Another proof of Derriennic's reverse maximal inequality for the supremum of ergodic ratios

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Abstract. Using the ratio ergodic theorem for a measure preserving transformation in a σ-finite measure space we give a straightforward proof of Derriennic's reverse maximal inequality for the supremum of ergodic ratios.

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1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T be a measure preserving transformation in $(X, \mathcal{F}, \mu)$. Given two measurable functions f and g on X such that $0 \le f,\, g \le \infty$ on X and $0 < \int_X g\, d\mu \le \infty$, let

$$s(f,g)(x) = \sup_{n \ge 0} \frac{\sum_{i=0}^{n} f(T^i x)}{\sum_{i=0}^{n} g(T^i x)}\,.$$

(Throughout this note we define $a/\infty = 0$ and $a/0 = \infty$ for any a, with $0 \le a \le \infty$.) In this note we use the ratio ergodic theorem to give a straightforward proof of the following reverse maximal inequality due to Derriennic [1] (cf. also Ornstein [5]). It is interesting to note that the author was inspired by reading Ephremidze's paper [3].

Theorem. *Suppose that T is conservative and ergodic, and that $\int_X f\, d\mu < \infty$. If $\alpha > \int_X f\, d\mu / \int_X g\, d\mu$, then, letting $E(\alpha) = \{x \mid s(f,g)(x) > \alpha\}$, we have*

$$\int_{E(\alpha)} f\, d\mu \le \alpha \int_{E(\alpha) \cup T^{-1} E(\alpha)} g\, d\mu.$$

Proof: We may assume that $\mu(E(\alpha)) > 0$. For $x \in X$, let $K(x) = \{n \ge 0 \mid T^n x \in E(\alpha)\}$ and $L(x) = \{0, 1, \dots\} \setminus K(x)$. Since T is conservative and ergodic, $K(x)$ is infinite for a.a. $x \in X$. To see that $L(x)$ is also infinite for a.a. $x \in X$, suppose there exists $k \ge 0$ such that $i \in K(x)$ for all $i \ge k$. Then clearly we have

$$\limsup_{l \to \infty} \frac{\sum_{i=k}^{l} f(T^i x)}{\sum_{i=k}^{l} g(T^i x)} \ge \alpha. \tag{1}$$

But this is a contradiction, since

$$(2)\qquad \lim_{l\to\infty}\frac{\sum_{i=k}^{l} f(T^i x)}{\sum_{i=k}^{l} g(T^i x)}=\frac{\int_X f\,d\mu}{\int_X g\,d\mu}<\alpha$$

for a.a. $x\in X$ by the ratio ergodic theorem (cf. Theorem 3.3.4 in [4]).

Since $K(x)$ and $L(x)$ are infinite for a.a. $x\in X$, we can write $K(x)=\bigcup_{n=1}^{\infty} I_n$ (disjoint union), where $I_n=[k_n,\ l_n]$ $(=\{i\mid k_n\le i\le l_n\})$ and $0\le k_n\le l_n<l_n+2\le k_{n+1}$ for each $n\ge 1$. Hence the set $J(x)=\{n\ge 0\mid T^n x\in E(\alpha)\cup T^{-1}E(\alpha)\}$ has the form

$$J(x)=\begin{cases}[0,\ l_1]\cup\bigcup_{n=2}^{\infty}[k_n-1,\ l_n] & \text{if } k_1=0,\\ \bigcup_{n=1}^{\infty}[k_n-1,\ l_n] & \text{if } k_1\ge 1.\end{cases}$$

Since $T^{k_n-1}x\notin E(\alpha)$ for $n\ge 2$, we have

$$(3)\qquad \frac{\sum_{i=k_n-1}^{l_n} f(T^i x)}{\sum_{i=k_n-1}^{l_n} g(T^i x)}\le\alpha \qquad (n\ge 2).$$

On the other hand, if h is a function in $L_1(\mu)$ such that $\int_X h\,d\mu=1$ and $0<h<\infty$ on X, then, by the ratio ergodic theorem,

$$(4)\qquad \lim_{n\to\infty}\frac{\sum_{i=0}^{n}(\chi_{E(\alpha)\cup T^{-1}E(\alpha)}f)(T^i x)}{\sum_{i=0}^{n} h(T^i x)}=\int_{E(\alpha)\cup T^{-1}E(\alpha)} f\,d\mu$$

and

$$(5)\qquad \lim_{n\to\infty}\frac{\sum_{i=0}^{n}(\chi_{E(\alpha)\cup T^{-1}E(\alpha)}g)(T^i x)}{\sum_{i=0}^{n} h(T^i x)}=\int_{E(\alpha)\cup T^{-1}E(\alpha)} g\,d\mu$$

for a.a. $x\in X$. Since $\sum_{i=0}^{\infty} h(T^i x)=\infty$ for a.a. $x\in X$, combining (3), (4) and (5) yields

$$(6)\qquad \int_{E(\alpha)\cup T^{-1}E(\alpha)} f\,d\mu\le\alpha\int_{E(\alpha)\cup T^{-1}E(\alpha)} g\,d\mu,$$

and this completes the proof, since $f\ge 0$ on X. □

2. Here we consider the case $g=1$ on X. Then it follows that $s(f,1)=f^*$, where $f^*(x)=\sup_{n\ge 1} n^{-1}\sum_{i=0}^{n-1} f(T^i x)$. In this case we have the following reverse maximal inequality.

Proposition. *If $\mu(X) = \infty$, T is ergodic (but not necessarily conservative), and f satisfies $\int_{\{f>t\}} f\,d\mu < \infty$ for all $t > 0$, then we have $\int_{\{f^*>\alpha\}} f\,d\mu \le 2\alpha\mu(\{f^* > \alpha\}) < \infty$ for all $\alpha > 0$.*

Proof: We first prove that $\mu(\{f^* > \alpha\}) < \infty$. To do this, let $f_1 = f\chi_{\{f\le\alpha/2\}}$ and $f_2 = f - f_1$. Then we have $f = f_1 + f_2$, $\|f_1\|_\infty \le \alpha/2$, and $\int_X f_2\,d\mu < \infty$. Since $f^* \le f_1^* + f_2^*$ and $\|f_1^*\|_\infty \le \alpha/2$, it follows that $\{f^* > \alpha\} \subset \{f_2^* > \alpha/2\}$, and by Hopf's maximal ergodic theorem (cf. Theorem 1.2.1 in [4])

$$\mu(\{f_2^* > \alpha/2\}) \le (2/\alpha)\int_{\{f_2^*>\alpha/2\}} f_2\,d\mu < \infty,$$

so that $\mu(\{f^* > \alpha\}) < \infty$. Putting $F = f - \alpha$, we then have $F^+ = (f-\alpha)^+ \in L_1(\mu)$ and $\{F^* > 0\} = \{f^* > \alpha\}$; furthermore $\int_X F\,d\mu = \int_X (f-\alpha)^+\,d\mu - \int_X (f-\alpha)^-\,d\mu = -\infty$ because $\mu(X) = \infty$. Hence by Theorem 1.4 in Ephremidze [2] we see that

$$\int_{\{f^*>\alpha\}\cup T^{-1}\{f^*>\alpha\}} (f-\alpha)\,d\mu \le 0.$$

Since $f \ge 0$ and $\mu(\{f^* > \alpha\}) < \infty$, we then have

$$\int_{\{f^*>\alpha\}} f\,d\mu \le \int_{\{f^*>\alpha\}\cup T^{-1}\{f^*>\alpha\}} f\,d\mu \le 2\alpha\mu(\{f^* > \alpha\}) < \infty,$$

completing the proof. □

Corollary. *If $\mu(X) = \infty$, and T is ergodic, then for any $\beta \ge 0$ we have*

$$\int_{\{f^*>t\}} f^*\left(\log\frac{f^*}{t}\right)^\beta d\mu < \infty \quad \text{for all } t > 0$$

if and only if

$$\int_{\{f>t\}} f\left(\log\frac{f}{t}\right)^{\beta+1} d\mu < \infty \quad \text{for all } t > 0.$$

Proof: See the proof of Theorem 2 in [6]. □

(Of course, as is known, this holds when $\mu(X) < \infty$, by the Theorem.)

References

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