Medial modes and rectangular algebras

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Abstract. Medial modes, a natural generalization of normal bands, were investigated by Płonka. Rectangular algebras, a generalization of rectangular bands (diagonal modes) were investigated by Pöschel and Reichel. In this paper we show that each medial mode embeds as a subreduct into a semimodule over a certain ring, and that a similar theorem holds for each Lallement sum of cancellative modes over a medial mode. Similar results are obtained for rectangular algebras. The paper generalizes earlier results of A. Romanowska, J.D.H. Smith and A. Zamojska-Dzienio.

Keywords: modes (idempotent and entropic algebras), cancellative modes, sums of algebras, embeddings, semimodules over semirings, idempotent subreducts of semimodules

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1. Introduction

Algebras called modes are investigated in the two monographs [9] and [11], where also further references can be found. They originated as a common generalization of affine spaces, convex sets and semilattices. In this paper, we are interested in the problem of embedding modes as subreducts into semimodules. One of the most efficient ways of describing the structure of an algebra is to embed it into another one, usually with a better known and richer structure. Such method appears to be quite successful in investigating the structure of modes. It is known that modes in many classes may be characterized as subreducts of semimodules over commutative semirings. By results of Ježek and Kepka proved in [1] one can deduce that each binary mode has this property. A similar result for so-called semilattice modes was obtained by Kearnes in [2]. In [12] A. Romanowska raised a question whether all modes are subreducts of semimodules over commutative semirings. Quite recently M. Stronkowski and D. Stanovský constructed negative examples but a general characterization of classes of modes embeddable into semimodules is still unclear. The paper continues earlier investigations on the subject conducted among others in the papers [10], [14] and [15], and in the doctoral dissertation [16].

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In [14], one introduced a certain special method of embedding semilattice sums of cancellative modes as subreducts into Płonka sums of affine spaces. As a corollary, one obtains an embedding of such semilattice sums into semimodules. A technique of embedding developed there is based on two facts: cancellative modes embed into appropriate modules (see Romanowska and Smith [10]), and then: a Płonka sum of modules over a certain ring is a semimodule over the same ring. It was shown that each so-called semilattice Lallement sum of cancellative modes embeds as a subreduct into a Płonka sum of certain affine spaces, and hence into a Płonka sum of the corresponding modules. Consequently, it embeds into a semimodule. We still do not know how far the assumptions of this method can be relaxed.

In [15], the above result was extended to the case of Lallement sums of cancellative modes over semigroup modes (i.e. normal bands). They also embed as subreducts into semimodules over certain rings. The proof involved certain new properties of functorial sums of algebras, and was done also by showing that the above mentioned Lallement sums are subalgebras of reducts of Płonka sums of modules.

In this paper we consider so-called medial modes, a certain generalization of normal bands, investigated by Płonka [6], and rectangular algebras, a generalization of rectangular bands (diagonal modes), investigated by Pöschel and Reichel [8], and also Lallement sums of cancellative modes over such algebras. We show that all such modes also embed into semimodules over some rings.

The paper is organized as follows. In Section 2, we recall basic definitions and results concerning modes, medial modes and rectangular algebras. Section 3 provides a brief survey of what we need about algebraic quasi-orders and sums of algebras. The main results concerning rectangular algebras and medial modes are proved in Sections 4 and 5. The last section provides results for Lallement sums of cancellative modes over medial modes and rectangular algebras.

The terminology and the notation of the paper is basically as in the books [9] and [11]. We refer the reader to those books for any otherwise undefined notions and further results. In particular, we use reverse Polish notation, i.e. terms (words) and operations are denoted by $x_1 \dots x_n f$ instead of $f(x_1, \dots, x_n)$ with the exception of traditional binary operations. It allows us to avoid writing too many brackets and makes formulas easier to read. The set of Ω-terms over X is denoted by $X\Omega$, the symbol $x_1 \dots x_n w$ means that $x_1, \dots, x_n$ are exactly the variables of w.

2. Medial modes and rectangular algebras

An algebra (A, Ω) of type $\tau : \Omega \longrightarrow \mathbb{Z}^+$ is called a *mode* if it is *idempotent* and *entropic*, i.e. each singleton in A is a subalgebra and each operation $\omega \in \Omega$ is actually a homomorphism from an appropriate power of the algebra. Both properties can also be expressed by the following identities:

(I) $\forall \omega \in \Omega,\ x \dots x\omega = x$

(E) $\forall \omega, \varphi \in \Omega$, with m-ary ω and n-ary φ,

$$(x_{11} \dots x_{1m}\omega) \dots (x_{n1} \dots x_{nm}\omega)\,\varphi = (x_{11} \dots x_{n1}\varphi) \dots (x_{1m} \dots x_{nm}\varphi)\,\omega,$$

satisfied in the algebra (A, Ω). A mode (A, Ω) is *cancellative* if for each (n-ary) ω in Ω, the algebra (A, Ω) satisfies the *cancellation law*

$$(a_1 \dots a_{i-1}\ x_i\ a_{i+1} \dots a_n\omega = a_1 \dots a_{i-1}\ y_i\ a_{i+1} \dots a_n\omega) \longrightarrow (x_i = y_i)$$

for each $i = 1, \dots, n$.

Let $\underline{\underline{\underline{M\tau}}}$ be the variety of all modes of a given type $\tau : \Omega \to (\mathbb{N} - \{0, 1\})$. Then the quotient ring

$$R(M\tau) = \mathbb{Z}[\{X_{\omega i} \mid \omega \in \Omega, 1 \le i \le \omega\tau\}]/\langle 1 - \sum_{i=1}^{\omega\tau} X_{\omega i} \mid \omega \in \Omega\rangle$$

is called the *affinization ring* for the variety $\underline{\underline{\underline{M\tau}}}$. For $\omega \in \tau^{-1}(n)$, the corresponding operation on an affine space over $R(M\tau)$ is

$$x_1 \dots x_n\omega = \sum_{i=1}^{n} x_i X_{\omega i}$$

for the indeterminates $X_{\omega 1}, \dots, X_{\omega n}$ pertaining to ω. The Ω-reducts of the affine $R(M\tau)$-spaces are in the variety $\underline{\underline{\underline{M\tau}}}$. Note that the ring $R(M\tau)$ is independent of the particular mode being embedded. It is the most general ring which can be used to embed all embeddable modes of the variety $\underline{\underline{\underline{M\tau}}}$ into corresponding affine spaces.

For cancellative modes the affine spaces over the ring $R(M\tau)$ play an essential role, since we have the following

Theorem 2.1 ([10], [11, Section 7.7])**.** *Each cancellative mode (C, Ω) of a fixed type $\tau : \Omega \to \mathbb{Z}^+$ embeds as an Ω-subreduct into an affine space $(G, P, \underline{R(M\tau)})$ over the ring $R(M\tau)$.*

In this paper we also consider Lallement sums of cancellative modes over medial modes and over rectangular algebras. (See Section 3.)

Definition 2.2 ([6])**.** An algebra (A, f) with one n-ary basic operation is called *medial*, if it satisfies the following identities:

$$x_{11} \dots x_{1n}f \dots x_{n1} \dots x_{nn}ff = x_{i_1 j_1} \dots x_{i_1 j_n}f \dots x_{i_n j_1} \dots x_{i_n j_n}ff$$

for every permutation $\{(i_1, j_1), (i_1, j_2), \ldots, (i_n, j_{n-1}), (i_n, j_n)\}$ of the set $\{(1,1), (1,2), \ldots, (n, n-1), (n, n)\}$ such that $(i_r, j_r) = (r, r)$ for $r = 1, \ldots, n$, and

$$(y_1 x_2 \ldots x_n f) y_2 \ldots y_n f = y_1 (x_2 y_2 x_3 \ldots x_n f) y_3 \ldots y_n f$$
$$= \ldots = y_1 \ldots y_{n-1} (x_2 x_3 \ldots x_n y_n f) f.$$

Note that in the case $n = 2$, the first identity coincides with the entropicity and the latter one reduces to the associativity. Note also that idempotent medial algebras are modes. They are sometimes called *medial modes* ([7]). For $n = 2$ medial modes are just normal bands. If we consider a reduct of normal band $(A, \cdot)$ with one n-ary operation f (for $n \geq 2$) defined as follows:

$$x_1 x_2 \ldots x_n f = x_1 \cdot x_2 \cdot \ldots \cdot x_n,$$

then the mode (A, f) (called normal $\{f\}$-band) is also a medial mode.

Medial modes are characterized by means of two other types of algebras.

Definition 2.3 ([4])**.** An idempotent algebra (A, d) with an n-ary operation d is called an *n-dimensional diagonal algebra* if it satisfies the diagonal identity

$$x_{11} \ldots x_{1n} d \ldots x_{n1} \ldots x_{nn} dd = x_{11} \ldots x_{nn} d.$$

For $n = 2$, diagonal algebras are precisely rectangular bands. Evidently diagonal algebras are modes.

Definition 2.4 ([5])**.** An algebra (A, r) with an n-ary operation r is called an *r_n-algebra* if (A, r) is the reduct of an abelian group $(A, +, -, 0)$ satisfying $(n-1)x = 0$ under the operation $x_1 \ldots x_n r = x_1 + \cdots + x_n$.

Note that each r_n-algebra (A, r) is a cancellative mode.

Theorem 2.5 ([6])**.** *An algebra (A, f) with one n-ary basic operation is a medial mode if and only if it is a Płonka sum of algebras, each of them being the direct product of one n-dimensional diagonal algebra and one r_n-algebra.*

Diagonal algebras are characterized by the following proposition.

Proposition 2.6 ([11, Section 5.2])**.** *Each n-ary diagonal mode (A, d) is a direct product of n projection subalgebras (A_i, d) satisfying the identity*

$$x_1 \ldots x_i \ldots x_n d = x_i.$$

For $n = 2$ Proposition 2.6 reduces to well known fact that each rectangular band is a direct product of a left-zero semigroup and a right-zero semigroup.

A further generalization of diagonal modes was considered by Pöschel and Reichel in [8] under the name of rectangular algebras. A mode (A, Ω) of any finite type $\tau : \Omega \to \mathbb{Z}^+$ is called a *rectangular algebra* if each operation ω in Ω satisfies the diagonal identity. A *projection algebra* is an algebra (B, Ω) for which every operation $\omega \in \Omega$ is a projection.

Theorem 2.7 ([8, Decomposition Theorem]). *Each rectangular algebra (A, Ω) is isomorphic to a finite direct product of projection algebras.*

Denote by $\underline{\underline{Re}}$ the variety of rectangular algebras of a finite type, where $\Omega = \{f_1, \dots, f_n\}$ for $n \geq 1$, and let $N = f_1\tau \cdot f_2\tau \cdot \dots \cdot f_n\tau$.

Theorem 2.8 ([8, Corollary 2.9]). *Up to isomorphism there are exactly N subdirectly irreducible algebras in the variety $\underline{\underline{Re}}$. Consequently $\underline{\underline{Re}}$ has 2^N subvarieties.*

3. Algebraic quasi-orders and sums of algebras

In [13] it was shown that each algebra (A, Ω) having a homomorphism h onto an idempotent, naturally quasi-ordered algebra (I, Ω) can be reconstructed as so-called (*generalized coherent*) *Lallement sum* of the corresponding fibres $h^{-1}(i)$ for $i \in I$ over (I, Ω). This construction generalizes the functorial (Agassiz) sum of algebras. We refer the reader to [11, Chapter 4] and [13], as well as to the paper [15], for definitions of various types of sums of algebras and their properties.

Let (I, Ω) be an algebra of type $\tau : \Omega \to \mathbb{N}$. The *algebraic quasi-order* of the algebra (I, Ω) is the quasi-order $\preceq$ defined on the set I as follows:

$$\begin{aligned} \preceq := \{(i, j) \mid \exists\, x_1 \dots x_n t \in X\Omega \ \text{ and } \ \exists\, i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \in I \\ \text{such that } \ j = i_1 \dots i_{k-1} i i_{k+1} \dots i_n t\}. \end{aligned}$$

(See e.g. [13] and [11, Chapter 4].) If additionally the algebra (I, Ω) satisfies the condition:

$$\text{if } \ a_i \preceq b_i, \ \text{ then } \ a_1 \dots a_{\omega\tau}\omega \preceq b_1 \dots b_{\omega\tau}\,\omega$$

for all ω in Ω and $a_1, \dots, a_{\omega\tau}, b_1, \dots, b_{\omega\tau}$ in I, then we say that it is *naturally quasi-ordered*. If $\preceq = I \times I$, then the algebra (I, Ω) has a *full* algebraic quasi-order. All idempotent (strongly) irregular algebras have full algebraic quasi-order and are naturally quasi-ordered. Note also that if (I, Ω) is an Ω-semilattice i.e. an Ω-reduct of a semilattice, then the algebraic quasi-order $\preceq$ of (I, Ω) coincides with the semilattice order $\leq$ defined by $x \leq y$ iff $xy = x$.

In the case of idempotent algebras, one can recognize whether they are naturally quasi-ordered also in a different way. On a quasi-ordered set $(I, \preceq)$ define a relation α as follows

$$(x, y) \in \alpha :\Leftrightarrow x \preceq y \ \text{ and } \ y \preceq x.$$

It is well known that α is an equivalence relation, and that the relation $x^\alpha \leq y^\alpha$ iff $x \preceq y$ is an ordering relation. Moreover the following holds.

Proposition 3.1 ([13], [11, Proposition 4.1.7.]). *An idempotent algebra (I, Ω) is naturally quasi-ordered iff the relation α is a congruence on (I, Ω) and (I^α, Ω) is an Ω-semilattice.*

Corollary 3.2 ([13, Examples 2.4, 2.5])**.** *The Płonka sum of algebras with full algebraic quasi-orders is naturally quasi-ordered.*

We are especially interested in Lallement sums embeddable into functorial sums. We will denote a Lallement sum of algebras (B_i, Ω) over an idempotent naturally quasi-ordered algebra (I, Ω) by $\mathfrak{L}_{i\in I}(B_i, \Omega)$ and a functorial sum (B, Ω) of (B_i, Ω) over (I, Ω) by $(B, \Omega) = \sum_{i\in I}(B_i, \Omega)$, similarly as in [15].

Theorem 3.3 ([13], [11, Theorem 7.4.2])**.** *Let a mode (B, Ω) be a Lallement sum of cancellative modes (B_i, Ω) over a naturally quasi-ordered mode (I, Ω). Then (B, Ω) is a subalgebra of a functorial sum (E, Ω) of cancellative envelopes (E_i, Ω) of (B_i, Ω) over (I, Ω)*

$$(B, \Omega) = \mathfrak{L}_{i\in I}(B_i, \Omega) \leq \sum_{i\in I}(E_i, \Omega).$$

The cancellative envelopes (E_i, Ω) are extensions of cancellative modes (B_i, Ω) built in a certain canonical way. (See [11, Section 7.4].)

In [14] the above result was used together with Theorem 2.1 to prove the following

Theorem 3.4 ([14])**.** *Let a mode (B, Ω) be a semilattice sum of cancellative modes (B_i, Ω) over a semilattice (I, Ω). Then (B, Ω) is a subreduct of a Płonka sum of affine $R(M\tau)$-spaces.*

Corollary 3.5 ([14])**.** *Let a mode (B, Ω) be a semilattice sum of cancellative modes. Then (B, Ω) embeds as a subreduct into a semimodule over a ring.*

Note that a direct product of an algebra (A, Ω) and an idempotent algebra (I, Ω) can always be considered as a functorial sum of isomorphic Ω-algebras $A_i = A \times \{i\}$, for $i \in I$, over the algebra I. On the other hand, assume that in a functorial sum $\sum_{i\in I}(A_i, \Omega)$ the indexing algebra (I, Ω) has a full algebraic quasi-order. Then for any two i, j in I, the summands (A_i, Ω) and (A_j, Ω) are isomorphic, and the functorial sum is isomorphic to the direct product of (A_i, Ω) and (I, Ω), i.e.

$$(A, \Omega) = \sum_{i\in I}(A_i, \Omega) \cong (A_i, \Omega) \times (I, \Omega). \tag{3.5.1}$$

(See [3].)

The functorial sums has a special property which resembles "associativity".

Theorem 3.6 ([15])**.** *Let (E, Ω) be a functorial sum of algebras (E_n, Ω) over an algebra (N, Ω). Let (N, Ω) be a functorial sum of algebras (N_s, Ω) over an algebra*

(S,Ω). Then (E,Ω) is a functorial sum of algebras (B_s,Ω) over the algebra (S,Ω), where (B_s,Ω) is a functorial sum of (E_n,Ω) over (N_s,Ω). Briefly:

$$\sum (E_n \mid n \in \sum_{s\in S} N_s) = \sum_{s\in S} (\sum (E_n \mid n \in N_s)). \tag{3.6.1}$$

4. Embedding rectangular algebras into modules

Let $\underline{\underline{Re}}$ be a variety of rectangular algebras of a given (finite) type, where $\Omega = \{f_1,\dots,f_n\}$ with $n \geq 1$, and $N = f_1\tau \cdot f_2\tau \cdot \dots \cdot f_n\tau$. By Corollary 2.8, $\underline{\underline{Re}}$ has N subvarieties of projection algebras. In fact, the sets of operations of algebras in each of these subvarieties differ only by the combination of projections, i.e. $\Omega = \{f_{ik} \mid i = 1,\dots,n,\ k = 1,\dots,f_i\tau\}$ with each operation defined as follows

$$f_{ik} : A^{f_i\tau} \longrightarrow A;\ \left(a_1,\dots,a_{f_i\tau}\right) \longmapsto a_k.$$

Obviously, for each k, we have $f_{ik}\tau = f_i\tau$. We denote by $\underline{\underline{P_{j_1j_2\dots j_n}}}$, where $j_i \in \{1,\dots,f_i\tau\}$, the variety of projection algebras with the set of operations $\{f_{ij_i} \mid i = 1,\dots n\}$, i.e. i-th operation is the projection on the j_i coordinate.

We start with constructing the affinization rings (see Section 2) for the variety of rectangular algebras and for its subvarieties of projection algebras.

Lemma 4.1. *The following rings are the affinization rings for the varieties $\underline{\underline{P_{j_1j_2\dots j_n}}}$ and $\underline{\underline{Re}}$:*

$$R(P_{j_1j_2\dots j_n}) = \mathbb{Z}\left[X_{11},\dots,X_{1f_1\tau},\dots,X_{n1},\dots,X_{nf_n\tau}\right] /$$
$$\langle X_{i1},\dots,1-X_{ij_i},\dots,X_{if_i\tau}, 1-\sum_{j=1}^{f_i\tau} X_{ij} \mid i = 1,\dots,n\rangle,$$
$$R(Re) = \mathbb{Z}\left[X_{11},\dots,X_{1f_1\tau},\dots,X_{n1},\dots,X_{nf_n\tau}\right] /$$
$$\langle X_{ij}X_{ik}, X_{ij}(1-X_{ij}), 1-\sum_{j=1}^{f_i\tau} X_{ij} \mid i = 1,\dots,n,\ j,k = 1,\dots,f_i\tau,\ j \neq k\rangle.$$

Proof: The rings are calculated as follows. First consider the variety $\underline{\underline{P_{j_1j_2\dots j_n}}}$ of projection algebras with j_i-th projections as the basic operations f_{ij_i}. Note that we can equate coefficients in each projection identity separately, so to simplify calculations assume that we consider i-th operation which is a projection on j-th coordinate. Let X_{ik}, for $k = 1,\dots,f_i\tau$ be the indeterminates pertaining to the operations f_{ij}. Equating coefficients in

$$x_1X_{i1} + x_2X_{i2} + \dots + x_{f_i\tau}X_{if_i\tau} = x_1x_2\dots x_{f_i\tau}f_{ij} = x_j$$

shows that $x_jX_{ij} = x_j$ and $x_kX_{ik} = 0$ for $k \neq j$ and $k = 1,\dots,f_i\tau$. By idempotency $\sum_{k=1}^{f_i\tau} X_{ik} = 1$. Whence the ring $R(P_{j_1j_2\dots j_n})$ is a quotient of

$$\mathbb{Z}\left[X_{11},\dots,X_{1f_1\tau},\dots,X_{n1},\dots,X_{nf_n\tau}\right] /$$
$$\langle X_{i1},\dots,1-X_{ij_i},\dots,X_{if_i\tau},1-\sum_{j=1}^{f_i\tau} X_{ij} \mid i=1,\dots,n\rangle.$$

Conversely, taking an affine space over the ring $R(P_{j_1j_2\dots j_n})$ for $j_i \in \{1,\dots,f_i\tau\}$, we obtain a projection algebra in $\underline{\underline{P_{j_1j_2\dots j_n}}}$ under the operations

$$x_1\dots x_{\omega_{ij}\tau}\omega_{ij} := x_1X_{i1} + \dots + x_{\omega_{ij}\tau}X_{i(\omega_{ij}\tau)} = x_j.$$

It follows that $R(P_{j_1j_2\dots j_n}) = \mathbb{Z}\left[X_{11},\dots,X_{1f_1\tau},\dots,X_{n1},\dots,X_{nf_n\tau}\right] / \langle X_{i1},\dots,1-X_{ij_i},\dots,X_{if_i\tau},1-\sum_{j=1}^{f_i\tau} X_{ij} \mid i=1,\dots,n\rangle$.

Now consider the variety $\underline{\underline{Re}}$ of rectangular algebras. The indeterminates are defined as before. Again, we can equate coefficients in diagonal identity for each operation separately. We obtain that $x_{jj}X_{ij}^2 = x_{jj}X_{ij}$ and $x_{jk}X_{ij}X_{ik} = 0$ for $k \neq j$. Then the ring $R(Re)$ is a quotient of

$$\mathbb{Z}\left[X_{11},\dots,X_{1f_1\tau},\dots,X_{n1},\dots,X_{nf_n\tau}\right] /$$
$$\langle X_{ij}X_{ik}, X_{ij}(1-X_{ij}), 1-\sum_{j=1}^{f_i\tau} X_{ij} \mid i=1,\dots,n,\ j,k=1,\dots,f_i\tau,\ j\neq k\rangle.$$

Similarly as in the case of projection algebras one shows that each affine space over the ring $R(Re)$ is a rectangular algebra. As the corresponding operation ω_i one takes the same operation as for projection algebras. □

Lemma 4.2. *Each rectangular algebra (R,Ω) embeds as a subreduct into a module over the ring $R(Re)$.*

PROOF: By Theorem 2.7, $(R,\Omega) = \prod_{s=1}^N (P_s,\Omega)$ where each (P_j,Ω) is a projection algebra. Since the structure of projection algebras is very simple we can consider each P_s as the set of free generators of a free module $M(P_s)$ over the appropriate affinization ring $R(P_{j_1j_2\dots j_n})$ (as well as over the ring $R(Re)$). In this way one obtains embedding of the rectangular algebra R into the $R(Re)$-module $M(R) := \prod_{s=1}^N M(P_s)$. □

Proposition 4.3. *The Płonka sum of rectangular algebras (R_i,Ω) over an Ω-semilattice (I,Ω) embeds as a subreduct into a semimodule over the ring $R(Re)$.*

PROOF: Let $(R_i,\Omega) = \prod_{s=1}^N (P_{s,i},\Omega)$ for $i \in I$. Each sum homomorphism $f_{i,j} : R_i \to R_j$ is uniquely determined by an N-tuple of functions $f_{i,j}^s : P_{s,i} \to P_{s,j}$

for $s = 1, \ldots, N$. By universality property for free modules each mapping $f^s_{i,j}$ extends to a (uniquely defined) module homomorphism $\overline{f}^s_{i,j} : M(P_{s,i}) \to M(P_{s,j})$ such that $\overline{f}^s_{i,j}|_{P_{s,i}} = f^s_{i,j}$.

Note that each free module obtained in the proof of the previous lemma, under the operations

$$x_1 \cdots x_{f_i\tau} f_i = \sum_{j=1}^{f_i\tau} x_j X_{ij},$$

for $i = 1, \cdots, n$, is a rectangular algebra, so R_i embeds into the rectangular algebra $M(R_i)$.

There exists a unique module homomorphism $\overline{f}_{i,j} : M(R_i) \to M(R_j)$, for $i \leq j$, determined by the N-tuple $\overline{f}^s_{i,j}$, uniquely extending the sum homomorphism $f_{i,j}$. The homomorphisms $\overline{f}_{i,j}$ are functorial (i.e. $\overline{f}_{i,j}\overline{f}_{j,k} = \overline{f}_{i,k}$ for all $i \leq j \leq k$ in (I, Ω)) and determine a Płonka sum structure on the disjoint union of the modules $M(R_i)$, each of them over the same ring $R(Re)$. Now the Płonka sum of these modules is a semimodule over the ring $R(Re)$. □

Lemma 4.4. *The affinization rings for the varieties of projection algebras are isomorphic to the ring $\mathbb{Z}$. The affinization ring $R(Re)$ is isomorphic to the ring $\mathbb{Z}^N$.*

Proof: We obtain these results by using the First Isomorphism Theorem for rings. For projection algebras in the subvariety $\underline{\underline{P_{j_1j_2\ldots j_n}}}$ define the ring homomorphism $h_{j_1j_2\ldots j_n} : \mathbb{Z}\left[X_{11}, \ldots, X_{1f_1\tau}, \ldots, X_{n1}, \ldots X_{nf_n\tau}\right] \to \mathbb{Z}$ by sending a polynomial w onto its value in $X_{ij_i} = 1$ for $i = 1, \ldots, n$ and $X_{ik} = 0$ for all $k \neq j_i$. Clearly, $\ker h_{j_1j_2\ldots j_n}$ is the ideal from the definition of $R(P_{j_1j_2\ldots j_n})$ and thus $R(P_{j_1j_2\ldots j_n})$ is isomorphic to $\mathbb{Z}$.

For rectangular algebras we define the ring homomorphism

$$h : \mathbb{Z}\left[X_{11}, \ldots, X_{1f_1\tau}, \ldots, X_{n1}, \ldots X_{nf_n\tau}\right] \to \mathbb{Z}^N$$

in such a way that every polynomial w maps to the N-tuple of coefficients $wh_{j_1j_2\ldots j_n}$ for all combinations of $j_i \in \{1, \ldots, f_i\tau\}$. □

Corollary 4.5. *Each rectangular algebra (R, Ω) embeds as a subreduct into a module over the ring $\mathbb{Z}^N$.*

Corollary 4.6. *The Płonka sum of rectangular algebras (R_i, Ω) over an Ω-semilattice (I, Ω) embeds as a subreduct into a semimodule over the ring $\mathbb{Z}^N$.*

Rectangular algebras with one n-ary operation are known as diagonal modes. The varieties of n-ary diagonal modes may be described similarly as the varieties of rectangular semigroup modes (see e.g. [11, Section 5.2]). First note that an algebra with one n-ary operation of i-th projection is a diagonal algebra. Let $\underline{\underline{D_n}}$

be the variety of n-ary diagonal algebras, and let $\underline{\underline{P_i}}$ be its subvariety of projection algebras with i-th projection as the basic operation. By Theorem 2.8, the lattice of subvarieties of $\underline{\underline{D_n}}$ is a Boolean lattice with the subvarieties $\underline{\underline{P_i}}$ being its atoms. For this special case we obtain the following

Corollary 4.7. *The following rings are the affinization rings for the varieties $\underline{\underline{P_i}}$ and $\underline{\underline{D_n}}$:*

$$\begin{aligned}
R(P_i) &= \mathbb{Z}\,[X_1, \dots, X_n]\,/\langle X_1, \dots, X_{i-1}, 1 - X_i, X_{i+1}, \dots, X_n, 1 - \sum_{j=1}^{n} X_j\rangle, \\
&\quad \textit{for} \ \ i = 1, \dots, n, \\
R(D_n) &= \mathbb{Z}\,[X_1, \dots, X_n]\,/ \\
&\quad\quad \langle X_i X_j, X_i(1 - X_i), 1 - \sum_{j=1}^{n} X_j \mid i, j = 1, \dots, n, \ i \neq j\rangle.
\end{aligned}$$

Corollary 4.8. *For each $i = 1, \dots, n$, the affinization ring $R(P_i)$ is isomorphic to the ring $\mathbb{Z}$. The affinization ring $R(D_n)$ is isomorphic to the ring $\mathbb{Z}^n$.*

To prove this result we use the same method as in the proof of Lemma 4.4. However in the case of one n-ary operation, the whole procedure is easier to describe and provides a good example. Let h be the ring homomorphism of the polynomial ring $\mathbb{Z}\,[X_1, \dots, X_n]$ onto the ring $\mathbb{Z}$ defined as follows

$$wh := w(0, 0, \dots, 1, 0, \dots 0)$$

(with 1 on the i-th position, for $i = 1, \dots, n$), where

$$w = a_0 + \sum_{i=1}^{n} a_i X_i + \sum_{i,j=1}^{n} a_{ij} X_i X_j + \sum_{i,j,k=1}^{n} a_{ijk} X_i X_j X_k + \dots$$

and all coefficients a_i are integers. It means that $wh = a_0 + a_i + a_{ii} + a_{iii} + \dots$.

In the case of diagonal algebras, the ring homomorphism g of the polynomial ring $\mathbb{Z}\,[X_1, \dots, X_n]$ onto the ring $\mathbb{Z}^n$ is defined as follows

$$\begin{aligned}
wg :&= (w(1, 0, \dots 0), \dots, w(0, \dots, 0, 1)) \\
&= (a_0 + a_1 + a_{11} + a_{111} + \dots, \dots, a_0 + a_n + a_{nn} + \dots).
\end{aligned}$$

Corollary 4.9. *Each n-ary diagonal mode (D, f) embeds as a subreduct into a module over the ring $\mathbb{Z}^n$.*

Corollary 4.10. *The Płonka sum of n-ary diagonal algebras (D_i, f) over the $\{f\}$-semilattice (I, f) embeds as a subreduct into a semimodule over the ring $\mathbb{Z}^n$.*

The situation described in Corollary 4.10 refers precisely to medial modes defined by certain additional identity.

Proposition 4.11 ([6]). *A medial mode (A, f) with n-ary operation f is the Płonka sum of diagonal algebras if and only if it satisfies the identity*

$$(x_1 \dots x_n f) x_2 \dots x_n f = x_1 \dots x_n f.$$

5. Embedding medial modes into semimodules

Return to the concept of r_n-algebras. By definition, such algebras are reducts of abelian groups in the variety defined by the identity $(n-1)x = 0$. This variety is equivalent to the variety of modules over the ring $\mathbb{Z}_{n-1}$. Note that modules over $\mathbb{Z}_{n-1}$ can also be considered as modules over the ring $\mathbb{Z}$ or the ring $\mathbb{Z}^n$.

Example 5.1. It is known (see [5]) that each symmetric medial mode (A, f) is the Płonka sum of r_n-algebras. Recall that a medial mode is symmetric if it satisfies the additional identity

$$x_1 \dots x_n f = x_{i_1} \dots x_{i_n} f,$$

for each permutation $\{i_1, \dots, i_n\}$ of the set $\{1, \dots, n\}$. As a reduct of a Płonka sum of modules, (A, f) is a reduct of a semimodule over the ring $\mathbb{Z}_{n-1}$. In this way we obtain another example of a class of modes embeddable into semimodules over a ring.

In what follows we will show that each medial mode embeds as a subreduct into a semimodule over the ring $\mathbb{Z}^n$.

Proposition 5.2. *Let (M, f) be a medial mode. Then (M, f) embeds into a semimodule over the ring $\mathbb{Z}^n$.*

Proof: By Theorem 2.5, $M = \sum_{i \in I}(D_i \times R_i)$, where D_i is a diagonal algebra, R_i is an r_n-algebra and I is an $\{f\}$-semilattice. Each summand can be considered as a functorial sum $\sum_{r \in R_i} D_r$ of pairwise isomorphic $D_r = D_i \times \{r\}$. Note that if in Theorem 3.6 we first assume that the algebra E is equal to the right hand side of the equality (3.6.1) instead of left one, then one can easily show, that the equality remains true, and the following holds

$$M = \sum_{i \in I}(D_i \times R_i) = \sum_{i \in I}\Big(\sum_{r \in R_i} D_r\Big) = \sum\Big(D_r \mid r \in \sum_{i \in I} R_i\Big).$$

It follows that there exist sum homomorphisms $\psi_{i,j} : R_i \to R_j$ for each pair (i, j) with $i \preceq j$ and $\varphi_{r,s} : D_r \to D_s$ for each pair (r, s) with $r \preceq s$. (Note that for $r, s \in R_i$ $\varphi_{r,s}$ is simply isomorphism.) Now the sum homomorphisms $h_{i,j}$: $\sum_{r\in R_i} D_r \to \sum_{s\in R_j} D_s$ can be defined as $xh_{i,j} := x\varphi_{r,r\psi_{i,j}}$ for each $r \in R_i$ and $x \in D_r$. By Corollary 4.9, each diagonal algebra D_i embeds as a subreduct into the module $M(D_i)$ over the ring $\mathbb{Z}^n$. Similarly, each r_n-algebra R_i is a reduct of the module R_i over the same ring. It follows that their product embeds as a subreduct into the $\mathbb{Z}^n$-module $M(D_i) \times R_i$. Now we need to extend the homomorphisms $h_{i,j}$ to functorial module homomorphisms $\overline{h}_{i,j}$. Again, we consider each direct product $M(D_i) \times R_i$ as a functorial sum $\sum_{r\in R_i} M(D_r)$. Similarly, as in the proof of Proposition 4.3 each homomorphism $\varphi_{r,s}$ extends to a unique module homomorphism $\overline{\varphi}_{r,s} : M(D_r) \to M(D_s)$ that satisfies the functoriality condition. Each homomorphism of r_n-algebras is also a module homomorphism. So take a (module) homomorphism $x\overline{h}_{i,j} := x\overline{\varphi}_{r,r\psi_{i,j}}$ for each $r \in R_i$ and $x \in M(D_r)$. Since both homomorphisms $\overline{\varphi}_{r,s}$ and $\psi_{i,j}$ are functorial, so is $\overline{h}_{i,j}$. It follows that $\sum_{i\in I}(M(D_i) \times R_i)$ is a Płonka sum of modules over the ring $\mathbb{Z}^n$ and hence a semimodule over the same ring. And the medial mode (M, f) embeds into this semimodule. □

6. Lallement sums of cancellative modes over medial modes and over rectangular algebras

In this section we consider an embedding of a Lallement sum of cancellative modes (A_m, f) over a medial mode (M, f) as a subreduct into a semimodule over the ring $R(M\tau) = \mathbb{Z}\left[X_1, \dots, X_{f\tau}\right] / \langle 1 - \sum_{i=1}^{f\tau} X_i\rangle = \mathbb{Z}\left[X_1, \dots, X_{f\tau-1}\right]$. First we will show that each medial mode (M, f) is naturally quasi-ordered so it satisfies the assumptions of Theorem 3.3. By Corollary 3.2 it is enough to show that each summand $M_i = D_i \times R_i$ has the full algebraic quasi-order. Indeed, it is easy to check that each diagonal algebra (D, f) satisfies the identities

$$x = xx\dots xyfx\dots xf = x(xx\dots xyf)x\dots xf = \dots = x\dots x(x\dots yxf)f,$$

and that each r_n-algebra (R, f) satisfies the identity

$$x = xy\dots yf.$$

Now for any two elements (x_1, y_1) and (x_2, y_2) in M_i we have the following

$$\begin{aligned}(x_2, y_2) &= (x_2(x_2\dots x_2x_1f)x_2\dots x_2f, y_2(y_1\dots y_1f)y_1\dots y_1f)\\ &= (x_2, y_2)(x_2\dots x_2x_1f, y_1\dots y_1f)(x_2, y_1)\dots(x_2, y_1)f\\ &= (x_2, y_2)((x_2, y_1)(x_2, y_1)\dots(x_2, y_1)(x_1, y_1)f)\dots(x_2, y_1)f.\end{aligned}$$

This shows that $(x_1, y_1) \preceq (x_2, y_2)$. Hence $\preceq$ is a full algebraic quasi-order on the summand M_i. It follows that each medial mode (M, f) is naturally quasi-ordered. The congruence α defined in Section 3 provides the decomposition of (M, f) into algebras (M_i, f), and the quotient M^α is an $\{f\}$-semilattice. (See Proposition 3.1.)

Proposition 6.1. *Let (A, f) be a Lallement sum of cancellative modes (A_m, f) over a medial mode (M, f). Then (A, f) embeds as a subreduct into a semimodule over the ring $\mathbb{Z}\left[X_1, \dots, X_{f\tau}\right] / \langle 1 - \sum_{i=1}^{f\tau} X_i \rangle$.*

Proof: By Theorem 3.3, the algebra A is a subalgebra of a functorial sum E of cancellative envelopes E_m of A_m over M with sum homomorphisms $g_{m,n}$. Let $M = \sum_{i \in I} M_i$, where $M_i = D_i \times R_i$ as defined above, with sum homomorphisms $h_{i,j}$. By Theorem 3.6 and the formula (3.5.1)

$$\sum (E_m \mid m \in \sum_{i \in I} M_i) = \sum_{i \in I} (E_m \times M_i).$$

By Theorem 2.1, each algebra E_m embeds as a subreduct into an $R(M\tau)$-module G_m. In this case $R(M\tau) = \mathbb{Z}\left[X_1, \dots, X_{f\tau}\right] / \langle 1 - \sum_{i=1}^{f\tau} X_i \rangle$. Each mode (M_i, f) embeds as a subreduct into an $\mathbb{Z}^{f\tau}$-module $M(M_i) = M(D_i) \times R_i$ which can be considered as an $R(M\tau)$-module since $\mathbb{Z}^{f\tau}$ is a homomorphic image of $R(M\tau)$ (see Corollary 4.8). In this way one obtains an embedding of the algebra $E_m \times M_i$ as a subreduct of the $R(M\tau)$-module $G_m \times M(M_i)$. Each homomorphism $g_{m,n} : E_m \to E_n$ extends to a homomorphism $\overline{g}_{m,n} : G_m \to G_n$ and each homomorphism $h_{i,j} : M_i \to M_j$ extends to a homomorphism $\overline{h}_{i,j} : M(M_i) \to M(M_j)$. And all $\overline{g}_{m,n}$ and $\overline{h}_{i,j}$ are functorial module homomorphisms. Now we consider each direct product $G_m \times M(M_i)$ as a functorial sum $\sum_{m_i \in M(M_i)} G_{m_i}$, and for each $m_i \in M(M_i)$ and $x \in G_{m_i}$ we define the mapping $\overline{f}_{i,j}$ as $x\overline{f}_{i,j} = x\overline{g}_{m_i, m_i\overline{h}_{i,j}}$. The mappings $\overline{f}_{i,j}$ are module homomorphisms satisfying the functoriality condition so they define the Płonka sum of $R(M\tau)$-modules $G_m \times M(M_i)$ over the $\{f\}$-semilattice I. In this way we obtain the semimodule over the ring $R(M\tau)$ and the algebra A is its subreduct. □

Note that the variety $\underline{\underline{Re}}$ is an idempotent irregular variety and hence the class of Płonka sums of algebras in $\underline{\underline{Re}}$ coincides with the regularization $\widetilde{\underline{\underline{Re}}}$ of $\underline{\underline{Re}}$. (See Płonka's Theorem in [7] or [11].) It follows also that each rectangular algebra has a full algebraic quasi-order. As a result we obtain the following

Proposition 6.2. *Let (A, Ω) be a Lallement sum of cancellative modes (A_r, Ω) over a rectangular algebra (R, Ω). Then (A, Ω) embeds as a subreduct into a module over an appropriate ring $R(M\tau)$.*

Proof: By Theorem 3.3 and the formula (3.5.1)

$$\mathcal{L}_{r \in R}(A_r, \Omega) \leq \sum_{r \in R} (E_r, \Omega) \cong (E_r, \Omega) \times (R, \Omega).$$

By Corollary 4.5, (R, Ω) embeds as a subreduct into a $\mathbb{Z}^N$-module which can be considered as an $R(M\tau)$-module. Together with Theorem 2.1 it gives an embedding of the direct product $(E_r, \Omega) \times (R, \Omega)$ into a module over the required ring. □

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