 = v.$$

Hence $\mathcal{V}^+$ is generating. □

**Definition 2.5.** Let $(\mathcal{V}, \mathcal{V}^+)$ be a positively generated non-degenerate ordered $\mathcal{F}$-bimodule. Let $\|\cdot\|$ be an $\mathcal{F}$-bimodule norm on $\mathcal{V}$. We say $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ if for any $v \in \mathcal{V}$,

$$\begin{aligned} \|v\| = \inf\{\max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ \text{for some } \ u_1, u_2 \in \mathcal{V}^+ \ \text{ and a suitable } \ N \in \mathbb{N}\}. \end{aligned}$$

In what follows we characterize $\mathcal{F}$-Riesz norms on a non-degenerate positively ordered $\mathcal{F}$-bimodule in the lines of Theorem 2.2.

**Definition 2.6.** Let $(\mathcal{V}, \mathcal{V}^+)$ be an ordered $\mathcal{F}$-bimodule and $\mathcal{A} \subset \mathcal{V}^+$. We define $\mathcal{S}(\mathcal{A})$ as follows:

$$\begin{aligned} \mathcal{S}(\mathcal{A}) = \{v \in \mathcal{V} \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \\ \text{for some } \ u_1, u_2 \in \mathcal{A} \ \text{ and a suitable } \ N \in \mathbb{N}\}. \end{aligned}$$

**Remarks.**

(a) $\mathcal{A} \subset \mathcal{S}(\mathcal{A})$.
(b) $v^* \in \mathcal{S}(\mathcal{A})$ whenever $v \in \mathcal{S}(\mathcal{A})$.

**Definition 2.7.** Let $\mathcal{A} \subset \mathcal{V}^+$. Then we say that $\mathcal{A}$ is *order absolutely $\mathcal{F}$-convex* if $\sum_{i=1}^k \alpha_i^* u_i \alpha_i \in \mathcal{A}$ whenever $u_1, u_2, \dots, u_k \in \mathcal{A}$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{F}$ with $\sum_{i=1}^k \|\alpha_i^* \alpha_i\| \leq 1$.

If the above condition holds only for $k = 1$ for some $\mathcal{A} \subset \mathcal{V}^+$, then we say $\mathcal{A}$ is *order $\mathcal{F}$-circled*.

**Definition 2.8.** $\mathcal{S} \subset \mathcal{V}^+$ is called *$\mathcal{F}$-absorbing* if for each $v \in \mathcal{V}$ there exist $\alpha, \beta \in \mathcal{F}$ such that $\alpha v \beta \in \mathcal{S}$.

**Definition 2.9.** $\mathcal{S} \subset \mathcal{V}^+$ is called *positively $\mathcal{F}$-absorbing* if for each $u \in \mathcal{V}^+$ there exists a $\alpha \in \mathcal{F}$ such that $\alpha^* u \alpha \in \mathcal{S}$.

**Lemma 2.10.** *Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex. Then $\mathcal{S}(\mathcal{A})$ is absolutely $\mathcal{F}$-convex.*

PROOF: Let $v_1, v_2, \dots, v_k \in \mathcal{S}(\mathcal{A})$ and let $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$ with $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$ and $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$. Then for each $i = 1, 2, \dots, k$ there exist $N_i \in \mathbb{N}$, $u_1^i, u_2^i \in \mathcal{A}$ with $1_{N_i} v_i 1_{N_i} = v_i$, $1_{N_i} u_1^i 1_{N_i} = u_1^i$, $1_{N_i} u_2^i 1_{N_i} = u_2^i$ with $\left(u_1^i, u_2^i\right)_{N_i}^+ \pm sa_{N_i}(v_i) \in \mathcal{V}^+$. Now $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{F}$. Therefore there exist $M_1, M_2, \dots, M_k \in \mathbb{N}$ such that $1_{M_i} \alpha_i 1_{M_i} = \alpha_i$, $i = 1, 2, \dots, k$. Also $\beta_1, \beta_2, \dots, \beta_k \in \mathcal{F}$. Therefore there exist $P_1, P_2, \dots, P_k \in \mathbb{N}$ such that $1_{P_i} \beta_i 1_{P_i} = \beta_i$, $i = 1, 2, \dots, k$. Let $N = \max\{N_1, N_2, \dots, N_k, M_1, \dots, M_k, P_1, \dots, P_k\}$. Then for each $i = 1, 2, \dots, k$ we have $\left(u_1^i, u_2^i\right)_N^+ \pm sa_N(v_i) \in \mathcal{V}^+$. Now $\left(\left(\alpha_i^*, \beta_i\right)_N^+\right)^* \left(\left(u_1^i, u_2^i\right)_N^+ \pm sa_N(v_i)\right) \left(\left(\alpha_i^*, \beta_i\right)_N^+\right) \in \mathcal{V}^+$ for all $i = 1, 2, \dots, k$. This means $\left(\alpha_i u_1^i \alpha_i^*, \beta_i^* u_2^i \beta_i\right)_N^+ \pm sa_N\left(\alpha_i v_i \beta_i\right) \in \mathcal{V}^+$ for each $i = 1, 2, \dots, k$. Adding $\left(\sum_{i=1}^k \alpha_i u_1^i \alpha_i^*, \sum_{i=1}^k \beta_i^* u_2^i \beta_i\right)_N^+ \pm sa_N\left(\sum_{i=1}^k \alpha_i v_i \beta_i\right) \in \mathcal{V}^+$. Since $\mathcal{A}$ is absolutely convex and $\sum_{i=1}^k \|\alpha_i\|^2 \leq 1$ and $\sum_{i=1}^k \|\beta_i\|^2 \leq 1$ we have $\sum_{i=1}^k \alpha_i u_1^i \alpha_i^* \in \mathcal{A}$ and $\sum_{i=1}^k \beta_i^* u_2^i \beta_i \in \mathcal{A}$. Therefore $\sum_{i=1}^k \alpha_i v_i \beta_i \in \mathcal{S}(\mathcal{A})$. Therefore $\mathcal{S}(\mathcal{A})$ is absolutely $\mathcal{F}$-convex. □

**Lemma 2.11.** *Let $\mathcal{V}^+$ be generating. Then $\mathcal{S}(\mathcal{A})$ is $\mathcal{F}$-absorbing if $\mathcal{A} \subset \mathcal{V}^+$ is positively $\mathcal{F}$-absorbing.*

PROOF: Let $\mathcal{A} \subset \mathcal{V}^+$ be positively $\mathcal{F}$-absorbing. Let $v \in \mathcal{V}$. Since $\mathcal{V}^+$ is generating, by Proposition 2.4, there exist $u_1, u_2 \in \mathcal{V}^+$ and a suitable $N \in \mathbb{N}$ such that $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$. Since $\mathcal{A}$ is positively $\mathcal{F}$-absorbing and $u_1, u_2 \in \mathcal{V}^+$ there exist $\alpha, \beta \in \mathcal{F}$ such that $\alpha^* u_1 \alpha \in \mathcal{A}$, $\beta^* u_2 \beta \in \mathcal{A}$. Find $M \in \mathbb{N}$ such that $1_M u_1 1_M = u_1$, $1_M u_2 1_M = u_2$, $1_M v 1_M = v$, $1_M \alpha 1_M = \alpha$,

$1_M\beta 1_M = \beta$. Then $\left((\alpha,\beta)^+_M\right)^* \left((u_1,u_2)^+_M \pm s a_M(v)\right)(\alpha,\beta)^+_M \in \mathcal{V}^+$. This gives $(\alpha^* u_1\alpha, \beta^* u_2\beta)^+_M \pm s a_M(\alpha^* v\beta) \in \mathcal{V}^+$. Since $\alpha^* u_1 \alpha \in \mathcal{A}$ and $\beta^* u_2\beta \in \mathcal{A}$, we get $\alpha^* v\beta \in \mathcal{S}(\mathcal{A})$. Hence $\mathcal{S}(\mathcal{A})$ is $\mathcal{F}$-absorbing. □

Some more concepts will be needed in the sequel.

**Definition 2.12.** Let $\mathcal{A} \subset \mathcal{V}^+$. $\mathcal{A}$ is called *positively bounded* if for any $v \in \mathcal{V}_{sa}$, $v + k_n a_n \in \mathcal{V}^+$ for all $n \in \mathbb{N}$ implies $v \in \mathcal{V}^+$, where $\{a_n\}$ is a sequence in $\mathcal{A}$ and $\{k_n\}$ is a sequence in $(0,\infty)$ with $\inf k_n = 0$.

**Definition 2.13.** Let $\mathcal{A} \subset \mathcal{V}^+$. $\mathcal{A}$ is called *almost positively bounded* if $\left(k_n u_1^n, k_n u_2^n\right)^+_{N_n} \pm s a_{N_n}(v) \in \mathcal{V}^+$ for all $n \in \mathbb{N}$ implies $v = 0$ where $\{u_1^n\}$, $\{u_2^n\}$ are sequences in $\mathcal{A}$ and $\{k_n\}$ is a sequence in $(0,\infty)$ with $\inf k_n = 0$, $\{N_n\}$ is a sequence in $\mathbb{N}$.

**Lemma 2.14.** *Let $\mathcal{V}^+$ be proper. Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex and positively bounded. Then $\mathcal{A}$ is almost positively bounded.*

Proof: Let $v \in \mathcal{V}$, sequences $\{u_1^n\}$, $\{u_2^n\}$ be in $\mathcal{A}$, $\{k_n\}$ be a sequence in $(0,\infty)$ with $\inf k_n = 0$ and $\{N_n\}$ be a sequence in $\mathbb{N}$ such that

$$Z_{N_n} = (k_n u_1^n, k_n u_2^n)^+_{N_n} \pm s a_{N_n}(v) \in \mathcal{V}^+$$

for all $n \in \mathbb{N}$. Then

$$(I_{N_n} + J_{N_n})\, Z_{N_n}\, (I_{N_n} + J_{N_n})^* = k_n u_1^n + k_n u_2^n \pm (v + v^*) \tag{1}$$

and

$$(I_{N_n} + iJ_{N_n})\, Z_{N_n}\, (I_{N_n} + iJ_{N_n})^* = k_n u_1^n + k_n u_2^n \pm i\,(v - v^*)\,. \tag{2}$$

Put $u_1^n + u_2^n = 2u_n$ for all $n \in \mathbb{N}$. From (1) and (2) we get

$$k_n u_n \pm \mathrm{Re}(v),\ \ k_n u_n \pm \mathrm{Im}(v) \in \mathcal{V}^+. \tag{3}$$

Since $\mathcal{A}$ is convex as it is order absolutely $\mathcal{F}$-convex, $u_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. As $\mathcal{A}$ is positively bounded, from (3) we get $\pm \mathrm{Re}\, v, \pm \mathrm{Im}\, v \in \mathcal{V}^+$. Finally as $\mathcal{V}^+$ is proper, we have $\mathrm{Re}\, v = 0$, $\mathrm{Im}\, v = 0$. That is $v = 0$. Hence $\mathcal{A}$ is almost positively bounded. □

**Remark.** It may be noted that the notion of (almost-)positively bounded sets is introduced to generalize the notion of (almost-)Archimedean property of the cone ([5]).

Now we are in a position to characterize $\mathcal{F}$-Riesz norms.

**Theorem 2.15.** *Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate positively generated ordered $\mathcal{F}$-bimodule. Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex, almost positively bounded and positively $\mathcal{F}$-absorbing. Also assume that $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$. Let $p(\cdot)$ be the gauge of $\mathcal{S}(\mathcal{A})$. Then $p(\cdot)$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$.*

*Conversely, let $\|\cdot\|$ be an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ where $(\mathcal{V}, \mathcal{V}^+)$ is a positively generated ordered $\mathcal{F}$-bimodule. Also let $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\} = \mathcal{U} \cap \mathcal{V}^+$, where $\mathcal{U}$ is the open unit ball of $(\mathcal{V}, \|\cdot\|)$. Then $\mathcal{U}^+$ is order absolutely $\mathcal{F}$-convex, almost positively bounded and positively $\mathcal{F}$-absorbing.*

Proof: First assume that $(\mathcal{V}, \mathcal{V}^+)$ is a non-degenerate positively generated ordered $\mathcal{F}$-bimodule. Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex, almost positively bounded and positively $\mathcal{F}$-absorbing. Also assume that $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$. Let $p(\cdot)$ be the gauge of $\mathcal{S}(\mathcal{A})$. We show that $p(\cdot)$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$. In the light of Theorem 2.3, Lemmas 2.10 and 2.11 we note that $p(\cdot)$ is a $\mathcal{F}$-semi-norm on $\mathcal{V}$. Let $v \in \mathcal{V}$. We show that

$$p(v) = \inf\{\max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

Since $\mathcal{S}(\mathcal{A})$ is $\mathcal{F}$-absorbing there exists some $\lambda > 0$ such that $\lambda v \in \mathcal{S}(\mathcal{A})$. This gives some $u_1, u_2 \in \mathcal{A}$ and a $N \in \mathbb{N}$ such that $(u_1, u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$. That is $\left(\lambda^{-1}u_1, \lambda^{-1}u_2\right)_N^+ \pm sa_N(v) \in \mathcal{V}^+$. Also $p(\lambda^{-1}u_1) = \lambda^{-1}p(u_1)$. Since $p(\cdot)$ is the gauge of $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$, we have $p(u_1) \leq 1$ and $p(u_2) \leq 1$. Therefore $p(\lambda^{-1}u_1) \leq \lambda^{-1}$, $p(\lambda^{-1}u_2) \leq \lambda^{-1}$. That is $\max\{p(\lambda^{-1}u_1), p(\lambda^{-1}u_2)\} \leq \lambda^{-1}$. Let $\epsilon > 0$. Then $(p(v) + \epsilon)^{-1}v \in \mathcal{S}(\mathcal{A})$. Replacing $\lambda$ by $(p(v) + \epsilon)$ in the above discussion, there exist $u_1, u_2 \in \mathcal{V}^+$ and some $N \in \mathbb{N}$ such that $(u_1, u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$ and $\max\{p(u_1), p(u_2)\} \leq (p(v) + \epsilon)$. That is,

$$p(v) \geq \inf\{\max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

Let $u_1, u_2 \in \mathcal{V}^+$ and $(u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+$ for some $N \in \mathbb{N}$. Find a $\lambda > 0$ such that $\lambda u_1, \lambda u_2 \in \mathcal{S}(\mathcal{A})$. This gives $(\lambda u_1, \lambda u_2)_N^+ \pm sa_N(\lambda v) \in \mathcal{V}^+$. Since $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$, we get $\lambda u_1, \lambda u_2 \in \mathcal{A}$. That is $\lambda v \in \mathcal{S}(\mathcal{A})$. Therefore $p(v) \leq \lambda^{-1}$. Let $\epsilon > 0$. Put $\lambda = (\max\{p(u_1), p(u_2)\} + \epsilon)^{-1}$. Then $\lambda u_1, \lambda u_2 \in \mathcal{S}(\mathcal{A})$ so that $p(v) \leq \max\{p(u_1), p(u_2)\} + \epsilon$. This gives

$$p(v) \leq \inf\{\max(p(u_1), p(u_2)) \mid (u_1, u_2)_N^+ \pm sa_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

Therefore $p(\cdot)$ is $\mathcal{F}$-Riesz semi-norm on $\mathcal{V}$. Now let $v \in \mathcal{V}$ be such that $p(v) = 0$. Then there is a sequence $\{k_n\}$ in $(0, \infty)$ with $\inf k_n = 0$ such that $k_n^{-1}v \in \mathcal{S}(\mathcal{A})$.

Thus for every $n \in \mathbb{N}$, there exist $u_1^n, u_2^n \in \mathcal{A}$ such that $(u_1^n, u_2^n)_{N_n}^+ \pm sa_{N_n}(k_n^{-1}v) \in \mathcal{V}^+$ for suitable $N_n \in \mathbb{N}$. This means that $\left(k_n u_1^n, k_n u_2^n\right)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$. Since $\mathcal{A}$ is almost positively bounded, we get $v = 0$. Hence $p(\cdot)$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$.

Conversely, let $\|\cdot\|$ be an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ where $\left(\mathcal{V}, \mathcal{V}^+\right)$ is a positively generated ordered $\mathcal{F}$-bimodule. Also let $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\} = \mathcal{U} \cap \mathcal{V}^+$, where $\mathcal{U}$ is the open unit ball of $(\mathcal{V}, \|\cdot\|)$. We show that $\mathcal{U}^+$ is order absolutely $\mathcal{F}$-convex, almost positively bounded and positively $\mathcal{F}$-absorbing.

Let $u \in \mathcal{U}$. Find an $\epsilon > 0$ such that $\|u\| + \epsilon < 1$. Since $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm there exist $u_1, u_2 \in \mathcal{V}^+$, a suitable $N \in \mathbb{N}$ such that $(u_1, u_2)_N^+ \pm sa_N(u) \in \mathcal{V}^+$ and $\max\{\|u_1\|, \|u_2\|\} < \|u\| + \epsilon < 1$. That is $\|u_1\| < 1, \|u_2\| < 1$. This means $u_1, u_2 \in \mathcal{U}^+$. That is $u \in \mathcal{S}(\mathcal{A})$. Thus $\mathcal{U} \subset \mathcal{S}(\mathcal{U}^+)$. Let $v \in \mathcal{S}(\mathcal{U}^+)$. Then there exist $u_1, u_2 \in \mathcal{U}^+$ and a suitable $N \in \mathbb{N}$ such that $(u_1, u_2,)_N^+ \pm sa_N(v) \in \mathcal{V}^+$. Since $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm, we have $\|v\| \leq \max\{\|u_1\|, \|u_2\|\} < 1$. Therefore $v \in \mathcal{U}$ or $\mathcal{S}(\mathcal{U}^+) \subset \mathcal{U}$. Therefore $\mathcal{S}(\mathcal{U}^+) = \mathcal{U}$. Next, let $u_1, u_2, \ldots, u_k \in \mathcal{U}^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{F}$ with $\sum_{i=1}^k \left\|\alpha_i^* \alpha_i\right\| \leq 1$. Put $u = \sum_{i=1}^k \alpha_i^* u_i \alpha_i$. Then $u \in \mathcal{V}$ and

$$\|u\| \leq \sum_{i=1}^{k} \|\alpha_i\|^2 \|u_i\| < \sum_{i=1}^{k} \|\alpha_i\|^2 \leq 1.$$

It follows $\mathcal{U}^+$ is order absolutely $\mathcal{F}$-convex. We now prove that $\mathcal{U}^+$ is almost positively bounded. Let $v \in \mathcal{V}$ and sequences $\{u_1^n\}$, $\{u_2^n\}$ be in $\mathcal{U}^+$ and $\{k_n\}$ in $(0, \infty)$ with $\inf k_n = 0$ and $\{N_n\}$ a sequence in $\mathbb{N}$ such that $\left(k_n u_1^n, k_n u_2^n\right)_{N_n}^+ \pm sa_{N_n}(v) \in \mathcal{V}^+$ for all $n \in \mathbb{N}$. We show that $\|v\| = 0$. Let $\epsilon > 0$. Since $\inf k_n = 0$ there exists a $n_0 \in \mathbb{N}$ such that $k_{n_0} < \epsilon$. As $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm and $\left\|u_1^{n_0}\right\| < 1$, $\left\|u_2^{n_0}\right\| < 1$, we have $\|v\| \leq \max\{\left\|k_{n_0} u_1^{n_0}\right\|, \left\|k_{n_0} u_2^{n_0}\right\|\} < k_{n_0} < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\|v\| = 0$. Since $\|\cdot\|$ is a norm, $v = 0$. Hence $\mathcal{U}^+$ is almost-positively bounded. Finally, let $v \in \mathcal{V}^+$ and $\epsilon > 0$. Put $\alpha = (\|v\| + \epsilon)^{-\frac{1}{2}} 1_n$ where $1_n v 1_n = v$. Then $\alpha^* v \alpha = \frac{1}{(\|v\| + \epsilon)} 1_n v 1_n = \frac{v}{(\|v\| + \epsilon)} \in \mathcal{U}^+$. Therefore $\mathcal{U}^+$ is positively $\mathcal{F}$-absorbing. □

**Theorem 2.16.** *Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Let $\mathcal{V}^+$ be proper and generating. Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex, positively bounded and $\mathcal{F}$-absorbing. Assume that $\mathcal{S}(\mathcal{A}) \cap \mathcal{V}^+ = \mathcal{A}$. Let $p(\cdot)$ be the gauge of $\mathcal{S}(\mathcal{A})$. Then $p(\cdot)$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ such that $\mathcal{V}^+$ is $p$-closed.*

*Conversely, let $(\mathcal{V}, \mathcal{V}^+)$ be an ordered $\mathcal{F}$-bimodule and $\mathcal{V}^+$ be generating. Let $\|\cdot\|$ be an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ such that $\mathcal{V}^+$ is closed. Let $\mathcal{U}^+ = \{v \in \mathcal{V}^+ \mid \|v\| < 1\}$. Then $\mathcal{U}^+$ is order absolutely $\mathcal{F}$-convex, positively bounded and positively $\mathcal{F}$-absorbing such that $\mathcal{S}(\mathcal{U}^+) \cap \mathcal{V}^+ = \mathcal{U}^+$. Moreover $\mathcal{V}^+$ is proper.*

Proof: First assume that $\mathcal{V}^+$ is proper and generating. Let $\mathcal{A} \subset \mathcal{V}^+$ be order absolutely $\mathcal{F}$-convex, positively bounded and $\mathcal{F}$-absorbing. Assume that $\mathcal{S}(\mathcal{A}) \cap$

$\mathcal{V}^+ = \mathcal{A}$. Let $p(\cdot)$ be the gauge of $\mathcal{S}(\mathcal{A})$. We show that $p(\cdot)$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}^+$ such that $\mathcal{V}^+$ is $p$-closed. In the light of Lemma 2.14 and Theorem 2.15 it suffices to prove that $\mathcal{V}^+$ is $p$-closed. We shall show that $\mathcal{V}_{sa}\backslash\mathcal{V}^+$ is $p$-open. Define for $v \in \mathcal{V}_{sa}$,

$$r(v) = \inf\{\alpha \in \mathcal{R} \mid v + \alpha a \in \mathcal{V}^+ \text{ for some } a \in \mathcal{A}\}.$$

We first show that $r(v) \leq 0$ if and only if $v \in \mathcal{V}^+$. Let $v \in \mathcal{V}^+$. Then $v + 0a \in \mathcal{V}^+$ for all $a \in \mathcal{A}$. That is $r(v) \leq 0$. To show the other way let $r(v) \leq 0$. Then for every $n \in \mathbb{N}$ there exists an $a_n \in \mathcal{A}$ such that $v + (r(v) + \frac{1}{n})a_n \in \mathcal{V}^+$. Also $v + (r(v) + \frac{1}{n})a_n \leq v + (\frac{1}{n})a_n$ as $r(v) \leq 0$. That is $v + (\frac{1}{n})a_n \in \mathcal{V}^+$ for every $n \in \mathbb{N}$. As $\mathcal{A}$ is positively bounded, $v \in \mathcal{V}^+$. We now show that $p(v) - r(v) \geq 0$ for all $v \in \mathcal{V}_{sa}$. Suppose $p(v) - r(v) < 0$ for some $v \in \mathcal{V}_{sa}$. Put $\epsilon = \frac{1}{2}(r(v) - p(v)) > 0$. Since $p(\cdot)$ is $\mathcal{F}$-Riesz norm on $\mathcal{V}$, there exists an $a \in \mathcal{A}$ such that $(p(v) + \epsilon)a \pm v \in \mathcal{V}^+$. Then $(r(v) - \epsilon)a \pm v \in \mathcal{V}^+$. In particular $(r(v) - \epsilon)a + v \in \mathcal{V}^+$. This contradicts the definition of $r(v)$. Thus $p(v) \geq r(v)$ for all $v \in \mathcal{V}_{sa}$. Finally we show that $\mathcal{V}_{sa}\backslash\mathcal{V}^+$ is $p$-open. Let $v \in \mathcal{V}_{sa}$, $v \notin \mathcal{V}^+$. Since $v \notin \mathcal{V}^+$, $r(v) > 0$. Let $\delta = \frac{1}{2}r(v)$. Let $\mathcal{D} = \{w \in \mathcal{V}_{sa} \mid p(v - w) < \delta\}$. Let $w \in \mathcal{D}$. Then $\delta > p(v - w) \geq r(v - w)$. So there exists an $a \in \mathcal{A}$ such that $\delta a + (v - w) \in \mathcal{V}^+$. If $w \in \mathcal{V}^+$, then $\delta a + v \in \mathcal{V}^+$. Thus $r(v) \leq \delta = \frac{r(v)}{2}$, which is a contradiction. Therefore $w \notin \mathcal{V}^+$. That is $\mathcal{V}_{sa}\backslash\mathcal{V}^+$ is $p$-open.

For the converse it suffices to prove that $\mathcal{U}^+$ is positively bounded and that $\mathcal{V}^+$ is proper in light of Theorem 2.15. We show that $\mathcal{U}^+$ is positively bounded. Let $v \in \mathcal{V}^+$ and $w_n = v + k_n u_n \in \mathcal{V}^+$ for all $n \in \mathbb{N}$, where $\{u_n\}$ is a sequence in $\mathcal{U}^+$ and $\{k_n\}$ is a sequence in $(0, \infty)$ with $\inf k_n = 0$. Without loss of generality we can take $\{k_n\}$ to be decreasing. Now $\{w_n\}$ is a convergent sequence because $\|v - w_n\| = \|k_n u_n\| < k_n \longrightarrow 0$. Therefore $w_n \longrightarrow v$. Since $\mathcal{V}^+$ is closed, $v \in \mathcal{V}^+$. Therefore $\mathcal{U}^+$ is positively bounded.

Finally we show that $\mathcal{V}^+$ is proper. Let $\pm v \in \mathcal{V}^+$. Then as $v$ is self-adjoint, $\|v\| = \inf\{\|u\| \mid u \in \mathcal{V}^+, u \pm v \in \mathcal{V}^+\}$. Also $0 \in \mathcal{V}^+$ and $0 \pm v \in \mathcal{V}^+$. That is $\|v\| \leq \|0\| = 0$. That is $v = 0$. Therefore $\mathcal{V}^+$ is proper. □

Now we move to the final result of the paper.

**Definition 2.17** ($\mathcal{F}$-Riesz normed bimodule)**.** Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule such that $\mathcal{V}^+$ is proper and generating. Assume that $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ such that $\mathcal{V}^+$ is norm closed. Then the triple $\big(\mathcal{V}, \mathcal{V}^+, \|\cdot\|\big)$ is called an $\mathcal{F}$-Riesz normed bimodule.

**Definition 2.18** (Matricially Riesz normed space)**.** Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space and suppose that $\{\|\cdot\|_n\}$ is a matrix norm on $V$. Then the triplet $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$ is called a *matricially normed space* if for each $n \in \mathbb{N}$, $\|\cdot\|_n$ is a Riesz norm on $M_n(V)$ and $M_n(V)^+$ is closed.

**Theorem 2.19.** *Let* $\left(V, \left\{M_n(V)^+\right\}, \{\|\cdot\|_n\}\right)$ *be a matricially Riesz normed space. Let* $(\mathcal{V}, \mathcal{V}^+)$ *be the matricial inductive limit of the matrix ordered space* $\left(V, \left\{M_n(V)^+\right\}\right)$ *and let* $(\mathcal{V}, \|\cdot\|)$ *be the matricial inductive limit of matrix normed space* $(V, \{\|\cdot\|_n\})$. *Then* $\left(\mathcal{V}, \mathcal{V}^+, \|\cdot\|\right)$ *is a non-degenerate* $\mathcal{F}$*-Riesz normed bimodule. Conversely, let* $\left(\mathcal{W}, \mathcal{W}^+, \|\cdot\|\right)$ *be a non-degenerate* $\mathcal{F}$*-Riesz normed bimodule. Let* $W = 1_1 \mathcal{W} 1_1$ *and* $M_n(W)^+ = 1_n \mathcal{W}^+ 1_n$ *and* $\|\cdot\|_n = \|\cdot\| \,|_{M_n(W)}$ *for all* $n \in \mathbb{N}$. *Then* $\left(W, \left\{M_n(W)^+\right\}, \{\|\cdot\|_n\}\right)$ *is a matricially Riesz normed space whose inductive limit is* $\left(\mathcal{W}, \mathcal{W}^+, \|\cdot\|\right)$.

Proof: Let $\left(V, \left\{M_n(V)^+\right\}, \{\|\cdot\|_n\}\right)$ be a matricially Riesz normed space. We show that $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$. Let $v \in \mathcal{V}$. Then there exists a smallest $n \in \mathbb{N}$ such that $1_n v 1_n = v$. Then

$$\|v\| = \|v\|_n = \inf\{\max(\|u_1\|_n, \|u_2\|_n) \mid (u_1, u_2)_n^+ \pm s a_n(v) \in M_{2n}(V)^+ \text{ for some } u_1, u_2 \in M_n(V)^+\}.$$

Let

$$p(v) = \inf\{\max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm s a_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

Then $p(v) \le \|v\|$. Let $\epsilon > 0$. Then there exist $u_1, u_2 \in \mathcal{V}^+$, $N \in \mathbb{N}$ such that $(u_1, u_2)_N^+ \pm s a_N(v) \in \mathcal{V}^+$ and $\max(\|u_1\|, \|u_2\|) < p(v) + \epsilon$. In this case $N \ge n$. Put $u_1' = 1_n u_1 1_n$, $u_2' = 1_n u_2 1_n$. Then $u_1', u_2' \in M_n(V)^+$. Also

$$\left((1_n, 1_n)_n^+\right)^* \left[(u_1, u_2)_N^+ \pm s a_N(v)\right] \left((1_n, 1_n)_n^+\right) = (u_1', u_2')_n^+ \pm s a_n(v) \in M_{2n}(V)^+$$

as $1_n v 1_n = v$. Next $\left\|u_1'\right\|_n \le \|u_1\|$, $\left\|u_2'\right\|_n \le \|u_2\|$ so that

$$\|v\| = \|v\|_n \le \max(\left\|u_1'\right\|_n, \left\|u_2'\right\|_n) \le \max(\|u_1\|, \|u_2\|) < p(v) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\|v\| \le p(v)$. Therefore $p(v) = \|v\|$. Hence $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$. We show that $\mathcal{V}^+$ is $\|\cdot\|$ closed. Let $v \in \bar{\mathcal{V}}^+$. Then there exists a sequence $\{v_k\} \subset \mathcal{V}^+$ such that $v_k \longrightarrow v$ in $\|\cdot\|$. Hence $v \in \mathcal{V}_{sa}$. Find an $n \in \mathbb{N}$ such that $1_n v 1_n = v$. Then $v_k' = 1_n v_k 1_n \longrightarrow 1_n v 1_n = v$ in $\|\cdot\|_n$. Since $M_n(V)^+$ is closed, we have $v \in M_n(V)^+ \subset \mathcal{V}^+$. Therefore $\mathcal{V}^+$ is closed.

For the converse it is enough to show that $\|\cdot\|_n$ is a Riesz norm on $M_n(W)$ for all $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$ and $w \in M_n(W)$. Let

$$r(w) = \inf\{\max(\|u_1\|_n, \|u_2\|_n) \mid (u_1, u_2)_n^+ \pm s a_n(w) \in M_{2n}(W)^+ \text{ for some } u_1, u_2 \in M_n(W)^+\}.$$

Recall that

$$\|w\|_n = \|w\| = \inf\{\max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm s a_N(w) \in \mathcal{W}^+ \text{ for some } u_1, u_2 \in \mathcal{W}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

Then $\|w\|_n \le r(w)$. Let $\epsilon > 0$. Then as above using $(1_n, 1_n)_n^+$, we may conclude that $r(w) \le \|w\|_n + \epsilon$. Therefore $r(w) = \|w\|_n$. That is $\|\cdot\|_n$ is a Riesz norm on $M_n(W)$. Also $M_n(W)^+$ is $\|\cdot\|_n$ closed. □

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